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SPECTRA OF THE LAPLACE OPERATOR ON GRASSMANN MANIFOLDS

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Abstract: In this paper, we explicitly compute the Laplace spectrum on the forms for Grassmann manifolds. This is a generalization of A. Ikeda–Y. Taniguchi and C. Tsukamoto calculations, based on the representation theory of compact Lie groups and on the "identification" of the Laplace operator with the Casimir operator in symmetric spaces.

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1. Introduction

Let M be a compact Riemannian manifold. We consider the Laplace operator Δ_p acting on the space of differential p-forms and its spectrum. An interesting problem is to compute explicitly the eigenvalues of the Laplace operator.

Many people work on this problem. We refer to [1], [6], [11] and [12]. In the case where $M = G/K$ is a compact symmetric space with G compact and

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semisimple, Ikeda and Taniguchi [10] propose a method to compute the spectrum on the forms using representations of G and K. The operator Δ_p is G-invariant when we consider the space of p-forms $C^\infty(\wedge^p M)$ as a G-module. They show that $\Delta_p = C$, the Casimir operator on G. On the other hand, the Freudenthal formula [2] gives the eigenvalues of C on irreducible G-modules. Then, it suffices to decompose $C^\infty(\wedge^p M)$ into irreducible G-submodules. Generally, this decomposition is not easy. Frobenius reciprocity law enables us to reduce the problem into the two followings: first, decomposing an irreducible G-module (as a K-module by restriction) into irreducible K-submodules, second, decomposing the p-th exterior power of the adjoint representation of the group K into irreducible K-submodules. C. Tsukamoto [13] uses this method to compute the spectra of the spaces $\mathrm{SO}\,(n+2)/\,\mathrm{SO}\,(2)\times \mathrm{SO}\,(n)$ and $\mathrm{Sp}\,(n+1)/\,\mathrm{Sp}\,(1)\times \mathrm{Sp}\,(n)$.

In this paper, we generalize the result of [13] to calculate the spectrum on the forms of the Grassmann manifolds. First, we give a branching law for $G = \mathrm{SO}\,(n)$ and $K = \mathrm{SO}\,(2q) \times \mathrm{SO}\,(n-2q)$. Next, we decompose the exterior powers of the adjoint representation into irreducible K-modules (see [4] for details and examples).

2. Preliminary

In this section, we introduce the definitions and notations that we will use later.

Let (G, K) be a compact symmetric pair with a compact connected semisimple Lie group G and $M = G/K$. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K respectively. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, with the properties $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. We identify $\mathfrak{m}$ with the tangent space of M at $o = \pi(e)$, where π is the projection $\pi : G \to M$. Restricting the Killing form sign changed to $\mathfrak{m}$, we can define a G-invariant Riemannian metric on M. We can extend this metric canonically to a Hermitian metric $\langle\ ,\ \rangle$ on $\wedge^p M$, the p-th exterior power of the complexified cotangent bundle of M. We defined the inner product $(\ ,\ )_M$ on $C^\infty(\wedge^p M)$ by

$$(\varphi, \psi)_M = \int_M \langle \varphi(m), \psi(m)\rangle\ dm, \quad \varphi, \psi \in C^\infty(\wedge^p M),$$

where dm is the smooth measure on M defined by the Riemannian metric.

$C^\infty(\wedge^p M)$ is considered as a G-module by $(g.\varphi)(x) = g.\varphi(g^{-1}.x)$ for $g \in G$, $\varphi \in C^\infty(\wedge^p M)$ and $x \in M$. It is clear that the inner product $(\ ,\ )_M$ is G-invariant.

We consider the Laplace-Beltrami operator Δ_p acting on $C^\infty(\wedge^p M)$. It is a self-adjoint, strongly elliptic differential operator and commutes with the G-action on $C^\infty(\wedge^p M)$. The set of eigenvalues of Δ_p is a discrete set of non-negative real numbers. Each eigenspace is a finite dimensional G-submodule of $C^\infty(\wedge^p M)$ and their sum is a dense subspace of $C^\infty(\wedge^p M)$. The collection of the eigenvalues with their multiplicities is called p-spectrum of Laplacian. Each irreducible G-submodule of $C^\infty(\wedge^p M)$ is included in some eigenspace of Δ_p and the sum of irreducible G-submodules of $C^\infty(\wedge^p M)$ equals to the sum of eigenspace of Δ_p.

Let $\{X_1, \dots, X_N\}$ be an orthonormal basis of $\mathfrak{g}$, with respect to the inner product induced from the Killing form B. We introduce the Casimir operator

$$\mathcal{C} = -\sum_{i=1}^{N} X_i^2 \,,$$

which is a two-sided invariant differential operator on G. We identify $\mathfrak{g}/\mathfrak{k}$ with $\mathfrak{m}$ by the adjoint action of K in $\mathfrak{g}$.

Let U a finite dimensional vector space. The space $C^\infty(G, U)$ is a G-module by $(g.f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C^\infty(G, U)$ and $x \in G$. If U is a K-module, we denote by $C^\infty(G, K, U)$ the G-submodule of $C^\infty(G, U)$ consisting of $f \in C^\infty(G, U)$ such that $f(gk) = k^{-1}f(g)$ for $k \in K$ and $g \in G$.

Proposition 1. *The G-modules $C^\infty(\wedge^p M)$ and $C^\infty(G, K, \wedge^p(\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}})$ are identified by the isomorphism:*

$$\alpha \mapsto \Phi(\alpha) = \tilde{\alpha},$$

where for all $g \in G$, $Y_1, Y_2, \dots, Y_p \in \mathfrak{g}/\mathfrak{k}$:

$$\tilde{\alpha}(g)(Y_1, Y_2, \dots, Y_p) = (\pi \circ L_g)^*(\alpha)(g)(Y_1, Y_2, \dots, Y_p).$$

Proposition 2. *Under the identification*

$$C^\infty(\wedge^p M) = C^\infty(G, K, \wedge^p(\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}}),$$

we have $\Delta = \mathcal{C}$, i.e:

$$\widetilde{\Delta\alpha} = \mathcal{C}\tilde{\alpha}, \quad \forall \alpha \in C^\infty(\wedge^p M)\,.$$

Let T be a maximal torus of G of Lie algebra $\mathfrak{t}$. We denote by $N = N(T)$ the normalizer of T in G and by $W_G = N/T$ the Weyl group of G associated to T. Let Δ_G be the set of real roots of G with respect to the maximal torus T and Δ_G^+ the set of positive roots of G. We choose a W-invariant inner product on $\mathfrak{t}$ and $\mathfrak{t}^*$. We denote by I^* the lattice of integral forms.

Definition 3. Let $\Lambda \in \mathfrak{t}^*$ be a linear form on $\mathfrak{t}$. We introduce the alternate sum of Λ

$$\xi(\Lambda) : \mathfrak{t} \to \mathbb{C}, \quad \text{with} \quad \xi(\Lambda)(H) = \sum_{w \in W_G} \det(w).e(\Lambda(w(H))), \ \forall H \in \mathfrak{t}.$$

Here, $\det(w) \in \{-1, 1\}$, is the determinant of the linear automorphism w of $\mathfrak{t}$.

We denote by $\delta = \delta_G = \sum_{\alpha \in \Delta_G^+} \alpha/2$, the half sum of positive roots,

$\rho = \prod_{\alpha \in \Delta_G^+}(e(\alpha/2) - e(-\alpha/2))$ and $c(\Lambda) = \dfrac{\xi(\Lambda)}{\rho}$ for $\Lambda \in \mathfrak{t}^*$

Proposition 4. (Weyl Character Formula, see [3]) *Let* $\mathrm{Irr}\,(G,T) \subset R(T)$ *denote the set of restrictions of irreducible characters of* G *on the maximal torus* T.

(i) If γ is an integral form and $\langle \gamma, \alpha \rangle \geq 0$ for all positive roots α, then $c(\gamma + \delta) \in \mathrm{Irr}\,(G,T)$ and the correspondence $\gamma \mapsto c(\gamma+\delta)$ is bijective.

(ii) We have $\rho = \xi(\delta)$.

