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TRANSFORMATION OPERATORS FOR STURM-LIOUVILLE OPERATORS WITH SINGULAR POTENTIALS

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Abstract: In this article, the existence of transformation operator is proved for a class of singular Sturm-Liouville differential operators. In addition, the classical relation between the potential of given operator and the kernel of transformation operator is given.

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1. Introduction

In interval (a, b), when the given interval is finite which is generated by the differential expression $l(y) := -y''(x)+q(x)y(x)$ in theory of the Sturm-Liouville operator, the function $q(x)$ satisfies the condition $q(x) \in L_1(a, b)$ in general. As in singular case, i.e., when interval (a, b) is infinite or the function $q(x)$ has nonintegrable singularity in extremity points of interval the condition of $q(x) \in L_{1,loc}(a, b)$ is given.

In study of [6] when $q(x)$ is a first order singular generalized function, Sturm-Liouville operator, i.e., singular Sturm-Liouville operator has been defined which has a potential as $q = u'$ by using concept of generalized derivative such that $u \in L_2(0, 1)$.

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Morever in this study, self-adjoint extensions of differential operators generated by differential expression $l(y)$ which has potential $q(x) = u'(x)$ such that $u \in L_2(0,1)$. When $\alpha \neq 2,4,6,...$ generalized functions can be corresponded to the functions $|x|^{-\alpha}\operatorname{sign} x$ by using the method of canonic regularization [5]. When $\alpha < \frac{3}{2}$, generalized functions which are obtained by this way can be shown as generalized derivative of functions from the space L_2 and therefore Sturm-Liouville operator which is given by the differential expression $l(y)$ and which has a potential like $q(x) = |x|^{-\alpha}\operatorname{sign} x$ can be defined. In [1] when $q(x) = Cx^{-\alpha}$ and $\alpha < \frac{3}{2}$, $C \in R$, a regularization of the constructing boundary value problems for Sturm-Liouville equation which has this type potential has been given.

As in the study of [4], when $q(x) = Cx^{-\alpha}$ and $\alpha \in [1,2)$, all self-adjoint extensions of operators generated by the differential expression $l(y)$ which has this type potential according to boundary value conditions and therefore when $\alpha \in [1,2)$, regularization of constructing boundary value problems for Sturm-Liouville equation which has this type potential has been given. Regularizations in the studies of [4] and [6] coincide only when $\alpha < \frac{3}{2}$.

Let us consider the differential expression

$$l(y) := -y''(x) + \frac{C}{x^{\alpha}}y(x) + q(x)y(x), \quad 0 < x < \pi, \tag{1}$$

where C is a real number, $q(x)$ is a real valued bounded function.

We shall define an operator L_0': $L_0'y = l(y)$, in the set of $D_0' = C_0^{\infty}(0,\pi)$. It is obvious that the operator L_0' is symmetric in the space of $L_2[0,\pi]$. We say that the operator L_0 which is the closure of L_0' is minimal operator generated by the differential expression (1). The conjugate L_0^* of the operator L_0 is said to be maximal operator generated by the differential expression (1).

In [4], all maximal dissipative and accumulative and also self-adjoint extensions of the operator L_0 have been studied according to the domain and boundary conditions of minimal and maximal operators generated by differential expression (1).

We denote that $(\Gamma_\alpha y)(x) = y'(x) - u(x)y(x)$, where $u(x) = C\frac{x^{1-\alpha}}{1-\alpha}$.

It has been shown, in [4], that if $y(x) \in D(L_0^*)$ then the function $(\Gamma_\alpha y)(x)$ has a limit as $x \longrightarrow +0$, i.e.,

$$\lim_{x \longrightarrow +0}(\Gamma_\alpha y)(x) = (\Gamma_\alpha y)(0).$$

Hence the domain $D(L_0)$ of minimal operator L_0 generated by differential expression (1) contains only functions $y(x) \in D(L_0^*)$ such that function $y(x)$ satisfies the conditions $y(0) = y(\pi) = (\Gamma_\alpha y)(0) = y'(\pi) = 0$.

In the present paper the construction method of transform operator is given for one class of singular Sturm-Liouville operator in the case $q = u'$. Here the differentiation is in the meaning of general differentiation. In the case when Sturm-Liouville operator has a singularity of Bessel type $\left(\frac{l(l-1)}{x^2},\ l \in Z_+\right)$ on the finite interval the transform operator was construct in [5], [1] and in the case $[0,\infty)$ it was given in [4]. When Sturm-Liouville operator has a singularity of Coulomb type $\left(\frac{A}{x}, A \in R - \{0\}\right)$ on the finite interval, the transformation operator was constructed in [7]. Note that transformation operator for the Sturm-Liouville operator with the potential $x\,|q(x)| \in L_1[0,\pi]$ related with $x = 0$ was constructed by Amirov. In the paper [3] the classical relation between the potential of given operator and the kernel of transformation operator was not given obviously, but in this study the classical relation between them is given.

2. Construction of the Integral Equation

Let us consider the boundary value problem $L = L(h, H)$:

$$l(y) := -y'' + u'(x)y = \lambda y, \quad 0 < x < \pi, \quad \lambda = \rho^2\,, \tag{2}$$

$$U(y) := (\Gamma_\alpha y)(0) - hy(0) = 0, \quad V(y) := (\Gamma_\alpha y)(\pi) + Hy(\pi) = 0\,, \tag{3}$$

where $\alpha \in (1, 3/2)$, h, H are real numbers, $q(x) = u'(x)$ is a real valued function in $L_2[0,\pi]$, λ is spectral parameter.

