A COMPARISON BETWEEN A NEW WAVELET PRECONDITIONER FOR FINITE DIFFERENCE OPERATORS AND SOME OTHER MULTILEVEL PRECONDITIONERS IN 1D

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Abstract: In this paper, we describe a new wavelet preconditioner for finite difference operators [5] and compare it to more classical multilevel preconditioners, precisely those constructed by J.H. Bramble, J.E. Pasciak and J. Xu [1] and by H. Yserentant [8]. Numerical comparisons are presented for mono-dimensional problems.

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1. Introduction

Finite difference operators, used classically in the approximation of differential operators during the approximation of partial differential equation solutions, very often suffer from ill conditioning (a matrix $A$ is said to be ill conditioned

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if its condition number $\kappa(A) = \| A \| \cdot \| A^{-1} \| \text{ satisfies } \kappa(A) >> 1$, where $\| \cdot \|$ is standing for a norm). This property induces error amplifications during the resolution of associated linear systems and poor convergence rates of iterative methods such as the conjugate gradient method.

To cure this ill conditioning property, so called preconditioners are usually employed to transform the initially badly conditioned problem into an other one, admitting the same solution, but being better conditioned (C is said to be a preconditioner for $A$ if $\kappa(CA) \ll \kappa(A)$).

In this paper, we compare several preconditioning techniques for finite difference approximations of variable coefficient elliptic operators. All these methods share a multilevel property in the sense that a hierarchical (or multi-scale) decomposition of functions or operators is involved in the construction of the preconditioner.

In Section 2, we first introduce a new wavelet preconditioner, which takes advantages of the good localization properties of wavelets as well as their Fourier transform. In this construction, the basic ingredients of a multi-scale construction of the inverse of a variable coefficient elliptic differential operator derived by P. Tchamitchian [7] have been used. However, since it is devoted to finite difference approximation preconditioning, wavelets are not used, in this construction, as generators of a space of approximation, but as tools for preconditioning a given finite difference problem.

In Section 3, we briefly compare this preconditioner to the BPX preconditioner constructed by J.H. Bramble, J.E. Pasciak and J. Xu [1], and to the one defined by H. Yserentant [8].

Section 4 is devoted to numerical results in 1D.

The construction of the new wavelet preconditioner is described in the framework of a finite difference approximation of the following basic problem:

\[
\text{Find } u \text{ such that } Lu = -\frac{d}{dx} \left( a(x) \frac{d}{dx} u \right) + u = f, \tag{1}
\]

where $f \in L^2(\mathbb{R})$. To ensure the ellipticity of the operator $L$, the coefficient $a(\cdot)$ is assumed to satisfy,

\[
\exists a_{\min}, a_{\max} \in \mathbb{R}; \quad \forall x \in \mathbb{R}, \quad 0 < a_{\min} \leq a(x) \leq a_{\max} < \infty. \tag{2}
\]

We call $L_J$ a finite difference approximation of $L$ on $N_J$, a regular grid of mesh size $h = 2^{-J}$ ($N_J = \{x_i = ih = \frac{i}{2^J}, i \in \mathbb{Z}\}$).

The resulting system of linear equations reads

\[
A_J u_J = f_J, \tag{3}
\]
where $A_J$ is an infinite matrix, $u_J$ is the vector of the approximations of \( \{ u(x_i), i \in \mathbb{Z} \} \) and $f_J$ is classically either \( \{ f(x_i), i \in \mathbb{Z} \} \) when $f$ is continuous or \( \{ f \ast \rho_J(x_i), i \in \mathbb{Z} \} \) with $\rho_J(\cdot)$ a suitable smoothing kernel.

Note that under the ellipticity condition (2), the operator $L$, as well as the operator $L_J$ is both invertible. Therefore, the matrix $A_J$ is non singular.

Classically, one gets that $\kappa(A_J) = O(4^J)$, with a constant depending on $a(\cdot)$.

### 2. A New Wavelet Multilevel Preconditioner

We first describe, following [5], the new wavelet preconditioner for $L_J$.