(iii) If γ is an integral form with $\langle \gamma, \alpha \rangle \geq 0$ for all positive roots α, the corresponding irreducible representation has dimension

$$\prod_{\alpha \in \Delta_G^+} \frac{\langle \alpha, \gamma + \delta \rangle}{\langle \alpha, \delta \rangle}.$$

For an irreducible representation V, we call dominant weight associated to V, the integral form defined by the bijection given by the Proposition 4, (i).

A decomposition into positive roots of an integral form $\tau \in I^*$ is a family of nonnegative integers $(n_\alpha)_{\alpha \in \Delta_G^+}$ such that $\tau = \sum_{\alpha \in \Delta_G^+} n_\alpha.\alpha$. Let $p(\tau)$ denote the cardinality of the set of all decompositions of τ into positive roots.

Proposition 5. (Steinberg Multiplicity Formula, see [3]) *Let $\chi(\gamma)$ denote the character of the irreducible reperesentation $V(\gamma)$, with dominant weight γ. Let $V(\gamma)$ and $V(\Lambda)$ be two irreducible representations with dominant weight γ and Λ respectively. The character of the tensor product $V(\gamma) \otimes V(\Lambda)$ is:*

$$\chi(\gamma).\chi(\Lambda) = \sum m(\gamma, \Lambda, \mu)\chi(\mu),$$

with

$$m(\gamma, \Lambda, \mu) = \sum_{v,w \in W_G} \det(v.w).p(v(\gamma+\delta) + w(\Lambda+\delta) - (\mu + 2\delta)),$$

and the summation is taken over all the integral forms μ such that $\langle \mu, \alpha \rangle \geq 0$ for all positive roots α.

Proposition 6. (Freudenthal Formula, see [2], [5]) *Let (ρ, V) be an irreducible representation of G over $\mathbb{C}$ with dominant weight Λ. Then, we have:*

$$\rho(C) = \langle \Lambda + 2\delta_G, \Lambda \rangle . 1_V,$$

where $\langle\ ,\ \rangle = -B$.

Theorem 7. (Frobenius Reciprocity, see [3]) *For any (real or complexe) finite dimensional K-module U and finite dimensional G-module V, we have a canonical isomorphism as vector spaces*

$$\mathrm{Hom}_G(V, C^\infty(G, K, U)) \cong \mathrm{Hom}_K(V, U).$$

Let $\mathcal{I}_G$ be a complete set of inequivalent irreducible representations of G over $\mathbb{C}$. For an element $(\rho, V) \in \mathcal{I}_G$, we define a G-homomorphism:

$$\begin{array}{rccc} i_\rho : & \mathrm{Hom}_G(V, C^\infty(\wedge^p M)) \otimes_{\mathbb{C}} V & \rightarrow & C^\infty(\wedge^p M), \\ & \phi \otimes u & \mapsto & \phi(u). \end{array}$$

i_ρ is injective because ρ is irreducible. We set $\mu_\rho = \dim_{\mathbb{C}} \mathrm{Hom}_G(V, C^\infty(\wedge^p M))$ and Γ^p_ρ the image of i_ρ. Then, Γ^p_ρ is isomorphic to the direct sum of μ_ρ copies of V. In the particular case, where $U = \wedge^p(\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}}$, we obtain $\mu_\rho = \dim_{\mathbb{C}} \mathrm{Hom}_K(V, \wedge^p(\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}})$. Then μ_ρ is finite.

Then, to decompose $C^\infty(G, K, \wedge^p(\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}})$ into irreducible G-submodules, it suffices to decompose any irreducible G-module into irreducible K-submodules, and the K-module $\wedge^p(\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}}$ into irreducible K-submodules.

3. Branching Laws

Let $G = \mathrm{SO}(n)$ and $K = \mathrm{SO}(2q) \times \mathrm{SO}(n-2q)$. Then $\mathfrak{g} = \mathfrak{so}(n)$ and $\mathfrak{k} = \mathfrak{so}(2q) \oplus \mathfrak{so}(n-2q)$. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ with:

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -{}^t X \\ X & 0 \end{pmatrix} \in \mathfrak{so}(n);\ X \in M(n-2q, 2q, \mathbb{R}) \right\}.$$

We will decompose any irreducible G-module into irreducible K-submodules. We denote by T_K a maximal torus of K with Lie algebra $\mathfrak{t}_{\mathfrak{k}}$, by W_K the Weyl group of K and by Δ_K the set of roots of K with respect to T_K. Let $V(\Lambda)$ (resp. $V'(\Lambda')$) be an irreducible G-module (resp. K-module) with highest weight Λ

(resp. Λ') and associated character $\chi_G(\Lambda)$ (resp. $\chi_K(\Lambda')$). We will determine the set E of highest weights of K such that:

$$(\chi_G(\Lambda))|_{\mathfrak{t}_{\mathfrak{k}}} = \sum_{\Lambda' \in E} \chi_K(\Lambda'). \tag{1}$$

Using the character Weyl formula, we obtain:

$$\frac{\xi_G(\Lambda + \delta_G)}{\xi_G(\delta_G)}|_{\mathfrak{t}_{\mathfrak{k}}} = \sum_{\Lambda' \in E} \frac{\xi_K(\Lambda' + \delta_K)}{\xi_K(\delta_K)}. \tag{2}$$

1. Case, where n is even, $n = 2m$:
We choose $T = T_K =$ SO (2) $\times \cdots \times$ SO (2). Then:

$$\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} = \left\{ \begin{pmatrix} R(\lambda_1) & & & \\ & R(\lambda_2) & & (0) \\ (0) & & \ddots & \\ & & & R(\lambda_m) \end{pmatrix} \in \mathfrak{g}; \right.$$

$$\left. \lambda_j \in \mathbb{R} \text{ and } R(\lambda) = 2\pi \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \right\},$$

where λ_j is considered to be an element of $\mathfrak{t}^*$.

- Any integral form for $(\mathfrak{g}, \mathfrak{t})$ is expressed as $h_1\lambda_1 + h_2\lambda_2 + ... + h_m\lambda_m;\ h_i \in \mathbb{Z}\}$.
- $\Delta_G = \{\pm\lambda_i \pm \lambda_j;\ 1 \leq i < j \leq m\}$.
- $\Delta_G^+ = \{\lambda_i \pm \lambda_j;\ 1 \leq i < j \leq m\}$.
- The simple roots:

$$\alpha_1 = \lambda_1 - \lambda_2,\ \alpha_2 = \lambda_2 - \lambda_3, ..., \alpha_{m-1} = \lambda_{m-1} - \lambda_m,\ \alpha_m = \lambda_{m-1} + \lambda_m.$$

- Any dominant weight for $(\mathfrak{g}, \mathfrak{t})$ which corresponds to an irreducible representation of G has the form $\Lambda = h_1\lambda_1 + h_2\lambda_2 + ... + h_m\lambda_m$, where the integers h_i satisfy $h_1 \geq h_2 \geq ... \geq h_{m-1} \geq |h_m|$. Equivalently:

$$\begin{cases} \Lambda = h_1\lambda_1 + h_2\lambda_2 + ... + \varepsilon h_m\lambda_m\,, \\ h_i \in \mathbb{Z} \quad \text{for all } 1 \leq i \leq m\,, \\ h_1 \geq h_2 \geq ... \geq h_m \geq 0\,, \\ \varepsilon = \pm 1\,. \end{cases} \tag{3}$$

• $W_G = \{\phi = (\varepsilon_1, ..., \varepsilon_m, \sigma)/\ \varepsilon_i = \pm 1, \sigma \in S_m,$ the number of ε_i equal to $-$ 1is even$\}$, with $\phi(a_1\lambda_1 + ... + a_m\lambda_m) = \sum_{i=1}^{m} \varepsilon_i a_i \sigma(\lambda_i)$, $\det(\phi) =$sign(σ) and S_m is the group of all permutations of $\{1, ..., m\}$.

• $\Delta_K = \{\pm\lambda_i \pm \lambda_j;\ 1 \leq i < j \leq q \text{ or } q+1 \leq i < j \leq m\}$.