We denote that $y_1(x) = y(x)$, $y_2(x) = (\Gamma_\alpha y)(x) = y'(x) - u(x)y(x)$ and let us write the expression of left hand side of the equation (2) as follows

$$l(y) := ((\Gamma_\alpha y)(x))' - u(x)(\Gamma_\alpha y)(x) - u^2(x)y(x) + y(x). \tag{4}$$

Then the equation (2) reduces to the system

$$\begin{cases} y_1' - u(x)y_1 = y_2, \\ y_2' + u(x)y_2 + u^2(x)y_1 - y_1 = -\lambda^2 y_1, \end{cases} \tag{5}$$

or in matrice form:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -\lambda^2 - u^2 & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{6}$$

The entries of the matrice

$$A(x) = \begin{pmatrix} u(x) & 1 \\ -\lambda^2 - u^2(x) & -u(x) \end{pmatrix},$$

are functions in $L_1[0,\pi]$.

For this reason [7], there exists only one solution of system (6) which satisfies the same initial conditions $y_1(\xi)=\alpha_1$, $y_2(\xi)=\alpha_2$ for each $\xi\in[0,\pi]$, $\alpha=(\alpha_1,\alpha_2)^T\in C^2$ especially the initial conditions $y_1(0)=1$, $y_2(0)=h$.

Definition 1. First component of the solution which satisfies the initial conditions $y(\xi)=\alpha_1$, $(\Gamma_\alpha y)(\xi)=\alpha_2$ of the system (6) is called as the solution of the equation (2) which satisfies the same initial conditions

Let us show the solution $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(x)$ of the system (5) which satisfies the initial conditions $y_1(0)=1$, $y_2(0)=i\lambda$ has a representation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(x,\lambda)=\begin{pmatrix} 1 \\ i\lambda \end{pmatrix}e^{i\lambda x}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}b(x)e^{i\lambda x} + \int_{-x}^{x} K(x,t)\begin{pmatrix} 1 \\ i\lambda \end{pmatrix}e^{i\lambda t}dt, \quad (7)$$

where $K(x,t)=\begin{pmatrix} K_{11}(x,t) & 0 \\ K_{21}(x,t) & K_{22}(x,t) \end{pmatrix}$.

If we write (7) with coordinates, we will get the following expressions:

$$y_1(x,\lambda)=e^{i\lambda x}+\int_{-x}^{x} K_{11}(x,t)e^{i\lambda t}dt\,, \quad (8)$$

$$y_2(x,\lambda)=e^{i\lambda x}+b(x)e^{i\lambda x}+\int_{-x}^{x} K_{21}(x,t)e^{i\lambda t}dt+i\lambda\int_{-x}^{x} K_{22}(x,t)e^{i\lambda t}dt. \quad (9)$$

The solution of the system (5) which satisfies the initial conditions $y_1(0)=1$, $y_2(0)=i\lambda$ satisfies the following system of integral equations:

$$\begin{aligned} y_1(x,\lambda)= \; & e^{i\lambda x}+\int_0^x u(t)\cos\lambda(x-t)y_1(t,\lambda)dt \\ & -\int_0^x \frac{\sin\lambda(x-t)}{\lambda}\{u^2(t)y_1(t,\lambda)+u(t)y_2(t,\lambda)\}dt, \end{aligned} \quad (10)$$

$$y_2(x,\lambda) = \quad i\lambda e^{i\lambda x} - \int_0^x U(t)\lambda \sin\lambda(x-t) y_1(t,\lambda)dt$$
$$\int_0^x \cos\lambda(x-t)\{u^2(t)y_1(t,\lambda) + u(t)y_2(t,\lambda)\}dt. \tag{11}$$

In order the function (7) to satisfy the system (6), it must be fulfilled the following equality

$$\int_{-x}^{x} K_{11}(x,t)e^{i\lambda t}dt = \int_0^x u(t)\cos\lambda(x-t)e^{i\lambda t}dt$$
$$+ \int_0^x u(t)\cos\lambda(x-t)\left(\int_{-t}^{t} K_{11}(t,s)e^{i\lambda s}ds\right)dt$$
$$- \int_0^x \frac{\sin\lambda(x-t)}{\lambda}U^2(t)e^{i\lambda t}dt$$
$$- \int_0^x \frac{\sin\lambda(x-t)}{\lambda}u(t)i\lambda e^{i\lambda t}dt - \int_0^x \frac{\sin\lambda(x-t)}{\lambda}u(t)b(t)e^{i\lambda t}dt$$
$$- \int_0^x \frac{\sin\lambda(x-t)}{\lambda}u^2(t)\left(\int_{-t}^{t} K_{11}(t,s)e^{i\lambda s}ds\right)dt$$
$$- \int_0^x \frac{\sin\lambda(x-t)}{\lambda}u(t)i\lambda\left(\int_{-t}^{t} K_{22}(t,s)e^{i\lambda s}ds\right)dt$$
$$- \int_0^x \frac{\sin\lambda(x-t)}{\lambda}u(t)\left(\int_{-t}^{t} K_{21}(t,s)e^{i\lambda s}ds\right)dt, \tag{12}$$