We first introduce $(V_j, \tilde{V}_j)_{j \in \mathbb{Z}}$, a biorthogonal multi-resolution analysis of $L^2(\mathbb{R})$, where the primal multi-resolution analysis $(V_j)_{j \in \mathbb{Z}}$, is of order $d > 2$ (i.e. reproduces polynomials of degree $d$), integer-interpolatory and associated to a compactly supported wavelet $\psi$ [4, 2]. We recall that in that framework their exists a wavelet $\psi$ such that

$$V_j = \text{span} \left\{ \psi_{l,k}(\cdot) = 2^l \psi(2^l \cdot - k), l \leq j - 1, k \in \mathbb{Z} \right\},$$

and that for any sequence $\{ \alpha_k \}_{k \in \mathbb{Z}}$, with $\alpha_k \in \mathbb{R}$, there exists a unique function $I_\alpha \in V_j$ such that $I_\alpha(k2^{-j}) = \alpha_k$. Moreover, if $I_\alpha = \sum_{l,k} d_{l,k} \psi_{l,k}$, the bijection $\alpha_k \mapsto d_{l,k}$ and its inverse can be implemented fast.

Using the interpolating operators on $V_J$, $L_J$ can be considered as a linear operator from $V_J$ to $V_J$. Then, there exists a bilinear and continuous form $A_J(\cdot, \cdot)$, defined on $V_J \times V_J$, such that for all $u, v \in V_J$, $A_J(u, v) = \langle L_J u, v \rangle^1$. This bilinear form is symmetric, definite and positive.

Choosing a positive integer $j_0$ ($j_0 < J$) we write, using $W_J$ the biorthogonal complement of $V_J$ in $V_{J+1}$,

$$V_J = V_{j_0} \oplus \left( \oplus_{j=j_0}^{J-1} W_J \right),$$

and the construction of our preconditioner is performed in two parts, searching for an approximation of the inverse of the operator $L_J$ on $V_{j_0}$, and on $\oplus_{j=j_0}^{J-1} W_J$.

To be precise, we will adopt, in the sequel, the following definition of a preconditioner for $L_J$. Let us note $(V_j)_J$, a set of embedded approximation spaces $(V_j \subset V_{j+1})$, and $L_j$ an operator from $V_j$ to $V_J$.

**Definition 2.1.** The family of operators $P_j : V_j \to V_j$ is said to be a family of preconditioners for $L_J$ if the operators $L_j P_j$ are perturbation of the $L^2$- scalar product.

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$^1 \langle \cdot, \cdot \rangle$ standing for the $L^2$- scalar product.
identity, i.e if, for each integer \( j \), there exists an operator \( Q_j : V_j \to V_j \), such that for all \( u \in V_j \), one has
\[
L_j P_j u = u - Q_j u, \\
\text{and } \| Q_j \|_{L^2(\mathbb{R})} \text{ is bounded uniformly with regards to } j,
\]
i.e there exists a constant \( C \) such that \( \| Q_j \|_{L^2(\mathbb{R})} < C. \) (5)

**Remark 2.2.** As smaller is the constant \( C \), as better is the preconditioner. Moreover, if \( C < 1 \), or if \( \exists p \in \mathbb{N}, \| Q_j^p \|_{L^2(\mathbb{R})} < 1 \), then, using a Neumann series, an explicit expression of \( L_j^{-1} \) can be given using \( P_j \).

**Remark 2.3.** If (5) holds only for \( u \in V_{j_0}, j_0 \leq j \), then the operator \( P_{j_0} : V_j \to V_j \), will be considered as an approximation of \( L_j^{-1} \) in the subspace \( V_{j_0} \).

### 2.1. Approximation of \( L_j^{-1} \) in \( V_{j_0} \)

For \( j < J \), let \( p_j : V_J \to V_j \) stands for the biorthogonal projector onto \( V_j \), and \( p_j^+ = Id_{V_j} - p_j \). Since \( L_J \) is associated to a definite and positive bilinear form, the operators
\[
p_j L_j p_j^{-1} : V_J \to V_j, \quad 0 \leq j \leq J,
\]
are invertible, and we can define in \( V_J \) the inverse of the Galerkin projection of \( L_J \) in \( V_j \),
\[
T_j = p_j^{-1} \left( p_j L_j p_j^{-1} \right)^{-1} p_j.
\]
Finally, the following proposition can be proved:

**Proposition 2.4.** (see [5]) The operators \( L_J T_j : V_J \to V_J \) \((0 \leq j \leq J)\) are uniformly bounded in the \( L^2 \)-norm.

### 2.2. Approximation of \( L_J^{-1} \) in \( \bigoplus_{j=j_0}^{J-1} W_j \)

If the family \( \{ \psi_{j,k}, j_0 \leq j \leq J - 1, k \in \mathbb{Z} \} \) is a wavelet basis of \( \bigoplus_{j=j_0}^{J-1} W_j \), following [7], one now focus on the construction of an approximation for each function \( L_J^{-1} \psi_{j,k}, j_0 \leq j \leq J - 1, k \in \mathbb{Z} \).