• $\Delta_K^+ = \{\lambda_i \pm \lambda_j;\ 1 \leq i < j \leq q \text{ or } q+1 \leq i < j \leq m\}$.

• The simple roots are:

$$\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, ..., \lambda_{q-1} - \lambda_q, \lambda_{q-1} + \lambda_q,$$
$$\lambda_{q+1} - \lambda_{q+2}, \lambda_{q+2} - \lambda_{q+3}, ..., \lambda_{m-1} - \lambda_m, \lambda_{m-1} + \lambda_m.$$

• Any dominant weight for $(\mathfrak{k}, \mathfrak{t})$ which corresponds to an irreducible representation of K can be written:

$$\begin{cases} \Lambda' = k_1\lambda_1 + ... + \varepsilon' k_q \lambda_q + k_{q+1}\lambda_{q+1} + ... + \varepsilon'' k_m \lambda_m\,, \\ k_i \in \mathbb{Z} \quad \text{for all } \ 1 \leq i \leq m\,, \\ k_1 \geq k_2 \geq ... \geq k_q \geq 0\,, \\ k_{q+1} \geq k_{q+2} \geq ... \geq k_m \geq 0\,, \\ \varepsilon' = \pm 1, \quad \varepsilon'' = \pm 1\,. \end{cases} \tag{4}$$

• $W_K = W_{\mathrm{SO}\,(2q)} \times W_{\mathrm{SO}\,(n-2q)}$.

2. Case where n is odd, $n = 2m+1$:

• $T = T_K = \mathrm{SO}\,(2) \times \cdots \times \mathrm{SO}\,(2) \times \{1\}$ and

$$\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}} = \left\{ \begin{pmatrix} R(\lambda_1) & & & & \\ & R(\lambda_2) & & (0) & \\ & & \ddots & & \\ & (0) & & R(\lambda_m) & \\ & & & & 0 \end{pmatrix} \in \mathfrak{g};\ \lambda_j \in \mathbb{R} \right\}.$$

• $\Delta_G = \{\pm\lambda_i \pm \lambda_j/\ 1 \leq i < j \leq m\} \cup \{\pm\lambda_i/\ 1 \leq i \leq m\}$.

• $\Delta_G^+ = \{\lambda_i \pm \lambda_j/\ 1 \leq i < j \leq m\} \cup \{\lambda_i/\ 1 \leq i \leq m\}$.

• The simple roots are:

$$\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3, ..., \alpha_{m-1} = \lambda_{m-1} - \lambda_m, \alpha_m = \lambda_m.$$

• Any dominant weight for $(\mathfrak{g}, \mathfrak{t})$ is given by:

$$\begin{cases} \Lambda = h_1\lambda_1 + h_2\lambda_2 + ... + h_m\lambda_m\,, \\ h_i \in \mathbb{Z} \quad \text{for all } \ 1 \leq i \leq m\,, \\ h_1 \geq h_2 \geq ... \geq h_m \geq 0\,. \end{cases} \tag{5}$$

• $W_G = \{\phi = (\varepsilon_1, ..., \varepsilon_m, \sigma)/\ \varepsilon_i = \pm 1, \sigma \in S_m\}$, with $\phi(a_1\lambda_1 + ... + a_m\lambda_m) = \sum_{i=1}^{m} \varepsilon_i a_i \sigma(\lambda_i)$ and $\det(\phi) = \varepsilon_1 ... \varepsilon_m.\ \mathrm{sign}(\sigma)$.

• $\Delta_K = \{\pm\lambda_i \pm \lambda_j/\ 1 \leq i < j \leq q \ \text{ or } \ q+1 \leq i < j \leq m\} \cup \{\pm\lambda_i/\ q+1 \leq i \leq m\}$.

• $\Delta_K^+ = \{\lambda_i \pm \lambda_j/\ 1 \leq i < j \leq q \ \text{ or } \ q+1 \leq i < j \leq m\} \cup \{\lambda_i/\ q+1 \leq i \leq m\}$.

• The simple roots are:

$$\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, ..., \lambda_{q-1} - \lambda_q, \lambda_{q-1} + \lambda_q,$$
$$\lambda_{q+1} - \lambda_{q+2}, \lambda_{q+2} - \lambda_{q+3}, ..., \lambda_{m-1} - \lambda_m, \lambda_m.$$

• Any dominant weight for $(\mathfrak{k}, \mathfrak{t})$ is expressed as:

$$\begin{cases} \Lambda' = k_1\lambda_1 + ... + \varepsilon k_q\lambda_q + k_{q+1}\lambda_{q+1} + ... + k_m\lambda_m\,, \\ k_i \in \mathbb{Z} \quad \text{for all } \ 1 \leq i \leq m\,, \\ k_1 \geq k_2 \geq ... \geq k_q \geq 0\,, \\ k_{q+1} \geq k_{q+2} \geq ... \geq k_m \geq 0\,, \\ \varepsilon = \pm 1\,. \end{cases} \tag{6}$$

• $W_K = W_{\mathrm{SO}\,(2q)} \times W_{\mathrm{SO}\,(n-2q)}$.

Notation. (i) We denote by:

$$e(\Lambda) = e^{2\pi i \Lambda}, \quad s(\Lambda) = e(\Lambda) - e(-\Lambda), \quad c(\Lambda) = e(\Lambda) + e(-\Lambda),$$
$$\alpha_{ij} = \frac{\lambda_i + \lambda_j}{2},\ \beta_{ij} = \frac{\lambda_i - \lambda_j}{2}.$$

(ii) For r and s integers such that $1 \leq r \leq s$, we designate by $[a_{ij}]_{r:s}$ a square matrix with i, j between r and s.

Remark 8. If r is integer, then we have $\frac{s(rx)}{s(x)} = \sum_{k=0}^{r-1} e((2k - r + 1)x)$.

Lemma 9. *Let $H_1, ..., H_m$ be integers ≥ 0 or integers $+\dfrac{1}{2} > 0$, verifying $H_1 > ... > H_m$. We have for all $q \in \{1, \ldots, m\}$:*

1. $$\frac{\det[c(H_i\lambda_j)]_{1:m}}{(\prod_{i=1}^{q}\prod_{j=i+1}^{m} s(\alpha_{ij})s(\beta_{ij}))}$$
$$= \sum_{K_{1,i}} ... \sum_{K_{q,i}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) H^r c(l_{r,m}\lambda_r) \right\} \det[c(K_{q,i}\lambda_j)]_{q+1:m},$$

2. $$\frac{\det[s(H_i\lambda_j)]_{1:m}}{(\prod_{i=1}^{q}\prod_{j=i+1}^{m} s(\alpha_{ij})s(\beta_{ij}))}$$
$$= \sum_{K_{1,i}} \cdots \sum_{K_{q,i}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) s(l_{r,m}\lambda_r) \right\} \det[s(K_{q,i}\lambda_j)]_{q+1:m},$$
with:

- $H^r = \begin{cases} 1/2 & \text{for} \quad K_{r,m} = 0\,, \\ 1 & \text{for} \quad K_{r,m} > 0\,. \end{cases}$
- *For all* $1 \leq r \leq q$ *and* $r \leq i \leq m$*, the* $l_{r,i}$ *are given by:*

$$\begin{cases} l_{r,r} = K_{r-1,r} - \max(K_{r-1,r+1}, K_{r,r+1})\,, \\ l_{r,i} = \min(K_{r-1,i}, K_{r,i}) - \max(K_{r-1,i+1}, K_{r,i+1})\,, \\ \text{for} \quad r+1 \leq i \leq m-1\,, \\ l_{r,m} = \min(K_{r-1,m}, K_{r,m})\,, \end{cases}$$

- *The summations are taken over all the sets of* $K_{r,i}$ *(*$1 \leq r \leq q$ *and* $r+1 \leq i \leq m$*) satisfying:*

$$(C_r) \quad \begin{cases} K_{r-1,i+1} < K_{r,i} < K_{r-1,i-1}, & \text{for} \quad r+1 \leq i \leq m-1, \\ K_{r,m} < K_{r-1,m-1}, \\ K_{r,m} < K_{r,m-1} < \ldots < K_{r,r+1}. \end{cases}$$