$$b(x)e^{i\lambda x} + \int_{-x}^{x} K_{11}(x,t)e^{i\lambda t}dt + i\lambda\int_{-x}^{x} K_{22}(x,t)e^{i\lambda t}dt$$
$$= -\int_0^x u(t)\lambda\sin\lambda(x-t)e^{i\lambda t}dt$$

$$-\int_0^x u(t)\lambda\sin\lambda(x-t)\left(\int_{-t}^t K_{11}(t,s)e^{i\lambda s}ds\right)dt$$

$$-\int_0^x u^2(t)\cos\lambda(x-t)e^{i\lambda t}dt$$

$$-\int_0^x u^2(t)\cos\lambda(x-t)\left(\int_{-t}^t K_{11}(t,s)e^{i\lambda s}ds\right)dt$$

$$-\int_0^x u(t)\cos\lambda(x-t)i\lambda e^{i\lambda t}dt$$

$$-\int_0^x u(t)\cos\lambda(x-t)b(t)e^{i\lambda t}dt$$

$$-\int_0^x u(t)\cos\lambda(x-t)\left(\int_{-t}^t K_{21}(t,s)e^{i\lambda s}ds\right)dt$$

$$-\int_0^x u(t)\cos\lambda(x-t)i\lambda\left(\int_{-t}^t K_{22}(t,s)e^{i\lambda s}ds\right)dt. \quad (13)$$

We put

$$I_1=\int_0^x u(t)\cos\lambda(x-t)e^{i\lambda t}dt,\ I_2=-\int_0^x \frac{\sin\lambda(x-t)}{\lambda}u^2(t)e^{i\lambda t}dt,$$

$$I_3=-\int_0^x \frac{\sin\lambda(x-t)}{\lambda}u(t)i\lambda e^{i\lambda t}dt,$$

$$I_4=-\int_0^x \frac{\sin\lambda(x-t)}{\lambda}u(t)b(t)e^{i\lambda t}dt,$$

$$I_1=\int_0^x u(t)\cos\lambda(x-t)e^{i\lambda t}dt$$

$$= \frac{1}{2} e^{i\lambda t} \int_0^x u(t)dt + \frac{1}{4} \int_{-x}^x u(\frac{x+s}{2}) e^{i\lambda s} ds,$$

$$I_3 = -\int_0^x \frac{\sin \lambda(x-t)}{\lambda} u(t) i\lambda e^{i\lambda t} dt$$

$$= -\frac{1}{2} e^{i\lambda x} \int_0^x u(t)dt + \frac{1}{4} \int_{-x}^x u(\frac{x+s}{2}) e^{i\lambda s} ds,$$

$$I_2 + I_4 = -\int_0^x \frac{\sin \lambda(x-t)}{\lambda} \{u^2(t) + u(t)b(t)\} e^{i\lambda t} dt$$

$$= -\frac{1}{2} \int_{-x}^x \left\{ \int_0^{(x+t)/2} \{u^2(s) + u(s)b(s)\} ds \right\} e^{i\lambda t} dt,$$

$$\int_0^x u(t) \cos \lambda(x-t) \left(\int_{-t}^t K_{11}(t,s) e^{i\lambda s} ds \right) dt$$

$$= \frac{1}{2} \int_{-x}^x \left(\int_{(x-t)/2}^x u(s) K_{11}(s, t+s-x) ds \right) e^{i\lambda t} dt$$

$$+ \frac{1}{2} \int_{-x}^x \left(\int_{(x+t)/2}^x u(s) K_{11}(s, t-s+x) ds \right) e^{i\lambda t} dt$$

$$- \int_0^x \frac{\sin \lambda(x-t)}{\lambda} u^2(t) \left(\int_{-t}^t K_{11}(t,s) e^{i\lambda s} ds \right) dt$$

$$= \frac{1}{2} \int_{-x}^x \left(\int_0^x u^2(t) \int_{t-x+s}^{t+x-s} K_{11}(s,\xi) d\xi ds \right) e^{i\lambda t} dt$$

$$- \int_0^x \frac{\sin \lambda(x-t)}{\lambda} u(t) i\lambda \left(\int_{-t}^t K_{22}(t,s) e^{i\lambda s} ds \right) dt$$

$$= -\frac{1}{2}\int_{-x}^{x}\left(\int_{(x-t)/2}^{x} u(s)K_{22}(t, t-x+s)ds\right)e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_{-x}^{x}\left(\int_{(x+t)/2}^{x} u(s)K_{22}(s, t-x+s)ds\right)e^{i\lambda t}dt.$$

Therefore,

$$-\int_{0}^{x}\frac{\sin\lambda(x-t)}{\lambda}u(t)\left(\int_{-t}^{t}K_{21}(t,s)e^{i\lambda s}ds\right)dt$$

$$=\frac{1}{2}\int_{-x}^{x}\left(\int_{0}^{x}u(t)\int_{t-x+s}^{t+x-s}K_{21}(s,\xi)dsd\xi\right)e^{i\lambda t}dt.$$