A centered finite difference approximation of the Laplace operator, \( \Delta_J \), on the grid \( N_J \), of order 2, is defined, associated to the symbol\(^2\)
\[
\sigma_J(\omega) = -\frac{4}{h^2} \left( \sin\left(\frac{h\omega}{2}\right)\right)^2.
\]
\(^2\forall u \in L^2(\mathbb{R}), \forall \omega \in \mathbb{R}, \quad \hat{L_J}u(\omega) = \sigma_J(\omega)\hat{u}(\omega), \) where \( \hat{u}(\omega) = \int_{\mathbb{R}} u(x)e^{-ix\omega}dx \) is the Fourier transform of \( u \).
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For each functions $\psi_{j,k}$, $j_0 \leq j \leq J - 1$, $k \in \mathbb{Z}$, the functions $g_{j,k}$ are defined through their Fourier transforms by

$$g_{j,k}(\omega) = \frac{1}{a_{j,k}} \frac{\hat{\psi}_{j,k}(\omega)}{\sigma_J(\omega)},$$

(7)

where $a_{j,k}$ is an approximation of the variable coefficient $a(\cdot)$ for the wavelet $\psi_{j,k}$ (in the implementation we will use $a_{j,k} = a(k2^{-j})$). Note that since $\hat{\psi}(\omega) \sim \omega \to 0 \omega^d$, and $\sigma_J(\omega) \sim \omega \to 0 \omega^2$, the function $g_{j,k}$ exists as soon as $d > 2$ ($d$ being as mentioned at the beginning of this section the order of the primal multi-resolution analysis).

Using these functions, the operator $E_{j_0}$: $V_J \to L^2(\mathbb{R})$, is defined as

$$E_{j_0}: \psi_{j,k} \mapsto \begin{cases} 0, & j < j_0, \\ g_{j,k}, & j_0 \leq j \leq J - 1, \\ k \in \mathbb{Z}. \end{cases}$$

(8)

If $\pi_J : L^2(\mathbb{R}) \to V_J$ stands for the biorthogonal projection onto $V_J$, and $Id_{V_J}$ the identity operator onto $V_J$, then the following theorem holds.

**Theorem 2.5.** (see [5]) $E_{j_0}$ is a continuous operator from $V_J$ to $L^2(\mathbb{R})$.

Moreover, if we define the operator $R_{j_0}$: $V_J \to V_J$ as

$$L_J(\pi_J E_{j_0}) = (Id_{V_J} - p_{j_0}) - R_{j_0}$$

then $R_{j_0}$ is bounded in the $L^2$-norm by $C2^{-j_0}$, the constant $C$ depending on the multi-resolution and on the Lipschitz constant of $a(\cdot)$.

Following Remark (2.3), the operator $(\pi_J E_{j_0})$ is then an approximation of $L_J^{-1}$ in the wavelet spaces $\oplus_{j=J_0}^{J-1} W_j$.

### 2.3. A Preconditioner for $L_J$

Gathering the two approximations, the following proposition holds.

**Proposition 2.6.** (see [5]) For any $j_0 \leq J$, the operator $T_{j_0} + \pi_J E_{j_0}$ is a preconditioner for $L_J$. Indeed, if we define the operator $U_{j_0}: V_J \to V_J$, as

$$L_J(T_{j_0} + \pi_J E_{j_0}) = Id_{V_J} - U_{j_0},$$

(9)

then there exists a constant $C$ depending on the multi-resolution and on the Lipschitz constant of $a(\cdot)$ such that $\|U_{j_0}\|_{L^2(\mathbb{R})} \leq C$.

Moreover, there exists a constant $C$ depending on the multi-resolution and on the Lipschitz constant of $a(\cdot)$ such that $\|U_{j_0}^2\|_{L^2(\mathbb{R})} \leq C2^{-j_0}$.
Due to the non homogeneity of the symbol associated to $\Delta_J$, the functions $a_{j,k}g_{j,k}$ are not the wavelets of a multi-resolution analysis. It follows that, the previous construction does not lead to an efficient numerical implementation. A technical modification is required.

The following function $\tilde{\theta}$, defined through its Fourier transform, by

$$\tilde{\theta}(\omega) = \frac{\hat{\psi}(\omega)}{\omega^2},$$  \hspace{1cm} (10)

is first introduced.