The $K_{r,i}$ *are integers* ≥ 0 *(resp. integers* $+\frac{1}{2} > 0$*) if the* H_i *are integers* ≥ 0 *(resp. integers* $+\frac{1}{2} > 0$*).*

Proof. We prove the first part when the H_i are integers. The case $q = 1$ is proved by Tsukamoto [13]. Using twice this case, we obtain:

$$\frac{\det[c(H_i\lambda_j)]_{1:m}}{\prod_{i=1}^{2}\prod_{j=i+1}^{m} s(\alpha_{ij})s(\beta_{ij})} = \sum_{K_{1,i}} \sum_{K_{2,i}} \left\{ \left(\prod_{i=1}^{m-1} \frac{s(l_{1,i}\lambda_1)}{s(\lambda_1)} \right) \left(\prod_{i=2}^{m-1} \frac{s(l_{2,i}\lambda_2)}{s(\lambda_2)} \right) \times H^1 H^2 c(l_{1,m}\lambda_1) c(l_{2,m}\lambda_2). \det[c(K_{2,i}\lambda_j)]_{3:m} \right\}.$$

The assertion is proved recursively. □

We introduce the integers $k_{r,i}$ for $1 \leq r \leq q-1$ and $r+1 \leq i \leq m$ like this:

- If $2r < 3q - m + 1$:

$$\begin{cases} \max(k_{r-1,i+1}, k_{q,i+q-r}) \leq k_{r,i} \leq k_{r-1,i-1}, \\ \qquad\qquad \text{for } r+1 \leq i \leq m-q+r, \\ k_{r-1,i+1} \leq k_{r,i} \leq k_{r-1,i-1}, \\ \qquad\qquad \text{for } m-q+r+1 \leq i \leq 2q-r, \\ k_{r-1,i+1} \leq k_{r,i} \leq \min(k_{r-1,i-1}, k_{q,i-q+r}), \\ \qquad\qquad \text{for } 2q-r+1 \leq i \leq m-1, \\ k_{r,m} \leq \min(k_{r-1,m-1}, k_{q,m-q+r}), \\ 0 \leq k_{r,m} \leq \ldots \leq k_{r,r+1}. \end{cases} \tag{7}$$

- If $2r \geq 3q - m + 1$:

$$\begin{cases} \max(k_{r-1,i+1}, k_{q,i+q-r}) \leq k_{r,i} \leq k_{r-1,i-1}, \\ \qquad\qquad \text{for} \quad r+1 \leq i \leq 2q-r, \\ \max(k_{r-1,i+1}, k_{q,i+q-r}) \leq k_{r,i} \leq \min(k_{r-1,i-1}, k_{q,i-q+r}), \\ \qquad\qquad \text{for} \quad 2q-r+1 \leq i \leq m-q+r, \\ k_{r-1,i+1} \leq k_{r,i} \leq \min(k_{r-1,i-1}, k_{q,i-q+r}) \quad \text{for,} \\ \qquad\qquad m-q+r+1 \leq i \leq m-1, \\ k_{r,m} \leq \min(k_{r-1,m-1}, k_{q,m-q+r}) \\ 0 \leq k_{r,m} \leq \ldots \leq k_{r,r+1}. \end{cases} \tag{8}$$

Theorem 10. *Let $G = \mathrm{SO}(2m)$, $K = \mathrm{SO}(2q) \times \mathrm{SO}(2m-2q)$ and V be an irreducible G-module of highest weight $\Lambda = h_1\lambda_1 + \ldots + \varepsilon h_m\lambda_m$. Then the irreducible decomposition of V as a K-module contains an irreducible K-submodule V' with the highest weight $\Lambda' = k_1\lambda_1 + \ldots + \varepsilon' k_q\lambda_q + k_{q+1}\lambda_{q+1} + \ldots + \varepsilon'' k_m\lambda_m$, if and only if:*

1. $\begin{cases} h_{i+q} \leq k_i \leq h_{i-q} & \text{for} \quad q+1 \leq i \leq m-q, \\ k_i \leq h_{i-q} & \text{for} \quad m-q+1 \leq i \leq m. \end{cases}$

2. The multiplicity $m_{\Lambda'}$ of $V' = V(\Lambda')$ in the decomposition does not vanish, and has the expression:

(i) If $k_m = 0$:

$m_{\Lambda'}$ is the multiplicity of $e((k_1 + q - 1)\lambda_1 + \ldots + \varepsilon' k_q\lambda_q)$ in:

$$\prod_{i=1}^{q-1} \prod_{j=i+1}^{q} s(\alpha_{ij}) s(\beta_{ij}) \times \sum_{k_{1,i}} \cdots \sum_{\substack{k_{q-1,i} \\ \text{un des } k_{r,m}=0}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) H^r c(l_{r,m}\lambda_r) \right\}.$$

(ii) If $k_m > 0$ and $\varepsilon'' = \eta\varepsilon$ with $\eta = \pm 1$:

$m_{\Lambda'}$ is the multiplicity of $e((k_1+q-1)\lambda_1 + ... + \varepsilon' k_q \lambda_q)$ in:

$$\prod_{i=1}^{q-1} \prod_{j=i+1}^{q} s(\alpha_{ij}) s(\beta_{ij}) \times \sum_{\substack{k_{1,i} \\ k_{1,m}>0}} ... \sum_{\substack{k_{q-1,i} \\ k_{q-1,m}>0}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) \right\}$$
$$\times \sum_{\substack{\varepsilon_1,...,\varepsilon_q \\ \varepsilon_1 ... \varepsilon_q = \eta}} e(\varepsilon_1 l_{1,m} \lambda_1 + ... + \varepsilon_q l_{q,m} \lambda_q),$$

where the summation is taken over $\varepsilon_1, ..., \varepsilon_q \in \{-1, 1\}$.

The $k_{r,i}$ verify conditions (7) et (8). The $l_{r,i}$ are given by:

$$\begin{cases} l_{r,r} = k_{r-1,r} - \max(k_{r-1,r+1}, k_{r,r+1}) + 1\,, \\ l_{r,i} = \min(k_{r-1,i}, k_{r,i}) - \max(k_{r-1,i+1}, k_{r,i+1}) + 1\,, \\ \qquad\qquad\qquad \text{for} \quad r+1 \le i \le m-1\,, \\ l_{r,m} = \min(k_{r-1,m}, k_{r,m}). \end{cases} \tag{9}$$

Proof. To decompose an irreducible G-module of highest weight Λ, into irreducible K-modules, we use the inequality (1) or (2), and we determine the set E such that:

$$\xi_G(\Lambda + \delta_G).\xi_K(\delta_K) = \xi_G(\delta_G). \sum_{\Lambda' \in E} \xi_K(\Lambda' + \delta_K).$$

We have:

$$\xi_G(\delta_G) = \prod_{\alpha \in \Delta_G^+} (e(\alpha/2) - e(-\alpha/2))\,, \quad \xi_K(\delta_K) = \prod_{\alpha \in \Delta_K^+} (e(\alpha/2) - e(-\alpha/2)).$$

Then

$$\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)} = \prod_{\alpha \in \Delta_G^+ - \Delta_K^+} (e(\alpha/2) - e(-\alpha/2)).$$

Writing Λ in the form (3), we have $\Lambda + \delta_G = H_1\lambda_1 + H_2\lambda_2 + ... + \varepsilon H_m \lambda_m$, where $\varepsilon = \pm 1$ and for all $1 \le i \le m$, $H_i = h_i + m - i$. The H_i are integers verifying $H_1 > H_2 > ... > H_m \ge 0$.