If we substitude the obtained expression in (12), we get

$$\int_{-x}^{x}K_{11}(x,t)e^{i\lambda t}dt = \frac{1}{2}\int_{-x}^{x}u\left(\frac{x+t}{2}\right)e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_{-x}^{x}\left(\int_{(x-t)/2}^{x}u(s)K_{11}(s, t+s-x)ds\right)e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_{-x}^{x}\left(\int_{(x+t)/2}^{x}u(s)K_{11}(s, t-s+x)ds\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^{x}\left(\int_{0}^{(x+t)/2}u^2(s)+u(s)b(s)ds\right)e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_{-x}^{x}\left(\int_{0}^{x}u^2(s)\int_{t-x+s}^{t+x-s}K_{11}(s,\xi)d\xi ds\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^{x}\left(\int_{(x-t)/2}^{x}u(s)K_{22}(s, t-x+s)ds\right)e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_{-x}^{x}\left(\int_{(x+t)/2}^{x}u(s)K_{22}(s,t-s+x)ds\right)e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_{-x}^{x}\left(\int_{0}^{x}u(t)\int_{t-x+s}^{t+x-s}K_{21}(s,\xi)d\xi ds\right)e^{i\lambda t}dt. \quad (12')$$

From $(12')$, we get the following integral equation for the function $K_{11}(x,t)$;

$$K_{11}(x,t)=\frac{1}{2}u\left(\frac{x+t}{2}\right)-\frac{1}{2}\int_{0}^{(x+t)/2}\left(u^2(s)+u(s)b(s)\right)ds$$

$$+\frac{1}{2}\int_{(x-t)/2}^{x}u(s)K_{11}(s,t+s-x)ds+\frac{1}{2}\int_{(x+t)/2}^{x}u(s)K_{11}(s,t-s+x)ds$$

$$+\frac{1}{2}\int_{0}^{x}u^2(s)\int_{t-x+s}^{t+x-s}K_{11}(s,\xi)d\xi ds+\frac{1}{2}\int_{0}^{x}u(s)\int_{t-x+s}^{t+x-s}K_{21}(s,\xi)d\xi ds$$

$$-\frac{1}{2}\int_{(x-t)/2}^{x}u(s)K_{22}(s,t-x+s)ds+\frac{1}{2}\int_{(x+t)/2}^{x}u(s)K_{22}(s,t-s+x)ds.$$

In the equality (13), we put

$$A_1=-\lambda\int_{0}^{x}u(t)\sin\lambda(x-t)e^{i\lambda t}dt$$

$$=\frac{-\lambda}{2i}e^{i\lambda x}\int_{0}^{x}u(t)dt+\frac{\lambda}{4i}\int_{-x}^{x}u\left(\frac{s+x}{2}\right)e^{i\lambda s}ds,$$

$$A_2=-\int_{0}^{x}u^2(t)\cos\lambda(x-t)e^{i\lambda t}dt$$

$$=\frac{-1}{2}e^{i\lambda x}\int_{0}^{x}u^2(t)dt-\frac{1}{4}\int_{-x}^{x}u^2\left(\frac{s+x}{2}\right)e^{i\lambda s}ds,$$

$$A_3 = -i\lambda \int_0^x u(t)\cos\lambda(x-t)e^{i\lambda t}dt$$

$$= \frac{-i\lambda}{2}e^{i\lambda x}\int_0^x u(t)dt - \frac{i\lambda}{4}\int_{-x}^x u\left(\frac{s+x}{2}\right)e^{i\lambda s}ds,$$

$$A_4 = -\int_0^x u(t)\cos\lambda(x-t)b(t)e^{i\lambda t}dt = \frac{-1}{2}e^{i\lambda x}\int_0^x u(t)b(t)dt$$

$$-\frac{1}{4}\int_{-x}^x u\left(\frac{s+x}{2}\right)b\left(\frac{s+x}{2}\right)e^{i\lambda s}ds,$$

$$A_5 = -\int_0^x u(t)\lambda\sin\lambda(x-t)\left(\int_{-t}^t K_{11}(t,s)e^{i\lambda s}ds\right)dt$$

$$= -\frac{\lambda}{2i}\int_{-x}^x\left(\int_{(x-t)/2}^x u(s)K_{11}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$+\frac{\lambda}{2i}\int_{-x}^x\left(\int_{(x+t)/2}^x u(s)K_{11}(s,t-s+x)ds\right)e^{i\lambda t}dt,$$

$$A_6 = -\int_0^x u^2(t)\cos\lambda(x-t)\left(\int_{-t}^t K_{11}(t,s)e^{i\lambda s}ds\right)dt$$

$$= -\frac{1}{2}\int_{-x}^x\left(\int_{(x-t)/2}^x u^2(s)K_{11}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^x\left(\int_{(x+t)/2}^x u^2(s)K_{11}(s,t-s+x)ds\right)e^{i\lambda t}dt,$$

$$A_7 = -\int_0^x u(t)\cos\lambda(x-t)\left(\int_{-t}^{t} K_{21}(t,s)e^{i\lambda s}ds\right)dt$$

$$= -\frac{1}{2}\int_{-x}^{x}\left(\int_{(x-t)/2}^{x} u(s)K_{21}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^{x}\left(\int_{(x+t)/2}^{x} u(s)K_{21}(s,t-s+x)ds\right)e^{i\lambda t}dt,$$

$$A_8 = -\int_0^x u(t)\cos\lambda(x-t)i\lambda\left(\int_{-t}^{t} K_{22}(t,s)e^{i\lambda s}ds\right)dt$$

$$= -\frac{i\lambda}{2}\int_{-x}^{x}\left(\int_{(x-t)/2}^{x} u(s)K_{22}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$-\frac{i\lambda}{2}\int_{-x}^{x}\left(\int_{(x+t)/2}^{x} u(s)K_{22}(s,t-s+x)ds\right)e^{i\lambda t}dt.$$