Following [6], a biorthogonal multi-resolution analysis generating the function $\tilde{\theta}$ can be constructed. Moreover, for $j_0 \leq j \leq J - 1$, and $k \in \mathbb{Z}$ the operator

$$D : \left\{ \begin{array}{l}
\text{span } \{a_{j,k}g_{j,k}, \; j,k \} \rightarrow \text{span } \{ \frac{1}{\sigma_{J}(2^j)} \tilde{\theta}_{j,k}, \; j,k \} \\
a_{j,k}g_{j,k} \mapsto \frac{1}{\sigma_{J}(2^j)} \tilde{\theta}_{j,k},
\end{array} \right. \hspace{1cm} (11)$$

and its inverse are continuous and uniformly bounded with regards to $J$. Then, the modification of the construction consists in replacing the operator $E_{j_0}$, defined in (8), by the operator $B_{j_0} : V_J \rightarrow L^2(\mathbb{R})$, defined as

$$B_{j_0} : \psi_{j,k} \mapsto \left\{ \begin{array}{l}
0, \hspace{1cm} j < j_0, \hspace{1cm} k \in \mathbb{Z}, \\
-\frac{1}{a_{j,k} \sigma_{J}(2^j)} \tilde{\theta}_{j,k}, \hspace{1cm} j_0 \leq j \leq J - 1, \hspace{1cm} k \in \mathbb{Z}.
\end{array} \right. \hspace{1cm} (12)$$

Finally, the operator

$$T_{j_0} + \pi_J B_{j_0},$$

is a preconditioner for the operator $L_J$. In the implementations, the matrix corresponding to the operator $T_{j_0} + \pi_J B_{j_0}$ will be noted $C_{1,j_0}$, and $\kappa(C_{1,j_0}A_J)$ will then be studied.

3. Analysis, Implementation and Comparison with Two Classical Multilevel Preconditioners

3.1. Analysis and Implementation

The new wavelet preconditioner uses extensively the properties of compactly supported regular wavelets that allow to cure the ill conditioning due to differentiation, as well as the ill conditioning due to multiplication by a variable coefficient.
More precisely, the separation between the small and the large scales (represented by the decomposition (4)) allows always to work with well localized wavelets, and then, using the term $\sigma_J(2^j)a_{j,k}$, to cure separately the problem related to the differentiation operator, and the problem related to the variable coefficient. The only price to pay for this simplicity is represented by the discriminating scale $j_0$ (we recall that from Proposition 2.6, $\|U^{2}_{j_0}\|_{L^2(\mathbb{R})} \leq \tilde{C}2^{-j_0}$).

As far as implementation is concerned, the definition of the new wavelet preconditioner is such that all the operators used in the preconditioner $T_{j_0} + \pi_JB_{j_0}$ involve biorthogonal wavelets, or are diagonal in suitable wavelet bases. As it has been said before, the specific structure of multi-resolution implies that summations of series of type

$$
\sum_{k \in \mathbb{Z}} c_{j_0,k}\phi_{j_0,k} + \sum_{j=j_0}^{J-1} \sum_{k \in \mathbb{Z}} d_{j,k}\psi_{j,k}
$$

(where $V_j = \text{span}\{\phi_{j,k}, k \in \mathbb{Z}\}$, and $W_j = \text{span}\{\psi_{j,k}, k \in \mathbb{Z}\}$) can be implemented fast, using tree algorithms. These algorithms use filters related to $\phi$ and $\psi$, and therefore various biorthogonal wavelets can be handled as soon as associated filters are computed (see [5] for details on this point and a precise description of the implementation).

Since this construction does not depend on the choice of the finite difference scheme used to define $L_J$, but only on the chosen multi-resolution analysis $(V_j)_j$, this preconditioner can be easily plugged in many finite difference softwares.

A key property of this construction is that it is fully explicit. It is therefore easy to adapt its expression to the form of $L$, using a suitable choice for $a_{j,k}$.