In the same way we have $\Lambda' + \delta_K = K_1\lambda_1 + ... + \varepsilon' K_q \lambda_q + K_{q+1}\lambda_{q+1} + ... + \varepsilon'' K_m \lambda_m$, where $K_i = k_i + q - i$ for all $1 \le i \le q$ and $K_i = k_i + m - i$ for all $q+1 \le i \le m$. The K_i are integers verifying:

$$K_1 > K_2 > ... > K_q \ge 0 \quad \text{and} \quad K_{q+1} > K_{q+2} > ... > K_m \ge 0.$$

Then we obtain:

$$\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)} = \prod_{i=1}^{q}\prod_{j=q+1}^{m} s(\alpha_{ij})s(\beta_{ij}).$$

On the other hand, we show that

$$\xi_G(\Lambda+\delta_G) = \frac{1}{2}\left\{\det\left[c(H_i\lambda_j)\right]_{1:m} + \varepsilon\det\left[s(H_i\lambda_j)\right]_{1:m}\right\}.$$

and

$$\xi_K(\Lambda'+\delta_K) = \frac{1}{4}\left\{\det[c(K_i\lambda_j)]_{1:q} + \varepsilon'\det[s(K_i\lambda_j)]_{1:q}\right\}$$
$$\times\left\{\det[c(K_i\lambda_j]_{q+1:m} + \varepsilon''\det[s(K_i\lambda_j)]_{q+1:m}\right\}.$$

To determine the set E such that

$$\frac{\xi_G(\Lambda+\delta_G)}{\xi_G(\delta_G)/\xi_K(\delta_K)} = \sum_{\Lambda'\in E}\xi_K(\Lambda'+\delta_K),$$

it suffices to determine the integers $K_1, ..., K_m$ such that

$$\frac{1}{2}.\frac{\det[c(H_i\lambda_j)]_{1:m} + \varepsilon\det[s(H_i\lambda_j)]_{1:m}}{\prod_{i=1}^{q}\prod_{j=q+1}^{m} s(\alpha_{ij})s(\beta_{ij})}$$
$$= \sum_{K_i}\frac{1}{4}\left\{\det[c(K_i\lambda_j)]_{1:q} + \varepsilon'\det[s(K_i\lambda_j)]_{1:q}\right\}$$
$$\times\left\{\det[c(K_i\lambda_j)]_{q+1:m} + \varepsilon''\det[s(K_i\lambda_j)]_{q+1:m}\right\},$$

where the summation is taken over the integers $K_1, ..., K_m$ verifying:

$$\begin{cases} K_1 > ... > K_q \geq 0, \\ K_{q+1} > ... > K_m \geq 0. \end{cases}$$

Equivalently, we search to determine the integers $K_1, ..., K_m$ such that

$$\prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij}) \times \sum_{K_{1,i}}...\sum_{K_{q,i}}\left\{\prod_{r=1}^{q}\left(\prod_{i=r}^{m-1}\frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)}\right)\right\}$$
$$\left\{H^1...H^q c(l_{1,m}\lambda_1)...c(l_{q,m}\lambda_q)\det[c(K_{q,i}\lambda_j]_{q+1:m}\right.$$
$$\left.+\varepsilon s(l_{1,m}\lambda_1)...s(l_{q,m}\lambda_q)\det[s(K_{q,i}\lambda_j)]_{q+1:m}\right\}$$
$$= \sum_{K_i}\frac{1}{2}\left\{\det[c(K_i\lambda_j)]_{1:q} + \varepsilon'\det[s(K_i\lambda_j)]_{1:q}\right\}$$
$$\times\left\{\det[c(K_i\lambda_j)]_{q+1:m} + \varepsilon''\det[s(K_i\lambda_j)]_{q+1:m}\right\}.$$

We permute successively the summations to get first this on $K_{q,i}$. After r steps, we obtain these new conditions:

$$\begin{cases} K_{q-r-1,i+r+1}+r<K_{q,i}<K_{q-r-1,i-r-1}-r\,, \\ \qquad\qquad\qquad \text{for } q+1\le i\le m-r-1\,, \\ K_{q,i}<K_{q-r-1,i-r-1}-r\,,\ \text{for } m-r\le i\le m\,, \\ 0\le K_{q,m}<...<K_{q,q+1}, \end{cases}$$

combined to (C_r), they give the next conditions

- If $2r>m-q-3$:

$$(C'_{q-r-1})\begin{cases} a_{q,r,i}<K_{q-r-1,i}<K_{q-r-2,i-1}\,, \\ \qquad\qquad \text{for } q-r\le i\le m-r-1\,, \\ K_{q-r-2,i+1}<K_{q-r-1,i}<K_{q-r-2,i-1}\,, \\ \qquad\qquad \text{for } m-r\le i\le q+r+1\,, \\ K_{q-r-2,i+1}<K_{q-r-1,i}<b_{q,r,i}\,, \\ \qquad\qquad \text{for } q+r+2\le i\le m-1\,, \\ K_{q-r-1,m}<b_{q,r,m}\,, \\ 0\le K_{q-r-1,m}<...<K_{q-r-1,q-r}. \end{cases}$$

- If $2r\le m-q-3$:

$$(C'_{q-r-1})\begin{cases} a_{q,r,i}<K_{q-r-1,i}<K_{q-r-2),i-1}\,, \\ \qquad\qquad \text{for } q-r\le i\le q+r+1\,, \\ a_{q,r,i}<K_{q-r-1,i}<b_{q,r,i}\,, \\ \qquad \text{for } q+r+2\le i\le m-r-1\,, \\ K_{q-r-2,i+1}<K_{q-r-1,i}<b_{q,r,i}\,, \\ \qquad\qquad \text{for } m-r\le i\le m-1\,, \\ K_{q-r-1,m}<b_{q,r,m}\,, \\ 0\le K_{q-r-1,m}<...<K_{q-r-1,q-r}. \end{cases}$$

Here:

$$\begin{aligned} a_{q,r,i} &= \max(K_{q-r-2,i+1},K_{q,i+r+1}+r)\,, \\ b_{q,r,i} &= \min(K_{q-r-2,i-1},K_{q,i-r-1}-r)\,. \end{aligned}$$

By $r=q-1$ steps, we have:

$$(C'_q)\quad\begin{cases} H_{i+q}+q\le K_{q,i}\le H_{i-q}-q, & \text{for}\quad q+1\le i\le m-q, \\ K_{q,i}\le H_{i-q}-q & \text{for,}\quad m-q+1\le i\le m, \\ 0\le K_{q,m}<...<K_{q,q+1}. \end{cases}$$

We set $K_{0,i}=H_i$, then we have to separate the following two cases:

1. There exists $0 \leq r \leq q$ such that $K_{r,m} = 0$, then $\det[s(K_{r,i}\lambda_j)]_{r+1:m} = 0$. Consequently, $\det[s(K_{q,i}\lambda_j)]_{q+1:m} = 0$.

2. For all $0 \leq r \leq q$, we have $K_{r,m} > 0$. Then $H^r = 1$ for all $1 \leq r \leq q$.

Using the formulas

$$\begin{aligned} c(l_{1,m}\lambda_1)...c(l_{q,m}\lambda_q) &= \sum_{\varepsilon_1,...,\varepsilon_q \in \{-1,1\}} e(\varepsilon_1 l_{1,m}\lambda_1 + ... + \varepsilon_q l_{q,m}\lambda_q), \\ s(l_{1,m}\lambda_1)...s(l_{q,m}\lambda_q) &= \sum_{\varepsilon_1,...,\varepsilon_q \in \{-1,1\}} \varepsilon_1...\varepsilon_q e(\varepsilon_1 l_{1,m}\lambda_1 + ... + \varepsilon_q l_{q,m}\lambda_q), \end{aligned}$$

we obtain:

$$\begin{aligned} c(l_{1,m}\lambda_1)...c(l_{q,m}\lambda_q) \det[c(K_{q,i}\lambda_j)]_{q+1:m} + \varepsilon s(l_{1,m}\lambda_1)...s(l_{q,m}\lambda_q) \\ \times \det[s(K_{q,i}\lambda_j)]_{q+1:m} = \sum_{\varepsilon_1,...,\varepsilon_q} e(\varepsilon_1 l_{1,m}\lambda_1 + ... + \varepsilon_q l_{q,m}\lambda_q) \\ \times \{\det[c(K_{q,i}\lambda_j)]_{q+1:m} + \varepsilon_1...\varepsilon_q.\varepsilon \det[s(K_{q,i}\lambda_j)]_{q+1:m}\}. \end{aligned}$$