If we write the obtained expression in the right hand side of the equation (13), we get

$$b(x)e^{i\lambda x} + \int_{-x}^{x} K_{21}(x,t)e^{i\lambda t}dt + i\lambda\int_{-x}^{x} K_{22}(x,t)e^{i\lambda t}dt$$

$$= -\frac{i\lambda}{4}\int_{-x}^{x} u\left(\frac{t+x}{2}\right)e^{i\lambda t}dt$$

$$+\frac{i\lambda}{2}\int_{-x}^{x}\left(\int_{(x-t)/2}^{x} u(t)K_{11}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$-\frac{i\lambda}{2}\int_{-x}^{x}\left(\int_{(x+t)/2}^{x} u(s)K_{11}(s,t-s+x)ds\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}e^{i\lambda x}\int_0^x u^2(t)dt-\frac{1}{4}\int_{-x}^x u^2\left(\frac{t+x}{2}\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^x\left(\int_{(x-t)/2}^x u^2(s)K_{11}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^x\left(\int_{(x+t)/2}^x u^2(s)K_{11}(s,t-s+x)ds\right)e^{i\lambda t}dt$$

$$-\frac{i\lambda}{4}\int_{-x}^x u\left(\frac{t+x}{2}\right)e^{i\lambda t}dt-\frac{1}{2}e^{i\lambda x}\int_0^x u(t)b(t)dt$$

$$-\frac{1}{4}\int_{-x}^x u\left(\frac{t+x}{2}\right)b\left(\frac{t+x}{2}\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^x\left(\int_{(x-t)/2}^x u(s)K_{21}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$-\frac{1}{2}\int_{-x}^x\left(\int_{(x+t)/2}^x u(s)K_{21}(s,t-s+x)ds\right)e^{i\lambda t}dt$$

$$-\frac{i\lambda}{2}\int_{-x}^x\left(\int_{(x-t)/2}^x u(s)K_{22}(s,t+s-x)ds\right)e^{i\lambda t}dt$$

$$-\frac{i\lambda}{2}\int_{-x}^x\left(\int_{(x-t)/2}^x u(s)K_{22}(s,t-s+x)ds\right)e^{i\lambda t}dt. \quad (13')$$

From the equation $(13')$, we get the following integral equations for the functions $K_{21}(x,t)$ and $K_{22}(x,t)$;

$$K_{21}(x,t)=-\frac{1}{4}u^2\left(\frac{t+x}{2}\right)-\frac{1}{4}u\left(\frac{t+x}{2}\right)b\left(\frac{t+x}{2}\right)$$

$$-\frac{1}{2}\int_{(x-t)/2}^x u^2(s)K_{11}(s,t+s-x)ds$$

$$- \frac{1}{2} \int_{(x+t)/2}^{x} u^2(s) K_{11}(s, t-s+x) ds$$

$$- \frac{1}{2} \int_{(x-t)/2}^{x} u(s) K_{21}(s, t+s-x) ds$$

$$- \frac{1}{2} \int_{(x+t)/2}^{x} u(s) K_{21}(s, t-s+x) ds,$$

$$K_{22}(x,t) = -\frac{1}{2} u\left(\frac{t+x}{2}\right) + \frac{1}{2} \int_{(x-t)/2}^{x} u(s) K_{11}(s, t+s-x) ds$$

$$- \frac{1}{2} \int_{(x+t)/2}^{x} u(s) K_{11}(s, t-s+x) ds - \frac{1}{2} \int_{(x-t)/2}^{x} u(s) K_{22}(s, t+s-x) ds$$

$$- \frac{1}{2} \int_{(x+t)/2}^{x} u(s) K_{22}(s, t-s+x) ds,$$

$$b(x) = -\frac{1}{2} \int_{0}^{x} u^2(s) e^{\frac{-1}{2} \int_{s}^{x} u(\xi) d\xi} ds.$$

3. The Existence of the Solution of the System of Integral Equations

Let us apply the successive approximation method for the existence of the solution of the system of integral equation. For this, we write the successive approximation as:

$$K_{11}^{(0)}(x,t) = \frac{1}{2} u\left(\frac{x+t}{2}\right) - \frac{1}{2} \int_{0}^{(x+t)/2} \left(u^2(s) + u(s) b(s)\right) ds,$$

$$K_{21}^{(0)}(x,t) = -\frac{1}{4} u^2\left(\frac{x+t}{2}\right) - \frac{1}{4} u\left(\frac{t+x}{2}\right) b\left(\frac{x+t}{2}\right),$$

$$K_{22}^{(0)}(x,t) = -\frac{1}{2}u\left(\frac{x+t}{2}\right),$$

$$K_{11}^{(n)}(x,t) = \frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{11}^{(n-1)}(s,t+s-x)ds$$
$$+\frac{1}{2}\int\limits_{(x+t)/2}^{x} u(s)K_{11}^{(n-1)}(s,t-s+x)ds$$
$$+\frac{1}{2}\int\limits_{0}^{x} u^2(s)\int\limits_{t-x+s}^{t+x-s} K_{11}^{(n-1)}(s,\xi)d\xi ds$$
$$+\frac{1}{2}\int\limits_{0}^{x} u(s)\int\limits_{t-x+s}^{t+x-s} K_{21}^{(n-1)}(s,\xi)d\xi ds$$
$$-\frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{22}^{(n-1)}(s,t-x+s)ds + \frac{1}{2}\int\limits_{(x+t)/2}^{x} u(s)K_{22}^{(n-1)}(s,t-s+x)ds,$$