### 3.2. Comparisons with Two Classical Multilevel Preconditioners

#### 3.2.1. Short Definitions of Two Classical Preconditioners

In [8], H. Yserentant constructed a preconditioner using a hierarchical decomposition of the initial grid $N_J$, and the corresponding interpolating operators $I_j$ defined as

$$
I_j : \begin{cases}
C(\mathbb{R}) & \rightarrow & V_j \\
u & \mapsto & I_ju,
\end{cases}
\text{ such that, } \forall x_i \in N_j, \ I_ju(x_i) = u(x_i),
$$

where

$$
N_j = \{x_i = \frac{i}{2^j}, i \in \mathbb{Z}\}, \ j_0 \leq j \leq J-1.
$$
Any function \( u \in V_J \) is written as

\[
  u = I_J u = I_{j_0} u + \sum_{j=1}^{J} (I_j u - I_{j-1} u),
\]

and one consider the discrete weighted norm defined by

\[
  ||| u |||_2 = ||I_0 u||^2 + \sum_{j=1}^{J} \sum_{x_k \in N_j \setminus N_{j-1}} d_{j,k} \left| (I_j u - I_{j-1} u)(x_k) \right|^2,
\]

where \( ||u||^2 = A_J(u, u) \). In the literature, the weights \( d_{j,k} \) are related to the value of the variable coefficient \( a(\cdot) \) around the point \( k2^{-j} \). The most common choice is

\[
  d_{j,k} = 4^j \langle 1, \phi_{j,k} \rangle, \quad \text{if} \quad V_J = \text{span} \{ \phi_{j,k}, k \in \mathbb{Z} \}.
\]

If \( C_{yser}^{-1} \) is the matrix inducing the previous norm, then the matrix \( C_{yser} \) is defined in [8] as the so-called Yserentant preconditioner for the matrix \( A_J \).

Parallely, the construction of the BPX preconditioner [1] is based on the existence of a hierarchical decomposition of the higher level approximation space \( V_J \) using the families of \( L^2 \)-biorthogonal projections

\[
  \pi_j : L^2(\mathbb{R}) \rightarrow V_j, \quad j_0 \leq j \leq J,
\]

such that, for all \( u \in V_J \),

\[
  u = \pi_J u = \pi_{j_0} u + \sum_{j=j_0}^{J-1} (\pi_{j+1} u - \pi_j u). \tag{14}
\]

If \( S_j : V_j \rightarrow V_j \) stands for the linear operator associated to the restriction of the bilinear form \( A_J(\cdot, \cdot) \) to each space \( V_j \),

\[
  \forall u, v \in V_j, \quad \langle S_j u, v \rangle = A_J(u, v),
\]

one notes \( \lambda_j = \rho(S_j) \) the spectral radius of \( S_j \).

Then, the BPX preconditioner is given by the operator

\[
  \tilde{C}_{bpx} = \sum_{j=0}^{J} \frac{1}{\lambda_j} \pi_j.
\]

The associated matrix will be noted \( C_{bpx} \).
3.2.2. Comparisons

Several common points can be found for these three multilevel preconditioners. The new wavelet preconditioner has been directly constructed in the framework of finite difference approximations, in opposite to the Yserentant and BPX preconditioners, which have been derived in the framework of variational formulations. Nevertheless, the two classical multilevel preconditioners can be transposed to finite differences in two easy ways. Either one can use interpolation operators $I$ and then associate to a finite difference operator $L_J$ a continuous operator $I^*L_JI$ (where $I^*$ stands for the adjoint of $I$) in the variational approximation space, either one can directly consider $L_J$ as a continuous operator in the variational approximation space. Both operators satisfy the suitable properties for applications of the preconditioners.

Each construction is based on a hierarchical decomposition of the space $V_J$, using interpolation operators for the Yserentant preconditioner (13), using projection operators on $V_j$ for the BPX preconditioner (14), and using projections on the detail spaces $W_j = V_{j+1} - V_j$ for the wavelet preconditioner.

One can remark that interpolating projectors lead to the so-called interpolating wavelet approach. Therefore, the framework of H. Yserentant is very closed from interpolating multi-resolution analysis, such as those constructed by A. Harten [3].

Nevertheless, the main difference between the wavelet preconditioner and the two other ones is that the wavelet preconditioner involves directly the detail spaces $W_j$, and exploits the properties of wavelets, that imply that differentiation operators, as well as multiplication operators can be easily preconditioned using diagonal operators.

In the two other constructions the detail spaces are not directly involved. It is the reason why these two methods are less explicit than the wavelet method.

3.3. Numerical Results

To compare these preconditioners for finite difference operator, we considered problem (1) with the following set of variable coefficients, represented on Figure 1.

$$a_1(x) = 2x(1 - x)^2 + 0.5 \quad a_2(x) = 1 + 1000 \sin^2(16\pi x)$$
$$a_3(x) = 1 + 10^{-4} - \sin(\pi x).$$

Function $a_1(\cdot)$ is rather smooth with a small Lipschitz constant; Function $a_2(\cdot)$ exhibits strong oscillations and the variation of the amplitude of function
$a_3(\cdot), \frac{(a_3)_{\text{max}}}{(a_3)_{\text{min}}}$, is of order $10^4$.