Then:

$$\begin{aligned} \prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij}) \times \sum_{\substack{K_{q,i} \\ K_{q,m}>0}} \sum_{\substack{K_{1,i} \\ K_{m,i}>0}} ... \sum_{\substack{K_{q-1,i} \\ K_{q-1,m}>0}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) \right\} \\ \sum_{\varepsilon_1,...,\varepsilon_q} e(\varepsilon_1 l_{1,m}\lambda_1 + ... + \varepsilon_q l_{q,m}\lambda_q) \\ \times \{\det[c(K_{q,i}\lambda_j)]_{q+1:m} + \varepsilon_1...\varepsilon_q.\varepsilon \det[s(K_{q,i}\lambda_j)]_{q+1:m}\} \\ = \sum_{K_i} \frac{1}{2} \left\{ \det[c(K_i\lambda_j)]_{1:q} + \varepsilon' \det[s(K_i\lambda_j)]_{1:q} \right\} \\ \times \left\{ \det[c(K_i\lambda_j)]_{q+1:m} + \varepsilon'' \det[s(K_i\lambda_j)]_{q+1:m} \right\}. \end{aligned}$$

By identification we find:

(i) $K_i = K_{q,i}$ for all $q+1 \le i \le m$ ($K_m = 0$), and

$$\sum_{K_1 > \ldots K_q \ge 0} \frac{1}{2}\left\{\det[c(K_i\lambda_j)]_{1:q} + \varepsilon' \det[s(K_i\lambda_j)]_{1:q}\right\}$$
$$= \prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij})$$
$$\times \underset{\text{un des } K_{r,m}=0}{\sum_{K_{1,i}} \cdots \sum_{K_{q-1,i}}} \left\{\prod_{r=1}^{q}\left(\prod_{i=r}^{m-1}\frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)}\right) H^r c(l_{r,m}\lambda_r)\right\},$$

(ii) $K_i = K_{q,i}$ for all $q+1 \le i \le m$ ($K_m > 0$), $\varepsilon'' = \eta\varepsilon$ ($\eta = \pm 1$) and

$$\sum_{K_1 > \ldots K_q \ge 0} \frac{1}{2}\left\{\det[c(K_i\lambda_j)]_{1:q} + \varepsilon' \det[s(K_i\lambda_j)]_{1:q}\right\}$$
$$= \prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij})$$
$$\times \sum_{\substack{K_{1,i} \\ K_{1,m}>0}} \cdots \sum_{\substack{K_{q-1,i} \\ K_{q-1,m}>0}} \left\{\prod_{r=1}^{q}\left(\prod_{i=r}^{m-1}\frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)}\right)\right\}$$
$$\sum_{\substack{\varepsilon_1,\ldots,\varepsilon_q \\ \varepsilon_1\ldots\varepsilon_q=\eta}} e(\varepsilon_1 l_{1,m}\lambda_1 + \ldots + \varepsilon_q l_{q,m}\lambda_q),$$

Here the conditions on $K_{r,i}$ for $1 \le r \le q$, are (C'_r). We find:

$$\begin{cases} h_{i+q} \le k_i \le h_{i-q}, & \text{for } \quad q+1 \le i \le m-q, \\ k_i \le h_{i-q}, & \text{for } \quad m-q+1 \le i \le m, \\ 0 \le k_m \le \ldots \le k_q. & \end{cases}$$

If we denote by:

$$k_{r,i} = K_{r,i} - m + i, \quad \text{for all } \ 0 \le r \le q-1 \ \text{ and } \ r+1 \le i \le m,$$

we verify the result. $\square$

Theorem 11. *Let $G = \mathrm{SO}\,(2m+1)$, $K = \mathrm{SO}\,(2q) \times \mathrm{SO}\,(2m-2q+1)$ and V be an irreducible G-module of highest weight $\Lambda = h_1\lambda_1 + \ldots + h_m\lambda_m$.*

An irreducible K-module of highest weight $\Lambda' = k_1\lambda_1 + ... + \varepsilon k_q\lambda_q + k_{q+1}\lambda_{q+1} + ... + k_m\lambda_m$ is a K-submodule of V if and only if:

$$1. \begin{cases} h_{i+q} \leq k_i \leq h_{i-q}, & \text{for} \quad q+1 \leq i \leq m-q, \\ k_i \leq h_{i-q}, & \text{for} \quad m-q+1 \leq i \leq m. \end{cases}$$

2. The multiplicity $m_{\Lambda'}$ of $V' = V(\Lambda')$ in $V = V(\Lambda)$ does not vanish and equals to the multiplicity of $e((k_1+q-1)\lambda_1 + ... + \varepsilon k_q\lambda_q)$ in the term:

$$\prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij}) \times \sum_{k_{1,i}} ... \sum_{k_{q-1,i}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) . \frac{s(l_{r,m}\lambda_r)}{s(\lambda_r/2)} \right\},$$

The integers $k_{r,i}$, for $1 \leq r \leq q$ and $r+1 \leq i \leq m$, verify the conditions (7) and (8). The $l_{r,i}$, for $1 \leq r \leq q$ and $r \leq i \leq m$, are given by:

$$\begin{cases} l_{r,r} = k_{r-1,r} - \max(k_{r-1,r+1}, k_{r,r+1}) + 1, \\ l_{r,i} = \min(k_{r-1,i}, k_{r,i}) - \max(k_{r-1,i+1}, k_{r,i+1}) + 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } r+1 \leq i \leq m-1\,, \\ l_{r,m} = \min(k_{r-1,m}, k_{r,m}) + \dfrac{1}{2}. \end{cases}$$

Proof. We have $\Lambda + \delta_G = H_1\lambda_1 + ... + H_m\lambda_m$, where $H_i = h_i + m - i + 1/2$ are integers $+\frac{1}{2}$ verifying $H_1 > H_2 > ... > H_m > 0$. In the same way, $\Lambda' + \delta_K = K_1\lambda_1 + ... + \varepsilon K_q\lambda_q + K_{q+1}\lambda_{q+1} + ... + K_m\lambda_m$, where $K_i = k_i + q - i$ for $1 \leq i \leq q$ are integers and $K_i = k_i + m - i + 1/2$ for $q+1 \leq i \leq m$ are integers $+\frac{1}{2}$, such that $K_1 > K_2 > \dots > K_q \geq 0$ and $K_{q+1} > K_{q+2} > \dots > K_m > 0$.

Following the same calculus like in Theorem 10, we obtain:

$$\begin{aligned} &\frac{\xi_G(\delta_G)}{\xi_K(\delta_K)} = s(\frac{\lambda_1}{2})...s(\frac{\lambda_q}{2}) \prod_{i=1}^{q}\prod_{j=q+1}^{m} s(\alpha_{ij})s(\beta_{ij})\,, \\ &\xi_G(\Lambda + \delta_G) = \det[s(H_i\lambda_j)]_{1:m}\,, \\ &\xi_K(\Lambda' + \delta_K) = \frac{1}{2}\left\{\det[c(K_i\lambda_j)]_{1:q} + \varepsilon \det[s(K_i\lambda_j)]_{1:q}\right\} \\ &\qquad \times \det[s(K_i\lambda_j)]_{q+1:m}, \end{aligned}$$

and

$$\frac{\det[s(H_i\lambda_j)]_{1:m}}{\prod_{i=1}^{q} s(\frac{\lambda_i}{2}) \prod_{i=1}^{q}\prod_{j=q+1}^{m} s(\alpha_{ij})s(\beta_{ij})} = \prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij})$$

$$\times \sum_{K_{1,i}} \cdots \sum_{K_{q,i}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) \frac{s(l_{r,m}\lambda_r)}{s(\lambda_r/2)} \right\} \det[s(K_{q,i}\lambda_j)]_{q+1:m}.$$