$$K_{21}^{(n)}(x,t) = -\frac{1}{2}\int\limits_{(x-t)/2}^{x} u^2(s)K_{11}^{(n-1)}(s,t+s-x)ds$$
$$-\frac{1}{2}\int\limits_{(x+t)/2}^{x} u^2(s)K_{11}^{(n-1)}(s,t-s+x)ds$$
$$-\frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{21}^{(n-1)}(s,t+s-x)ds$$
$$-\frac{1}{2}\int\limits_{(x+t)/2}^{x} u(s)K_{21}^{(n-1)}(s,t-s+x)ds,$$

$$K_{22}^{(n)}(x,t) = \frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{11}^{(n-1)}(s,t+s-x)ds$$

$$-\frac{1}{2}\int\limits_{(x+t)/2}^{x} u(s)K_{11}^{(n-1)}(s,t-s+x)ds$$

$$-\frac{1}{2}\int\limits_{(x-t)/2}^{x} u(s)K_{22}^{(n-1)}(s,t+s-x)ds$$

$$-\frac{1}{2}\int\limits_{(x+t)/2}^{x} u(s)K_{22}^{(n-1)}(s,t-s+x)ds.$$

We put

$$\sigma(x)=\frac{1}{2}\int\limits_{0}^{x}(|u(s)|+|u(s)|^2)ds.$$

We get the following inequalities from the expressions of the functions $K_{11}^{(0)}(x,t)$, $K_{21}^{(0)}(x,t)$, and $K_{22}^{(0)}(x,t)$:

$$\int\limits_{-x}^{x}\left|K_{11}^{(0)}(x,t)\right|dt\le\frac{1}{2}\int\limits_{0}^{x}\left\{|u(\xi)|+|u(\xi)|^2\right\}d\xi,$$

$$\int\limits_{-x}^{x}\left|K_{21}^{(0)}(x,t)\right|dt\le\frac{1}{2}\int\limits_{0}^{x}\left\{|u(\xi)|^2+|u(\xi)|\right\}d\xi,$$

$$\int\limits_{-x}^{x}\left|K_{22}^{(0)}(x,t)\right|dt\le\frac{1}{2}\int\limits_{0}^{x}\left\{|u(\xi)|^2+|u(\xi)|\right\}d\xi,$$

$$\int\limits_{-x}^{x}\left|K_{22}^{(1)}(x,t)\right|dt\le\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x-t)/2}^{x}|u(s)|\left|K_{11}^{(0)}(s,t+s-x)\right|ds$$

$$+\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x+t)/2}^{x}|u(s)|\left|K_{11}^{(0)}(s,t-s+x)\right|ds$$

$$+\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x-t)/2}^{x}|u(s)|\left|K_{22}^{(0)}(s,t+s-x)\right|ds$$

$$+\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x+t)/2}^{x}|u(s)|\,K_{22}^{(0)}(s,t-s+x)ds,$$

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x-t)/2}^{x}|u(s)|\left|K_{11}^{(0)}(s,t+s-x)\right|ds$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{x-2s}^{x}\left|K_{11}^{(0)}(s,t+s-x)\right|dt$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-s}^{s}\left|K_{11}^{(0)}(s,\xi)\right|d\xi\leq\frac{\sigma^2(x)}{2!}$$

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x+t)/2}^{x}|u(s)|\left|K_{11}^{(0)}(s,t-s+x)\right|ds$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-x}^{2s-x}\left|K_{11}^{(0)}(s,t-s+x)\right|dt$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-s}^{s}\left|K_{11}^{(0)}(s,\xi)\right|d\xi\leq\frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x-t)/2}^{x}|u(s)|\left|K_{22}^{(0)}(s,t+s-x)\right|ds$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-s}^{s}\left|K_{22}^{(0)}(s,\xi)\right|d\xi\leq\frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x+t)/2}^{x}|u(s)|\left|K_{22}^{(0)}(s,t-s+x)\right|ds$$

$$= \frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-s}^{s} \left|K_{22}^{(0)}(s,\xi)\right| d\xi \le \frac{\sigma^2(x)}{2!}$$

$$\int_{-x}^{x} \left|K_{21}^{(1)}(x,t)\right| dt \le \frac{1}{2}\int_{-x}^{x} dt \int_{(x-t)/2}^{x} |u(s)|^2 \left|K_{11}^{(0)}(s,t+s-x)\right| ds$$

$$+ \frac{1}{2}\int_{-x}^{x} dt \int_{(x+t)/2}^{x} |u(s)|^2 \left|K_{11}^{(0)}(s,t-s+x)\right| ds$$

$$+ \frac{1}{2}\int_{-x}^{x} dt \int_{(x-t)/2}^{x} |u(s)| \left|K_{21}^{(0)}(s,t+s-x)\right| ds$$

$$+ \frac{1}{2}\int_{-x}^{x} dt \int_{(x+t)/2}^{x} |u(s)| \left|K_{21}^{(0)}(s,t-s+x)\right| ds$$

$$\frac{1}{2}\int_{-x}^{x} dt \int_{(x-t)/2}^{x} |u(s)|^2 \left|K_{11}^{(0)}(s,t+s-x)\right| ds$$

$$= \frac{1}{2}\int_{0}^{x} |u(s)|^2\, ds \int_{x-2s}^{x} \left|K_{11}^{(0)}(s,t+s-x)\right| dt$$