We implemented the three multi-level preconditioners defined in Section 2 and Section 3. Here, the approximation $a_{j,k}$ is taken as the value of the variable coefficient $a(\cdot)$ at the middle of the wavelet support, and the operator $L$ is discretized with a centered finite difference scheme of order two.

The value of the parameter $j_0$ is set to $j_0 = 0$ for function $a_1(\cdot)$, and to $j_0 = 4$ for functions $a_2(\cdot)$ and $a_3(\cdot)$.

Moreover, the finite difference approximation of the Laplace operator is given for all $u \in L^2$ by

$$\Delta_J u(x) = \frac{1}{h^2} \left( u(x + h) - 2u(x) + u(x - h) \right),$$

for $x \in [0,1]$ and $h = 2^{-J}$.

The corresponding results are given in Tables 1, 2, and 3.

It appears from the Tables that the three preconditioners are very efficient.
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<table>
<thead>
<tr>
<th>Scale $J$</th>
<th>$\kappa(A_J)$</th>
<th>$\kappa(C_{J,j_0}A_J)$ for $j_0 = 0$</th>
<th>$\kappa(C_{yser}A_J)$</th>
<th>$\kappa(C_{bpx}A_J)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.18e+03</td>
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<td>43.01</td>
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<tr>
<td>9</td>
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</tr>
</tbody>
</table>

Table 1: $\kappa(A_J), \kappa(C_{J,j_0}A_J), \kappa(C_{yser}A_J), \kappa(C_{bpx}A_J)$ versus $J$, for $a_1(\cdot)$.

<table>
<thead>
<tr>
<th>Scale $J$</th>
<th>$\kappa(A_J)$</th>
<th>$\kappa(C_{J,j_0}A_J)$ for $j_0 = 4$</th>
<th>$\kappa(C_{yser}A_J)$</th>
<th>$\kappa(C_{bpx}A_J)$</th>
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<td>4.85e+05</td>
<td>4.76e+05</td>
</tr>
</tbody>
</table>

Table 2: $\kappa(A_J), \kappa(C_{J,j_0}A_J), \kappa(C_{yser}A_J), \kappa(C_{bpx}A_J)$ versus $J$, for $a_2(\cdot)$.

<table>
<thead>
<tr>
<th>Scale $J$</th>
<th>$\kappa(A_J)$</th>
<th>$\kappa(C_{J,j_0}A_J)$ for $j_0 = 4$</th>
<th>$\kappa(C_{yser}A_J)$</th>
<th>$\kappa(C_{bpx}A_J)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.53e+03</td>
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<td>97.44</td>
<td>6.40e+04</td>
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</tr>
</tbody>
</table>

Table 3: $\kappa(A_J), \kappa(C_{J,j_0}A_J), \kappa(C_{yser}A_J), \kappa(C_{bpx}A_J)$ versus $J$, for $a_3(\cdot)$.  

for the smooth coefficient $a_1(\cdot)$. Indeed, for such a coefficient, the multilevel splitting definitively cure the bad conditioning that is essentially connected to the differentiation operator.

For $a(\cdot) = a_2(\cdot)$ and $a(\cdot) = a_3(\cdot)$, the oscillations and the large amplitude of $a(\cdot)$ on $[0, 1]$ deteriorate the condition number. As predicted by Theorem 2.5, a larger value of $j_0$ is required to get a satisfactory preconditioning. Note that BPX, as well as Yserentant preconditioner can also be implemented using a splitting of type $V_J = V_{j_0} + (V_J - V_{j_0})$.

4. Conclusion

In this paper, we have compared three different multi-scale preconditioners in the framework of finite difference approximation of second order elliptic operators, focusing on their capability to cure the ill conditioning connected to variable coefficients.

The results obtained with the three preconditioners are comparable. One should emphasize that the wavelet preconditioner has a low computational cost thanks to the fast wavelet transform algorithms. Moreover, due to its explicit definition, it can be adapted to many situations, including for instance problems of the type $b(x)U + \frac{\partial}{\partial x} (a(x) \frac{\partial U}{\partial x}) = f$ with a locally dominant value of $b$.

References