Then determining the set E verifying:

$$\xi_G(\Lambda+\delta_G).\xi_K(\delta_K) = \xi_G(\delta_G).\sum_{\Lambda'\in E} \xi_K(\Lambda'+\delta_K),$$

is equivalent to calculate the integers $K_1, ..., K_q$ and the integers $+\frac{1}{2}, K_{q+1}, ..., K_m$ such that:

$$K_1 > ... > K_q \geq 0 \quad \text{et} \quad K_{q+1} > ... > K_m > 0,$$

and

$$\frac{1}{2}\sum_{K_i} \{\det[c(K_i\lambda_j)]_{1:q} + \varepsilon \det[s(K_i\lambda_j)]_{1:q}\} \times \det[s(K_i\lambda_j)]_{q+1:m}$$

$$= \prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij})$$

$$\times \sum_{K_{1,i}} \cdots \sum_{K_{q,i}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) \frac{s(l_{r,m}\lambda_r)}{s(\lambda_r/2)} \right\} \det[s(K_{q,i}\lambda_j)]_{q+1:m}.$$

After permutation of the summation and identification of the terms, we find that $K_i = K_{q,i}$, for $q+1 \leq i \leq m$, with the conditions:

$$\begin{cases} H_{i+q} + q \leq K_{q,i} \leq H_{i-q} - q, & \text{for} \quad q+1 \leq i \leq m-q, \\ K_{q,i} \leq H_{i-q} - q, & \text{for} \quad m-q+1 \leq i \leq m, \\ 0 < K_{q,m} < ... < K_{q,q+1}, & \end{cases}$$

and

$$\frac{1}{2}\sum_{K_1>...>K_q>0} (\det[c(K_i\lambda_j)]_{1:q} + \varepsilon \det[s(K_i\lambda_j)]_{1:q})$$
$$= \prod_{i=1}^{q-1}\prod_{j=i+1}^{q} s(\alpha_{ij})s(\beta_{ij}) \times \sum_{K_{1,i}} \cdots \sum_{K_{q-1,i}} \left\{ \prod_{r=1}^{q} \left(\prod_{i=r}^{m-1} \frac{s(l_{r,i}\lambda_r)}{s(\lambda_r)} \right) \frac{s(l_{r,m}\lambda_r)}{s(\lambda_r/2)} \right\}.$$

$K_{r,i}$ verify the conditions (C'_r) (with $K_{r,m} > 0$). If we set $k_{r,i} = K_{r,i} - m + i - \frac{1}{2}$, for $0 \le r \le q-1$ and $r+1 \le i \le m$, we obtain the result of the theorem. $\square$

4. Decomposition of $\wedge^p(\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}}$

We identify the cotangent space $M = G/K$ at $o = [K]$ with $(\mathfrak{g}/\mathfrak{k})^*$, the dual space of $\mathfrak{g}/\mathfrak{k}$. We have:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_G} \mathfrak{g}_\alpha, \quad \mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_K} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{m}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta_G \setminus \Delta_K} \mathfrak{g}_\alpha.$$

The highest weights of the irreducible K-submodules of $\mathfrak{m}_{\mathbb{C}}$ are some of the roots $\alpha \in \Delta_G \setminus \Delta_K$. A dominant weight Λ' corresponding to an irreducible K-submodule of $\mathfrak{m}_{\mathbb{C}}$ can be written

$$\begin{aligned} &\Lambda' = \lambda_1 + \lambda_{q+1}, && \text{if} \quad n = 2m, \\ &\Lambda' = \lambda_1 + \lambda_{q+1} \quad \text{or} \quad \Lambda' = \lambda_1, && \text{if} \quad n = 2m+1. \end{aligned}$$

On the other hand, $\dim_{\mathbb{C}} \mathfrak{m}_{\mathbb{C}} = k(n-k)$ and the restriction of $\mathfrak{m}_{\mathbb{C}}$ to $\mathrm{SO}(k) \times \{I_{n-k}\}$ (resp. $\{I_k\} \times \mathrm{SO}(n-k)$) is isomorphic to $n-k$ (resp. k) copies of the standard representation $\mathbb{C}^k$ (resp. $\mathbb{C}^{n-k}$) of $\mathrm{SO}(k)$ (resp. $\mathrm{SO}(n-k)$). We have the same thing for the restriction of the K-module $V(\lambda_1 + \lambda_{q+1})$. Then, $\mathfrak{m}_{\mathbb{C}}$ is isomorphic to $V(\lambda_1 + \lambda_{q+1})$, i.e. $\mathfrak{m}^*_{\mathbb{C}}$ is an irreducible K-module.

Notation. Let H and L be two groups, V a H-module and W a L-module. The space $V \otimes W$ has a structure of $H \times L$-module by the action of H on V and L on W. We denote by $V \boxtimes W$ the obtained $H \times L$-module. Thus, the $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$-module $\mathfrak{m}_{\mathbb{C}}$ is isomorphic to $V(\lambda_1) \boxtimes V(\lambda_{q+1})$.

4.1. Particular Case $K = \mathrm{SO}(4) \times \mathrm{SO}(n-4)$

Let H be the subgroup $\mathrm{SO}(2) \times \mathrm{SO}(2)$ of $\mathrm{SO}(4)$. We begin by decomposing the restriction of $\wedge^p \mathfrak{m}^*_{\mathbb{C}}$ to $H \times \mathrm{SO}(n-4)$. To restrict $\mathfrak{m}^*_{\mathbb{C}}$ or $\mathfrak{m}_{\mathbb{C}}$, i.e. $V(\lambda_1) \boxtimes V(\lambda_3)$ to $H \times \mathrm{SO}(n-4)$, we restrict the $\mathrm{SO}(4)$-module $V(\lambda_1)$ to H.

Using a particular case of the Theorem 10, we get the decomposition of the $\mathrm{SO}(4)$-module $V(\lambda_1)$ into irreducible H-submodules:

$$V(\lambda_1)|_H \cong V(\lambda_1) \oplus V(-\lambda_1) \oplus V(\lambda_2) \oplus V(-\lambda_2).$$

We denote by $V_1 = V(\lambda_1) \boxtimes V(\lambda_3)$, $V_2 = V(-\lambda_1) \boxtimes V(\lambda_3)$, $V_3 = V(\lambda_2) \boxtimes V(\lambda_3)$ and $V_4 = V(-\lambda_2) \boxtimes V(\lambda_3)$. Then

$$\mathfrak{m}_{\mathbb{C}}^* \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4 \quad \text{(irreducible } H \times \text{ SO}\,(n-4)\text{-modules)}.$$

Using the notation $\wedge^{a,b,c,d} = \wedge^a V_1 \otimes \wedge^b V_2 \otimes \wedge^c V_3 \otimes \wedge^d V_4$ ($H \times$ SO$\,(n-4)$-module), we get the decomposition

$$\wedge^p \mathfrak{m}_{\mathbb{C}}^* \cong \sum \wedge^{a,b,c,d} \; a+b+c+d=p, \; (H \times \text{ SO}\,(n-4)\text{-modules}). \tag{10}$$

On the other hand, the restriction to SO$\,(n-4)$ of V_1, V_2, V_3 or V_4 (as $H \times$ SO$\,(n-4)$-modules) is isomorphic to $V = V(\lambda_3)$. Also, the SO$\,(2) \times$ SO$\,(2) \times$ SO$\,(n-4)$-module, $\wedge^{a,b,c,d}$, is isomorphic to:

$$V((a-b)\lambda_1) \boxtimes V((c-d)\lambda_2) \boxtimes (\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V). \tag{11}$$

It means that it suffices to decompose the SO$\,(n-4)$-module $(\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V)$ into irreducible SO$\,(n-4)$-submodules to obtain the decomposition of $H \times$ SO$\,(n-4)$-module, $\wedge^{a,b,c,d}$. We suppose that:

$$\wedge^a V \otimes \wedge^b V \otimes \wedge^c V \otimes \wedge^d V \cong \sum V(\mu), \quad \text{(irreducible SO}\,(n-4)\text{-modules)}. \tag{12}$$

We obtain:

$$\wedge^{a,b,c,d} \cong \sum_{\mu} V((a-b)\lambda_1) \boxtimes V((c-d)\lambda_2) \boxtimes V(\mu), \quad (\text{SO}\,(2) \times \text{SO}\,(2) \times \text{SO}\,(n-4)\text{-modules}). \tag{13}$$

Notation. We set $\gamma_{j-2} = \lambda_j$ for $3 \leq j \leq m$.