$$= \frac{1}{2}\int_{0}^{x} |u(s)|^2\, ds \int_{-s}^{s} \left|K_{11}^{(0)}(s,\xi)\right| d\xi \le \frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int_{-x}^{x} dt \int_{(x+t)/2}^{x} |u(s)|^2 \left|K_{11}^{(0)}(s,t-s+x)\right| ds$$

$$= \frac{1}{2}\int_{0}^{x} |u(s)|^2\, ds \int_{-x}^{2s-x} \left|K_{11}^{(0)}(s,t-s+x)\right| dt$$

$$= \frac{1}{2}\int_{0}^{x} |u(s)|^2\, ds \int_{-s}^{s} \left|K_{11}^{(0)}(s,\xi)\right| d\xi \le \frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int_{-x}^{x} dt \int_{(x-t)/2}^{x} |u(s)| \left|K_{21}^{(0)}(s, t+s-x)\right| ds$$
$$= \frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{x-2s}^{x} \left|K_{21}^{(0)}(s, t+s-x)\right| dt$$
$$= \frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-s}^{s} \left|K_{21}^{(0)}(s, \xi)\right| d\xi \le \frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int_{-x}^{x} dt \int_{(x+t)/2}^{x} |u(s)| \left|K_{21}^{(0)}(s, t-s+x)\right| ds$$
$$= \frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-x}^{2s-x} \left|K_{21}^{(0)}(s, t-s+x)\right| dt$$
$$= \frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-s}^{s} \left|K_{21}^{(0)}(s, \xi)\right| d\xi \le \frac{\sigma^2(x)}{2!}.$$

Now, let us get the similar evaluations for the function $K_{11}^{(1)}(x,t)$. From the expression of $K_{11}^{(1)}(x,t)$. We get:

$$\int_{-x}^{x} \left|K_{11}^{(1)}(x,t)\right| dt \le \frac{1}{2}\int_{-x}^{x} dt \int_{(x-t)/2}^{x} |u(s)| \left|K_{11}^{(0)}(s, t+s-x)\right| ds$$
$$+ \frac{1}{2}\int_{-x}^{x} dt \int_{(x+t)/2}^{x} |u(s)| \left|K_{11}^{(0)}(s, t-s+x)\right| ds$$
$$+ \frac{1}{2}\int_{-x}^{x} dt \int_{0}^{x} |u(s)|^2 \int_{t-x+s}^{t+x-s} \left|K_{11}^{(0)}(s, \xi)\right| d\xi ds$$
$$+ \frac{1}{2}\int_{-x}^{x} dt \int_{0}^{x} |u(s)| \int_{t-x+s}^{t+x-s} \left|K_{21}^{(0)}(s, \xi)\right| d\xi ds$$

$$+\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x-t)/2}^{x}|u(s)|\left|K_{22}^{(0)}(s,t+s-x)\right|ds$$

$$+\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x+t)/2}^{x}|u(s)|\left|K_{22}^{(0)}(s,t-s+x)\right|ds.$$

Using a similar method in $K_{21}^{(1)}(x,t)K_{22}^{(1)}(x,t)$. If we evaluate the integral in the right hand side of the last inequality, we get the following inequalities;

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x-t)/2}^{x}|u(s)|\left|K_{11}^{(0)}(s,t+s-x)\right|ds$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-s}^{s}\left|K_{11}^{(0)}(s,\xi)\right|d\xi\leq\frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x+t)/2}^{x}|u(s)|\left|K_{11}^{(0)}(s,t-s+x)\right|ds$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-s}^{s}\left|K_{11}^{(0)}(s,\xi)\right|d\xi\leq\frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x-t)/2}^{x}|u(s)|\left|K_{22}^{(0)}(s,t+s-x)\right|ds$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-s}^{s}\left|K_{22}^{(0)}(s,\xi)\right|d\xi\leq\frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int\limits_{-x}^{x}dt\int\limits_{(x+t)/2}^{x}|u(s)|\left|K_{22}^{(0)}(s,t-s+x)\right|ds$$
$$=\frac{1}{2}\int\limits_{0}^{x}|u(s)|\,ds\int\limits_{-s}^{s}\left|K_{22}^{(0)}(s,\xi)\right|d\xi\leq\frac{\sigma^2(x)}{2!}$$

$$\frac{1}{2}\int_{-x}^{x} dt \int_{0}^{x} |u(s)|^2 \int_{t-x+s}^{t+x-s} \left|K_{11}^{(0)}(s,\xi)\right| d\xi ds$$
$$= \frac{1}{2}\int_{0}^{x} |u(s)|^2 \left(\int_{-x}^{x} dt \int_{t-x+s}^{t+x-s} \left|K_{11}^{(0)}(s,\xi)\right| d\xi\right) ds$$
$$= x\int_{0}^{x} |u(s)|^2 \sigma(s) ds \leq \pi \frac{\sigma^2(x)}{2!},$$

$$\frac{1}{2}\int_{-x}^{x} dt \int_{0}^{x} |u(s)| \int_{t-x+s}^{t+x-s} \left|K_{21}^{(0)}(s,\xi)\right| d\xi ds$$
$$= \frac{1}{2}\int_{0}^{x} |u(s)| \left(\int_{-x}^{x} dt \int_{t-x+s}^{t+x-s} \left|K_{21}^{(0)}(s,\xi)\right| d\xi\right) ds$$
$$= x\int_{0}^{x} |u(s)| \sigma(s) ds \leq \pi \frac{\sigma^2(x)}{2!}.$$

Since, $\int_{-x}^{x} \left|K_{ij}^{0}(x,t)\right| dt \leq \sigma(x)$ we will show that the following inequality is true. For this, let us use the induction method:

$$\int_{-x}^{x} \left|K_{ij}^{(m)}(x,t)\right| dt \leq \frac{\{\sigma(x)\}^{m+1}}{(m+1)!}, \quad m = 0, 1, \ldots. \tag{14}$$

For $m = 0, 1$ it is clear that the above inequality is true. We assume that it is true for $(m-1)$ and we show it for m:

$$\int_{-x}^{x} \left|K_{11}^{(m)}(x,t)\right| dt \leq \frac{1}{2}\int_{-x}^{x} dt \int_{(x-t)/2}^{x} |u(s)| \left|K_{11}^{(m-1)}(s,t+s-x)\right| ds$$
$$+ \frac{1}{2}\int_{-x}^{x} dt \int_{(x+t)/2}^{x} |u(s)| \left|K_{11}^{(m-1)}(s,t-s+x)\right| ds$$

$$+\frac{1}{2}\int_{-x}^{x} dt \int_{(x-t)/2}^{x} |u(s)| \left|K_{22}^{(m-1)}(s, t+s-x)\right| ds$$

$$+\frac{1}{2}\int_{-x}^{x} dt \int_{(x+t)/2}^{x} |u(s)| \left|K_{22}^{(m-1)}(s, t-s+x)\right| ds$$

$$+\frac{1}{2}\int_{-x}^{x} dt \int_{0}^{x} \left|u^2(s)\right| \int_{t-x+s}^{t+x-s} \left|K_{11}^{(m-1)}(s,\xi)\right| d\xi ds$$

$$+\frac{1}{2}\int_{-x}^{x} dt \int_{0}^{x} |u(s)| \int_{t-x+s}^{t+x-s} \left|K_{21}^{(m-1)}(s,\xi)\right| d\xi ds$$

$$=\frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-s}^{s} \left|K_{11}^{(m-1)}(s,\xi)\right| d\xi$$

$$+\frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-s}^{s} \left|K_{11}^{(m-1)}(s,\xi)\right| d\xi \frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-s}^{s} \left|K_{22}^{(m-1)}(s,\xi)\right| d\xi$$

$$+\frac{1}{2}\int_{0}^{x} |u(s)|\, ds \int_{-s}^{s} \left|K_{22}^{(m-1)}(s,\xi)\right| d\xi$$

$$+\frac{1}{2}\int_{0}^{x} |u(s)|^2 \left(\int_{-x}^{x} dt \int_{t-x+s}^{t+x-s} \left|K_{11}^{(m-1)}(s,\xi)\right| d\xi\right) ds$$

$$+\frac{1}{2}\int_{0}^{x} |u(s)| \left(\int_{-x}^{x} dt \int_{t-x+s}^{t+x-s} \left|K_{21}^{(m-1)}(s,\xi)\right| d\xi\right) ds \le \frac{\{\sigma(x)\}^{m+1}}{(m+1)!}.$$

From inequalities (14) it is obvious that the series $\sum\limits_{m=0}^{\infty} K_{ij}^{(m)}(x,t)$ are uniformly convergent in the space $L_1[0,\pi]$ and the functions $K_{ij}(x,.) \in L_1[0,\pi]$ which are the sum of that series, which satisfies the following inequality

$$\int_{-x}^{x} |K_{ij}(x,t)|\, dt \le e^{\sigma(x)} - 1.$$

Thus, we proved the following theorem.

Theorem 2. *For every solutions of the problem L which satisfies the initial conditions*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ i\lambda \end{pmatrix},$$

the following expression is true:

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ i\lambda \end{pmatrix} e^{i\lambda x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} b(x) e^{i\lambda x} + \int\limits_{-x}^{x} K(x,t) \begin{pmatrix} 1 \\ i\lambda \end{pmatrix} e^{i\lambda t} dt\,,$$

where,

$$b(x) = -\frac{1}{2}\int\limits_0^x u^2(s) e^{\frac{-1}{2}\int\limits_s^x U(\xi)d\xi} ds,$$

$$K(x,t) = \begin{pmatrix} K_{11}(x,t) & 0 \\ K_{21}(x,t) & K_{22}(x,t) \end{pmatrix},$$

$$K_{11}(x,x) = \frac{1}{2}u(x),$$

$$K_{21}(x,x) = \frac{1}{2}b'(x) - \frac{1}{2}\int\limits_0^x u^2(s) K_{11}(s,s) ds - \frac{1}{2}\int\limits_0^x u(s) K_{21}(s,s) ds$$

and

$$K_{22}(x,x) = -\frac{1}{2}u(x) - b(x).$$

Corollary 3. *For every solution*

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \dfrac{y_1 + \overline{y_1}}{2} + \dfrac{h}{\lambda}\dfrac{y_1 - \overline{y_1}}{2i} \\ \dfrac{y_2 + \overline{y_2}}{2} + \dfrac{h}{\lambda}\dfrac{y_2 - \overline{y_2}}{2i} \end{pmatrix}$$

of the system (6) which satisfies the initial conditions $\varphi_1(0,\lambda) = 1$, $\varphi_2(0,\lambda) = h$ *the expression (7) is true.*

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