1. In the case when $n = 2m$, we set:

$$\begin{aligned}
&\Gamma_0 = 0\,,\\
&\Gamma_j = \gamma_1 + ... + \gamma_j \quad \text{for } 1 \leq j \leq m-4,\\
&\Gamma_{m-3} = \frac{1}{2}(\gamma_1 + ... + \gamma_{m-3} - \gamma_{m-2}),\\
&\Gamma_{m-2} = \frac{1}{2}(\gamma_1 + ... + \gamma_{m-3} + \gamma_{m-2})\,,
\end{aligned}$$

and

$$\begin{aligned}
&V_{i,j} = V(\Gamma_i + \Gamma_j), \text{ for } 0 \le i \le j \le m-4,\\
&V_{i,m-3} = V(\Gamma_i + \Gamma_{m-3} + \Gamma_{m-2}), \text{ for } 0 \le i \le m-4,\\
&V_{m-3,m-3} = V(2\Gamma_{m-3} + 2\Gamma_{m-2}),\\
&V_{i,m-2} = V(\Gamma_i + 2\Gamma_{m-3}) \oplus V(\Gamma_i + 2\Gamma_{m-2}), \text{ for } 0 \le i \le m-4,\\
&V_{m-3,m-2} = V(3\Gamma_{m-3} + \Gamma_{m-2}) \oplus V(\Gamma_{m-3} + 3\Gamma_{m-2}),\\
&V_{m-2,m-2} = V(4\Gamma_{m-3}) \oplus V(4\Gamma_{m-2}),\\
&V_{i,j} = V_{i,n-4-j}, \text{ for } m-1 \le j \le n-4-i.
\end{aligned}$$

2. In the case when $n = 2m+1$, we set:

$$\begin{aligned}
&\Gamma_0 = 0\,,\\
&\Gamma_j = \gamma_1 + ... + \gamma_j\,, \quad \text{for} \quad 1 \le j \le m-3\,,\\
&\Gamma_{m-2} = \frac{1}{2}(\gamma_1 + ... + \gamma_{m-3} + \gamma_{m-2})\,,
\end{aligned}$$

and

$$\begin{aligned}
&V_{i,j} = V(\Gamma_i + \Gamma_j)\,, \quad \text{for} \quad 0 \le i \le j \le m-3\,,\\
&V_{i,m-2} = V(\Gamma_i + 2\Gamma_{m-2})\,, \quad \text{for} \quad 0 \le i \le m-3\,,\\
&V_{m-2,m-2} = V(4\Gamma_{m-2})\,,\\
&V_{i,j} = V_{i,n-4-j}\,, \quad \text{for} \quad m-1 \le j \le n-4-i.
\end{aligned}$$

Γ_j for $1 \le j \le m-2$ are the fundamental weight of the group $\mathrm{SO}\,(n-4)$. With these notations, the restriction of $\wedge^{a,b,c,d}$ to $\mathrm{SO}\,(n-4)$ is isomorphic to:

$$\wedge^a V(\Gamma_1) \otimes \wedge^b V(\Gamma_1) \otimes \wedge^c V(\Gamma_1) \otimes \wedge^d V(\Gamma_1).$$

Proposition 12. (see [13]) *For $0 \le r \le s \le m-2$, the $\mathrm{SO}\,(n-4)$-module $\wedge^r V(\Gamma_1) \otimes \wedge^s V(\Gamma_1)$ can be decomposed into $\mathrm{SO}\,(n-4)$-submodules like this:*

$$\wedge^{r,s} = \wedge^r V(\Gamma_1) \otimes \wedge^s V(\Gamma_1) \cong \sum_{i,j} V_{i,j},$$

where the indices of the summation (i,j) are positive or null and verify:

$$\begin{cases} j-i & \ge & s-r,\\ i+j & \le & r+s,\\ i+j & \equiv & r+s \pmod 2).\end{cases}$$

We obtain then the next proposition.

Proposition 13. *(i) The restriction of* $\wedge^{a,b,c,d}$ *to* $\mathrm{SO}\,(n-4)$ *can be decomposed like this:*

$$\wedge^{a,b,c,d}|_{\mathrm{SO}\,(n-4)} \cong \sum_{\substack{(i,j)\in S_1\\(k,l)\in S_2}} V_{i,j}\otimes V_{k,l},$$

where S_1 *is the set of* (i,j) *such that:*

$$\begin{cases} j-i & \geq & b-a\,,\\ i+j & \leq & a+b\,,\\ i+j & \equiv & a+b \pmod 2\,,\end{cases}$$

and S_2 *is the set of* (k,l) *such that:*

$$\begin{cases} l-k & \geq & d-c\,,\\ k+l & \leq & c+d\,,\\ k+l & \equiv & c+d \pmod 2\,.\end{cases}$$

(ii) The Steinberg multiplicity formula allows us to decompose $V_{i,j}\otimes V_{k,l}$ *into irreducible* $\mathrm{SO}\,(n-4)$*-submodules.*

(iii) To get the decomposition of $\wedge^{a,b,c,d}$ *into* $\mathrm{SO}\,(4)\times\mathrm{SO}\,(n-4)$*-modules, we regroup the irreducible* $\mathrm{SO}\,(2)\times\mathrm{SO}\,(2)$*-modules (*$V((a-b)\lambda_1+(c-d)\lambda_2)$*) into irreducible* $\mathrm{SO}\,(4)$*-modules (see (11)).*

4.2. General Case

We consider now the case, where $k=2q$. To decompose the K-module $\wedge^p\mathfrak{m}_{\mathbb{C}}^*$ into irreducible K-submodules, we begin by decomposing the restriction of $\mathfrak{m}_{\mathbb{C}}^*$ to $\mathrm{SO}\,(2)\times\mathrm{SO}\,(2q-2)\times\mathrm{SO}\,(n-2q)$, then the restriction of $\wedge^p\mathfrak{m}_{\mathbb{C}}^*$ to $\mathrm{SO}\,(2)\times\mathrm{SO}\,(2q-2)\times\mathrm{SO}\,(n-2q)$ and finally, we come back to K as the case $q=2$.

As $\mathfrak{m}_{\mathbb{C}}^*\cong V(\lambda_1+\lambda_{q+1})$, it suffices to study the restriction of $\mathrm{SO}\,(2q)$-module $V(\lambda_1)$ to $\mathrm{SO}\,(2)\times\mathrm{SO}\,(2q-2)$. Using a particular case of the Theorem 10, we get:

$$V(\lambda_1)|_{\mathrm{SO}\,(2)\times\mathrm{SO}\,(2q-2)}\cong V(\lambda_1)\oplus V(-\lambda_1)\oplus V(\lambda_2),$$

where $V(\pm\lambda_1)$ is trivial and $V(\lambda_2)$ is the standard representation of $\mathrm{SO}\,(2q-2)$. Then:

$$\begin{aligned} &V(\lambda_1+\lambda_{q+1})|_{\mathrm{SO}\,(2)\times\mathrm{SO}\,(2q-2)\times\mathrm{SO}\,(n-2q)}\\ &\qquad\cong V(\lambda_1+\lambda_{q+1})\oplus V(-\lambda_1+\lambda_{q+1})\oplus V(\lambda_2+\lambda_{q+1}),\end{aligned}$$

as $\mathrm{SO}(2) \times \mathrm{SO}(2q-2) \times \mathrm{SO}(n-2q)$-modules.

We denote by $U_1 = V(\lambda_1 + \lambda_{q+1})$, $U_2 = V(-\lambda_1 + \lambda_{q+1})$ and $U_3 = V(\lambda_2 + \lambda_{q+1})$.

Proposition 14. *The decomposition of $\wedge^p \mathfrak{m}_{\mathbb{C}}^*$ into irreducible K-submodules can be made recursively like this:*

(i) The first step follows the Proposition 13.

(ii) The restriction of $\wedge^p \mathfrak{m}_{\mathbb{C}}^*$ to $\mathrm{SO}(2) \times \mathrm{SO}(2q-2) \times \mathrm{SO}(n-2q)$ can be decomposed as follow:

- $\wedge^p \mathfrak{m}_{\mathbb{C}}^* \cong \sum_{i+j+k=p} \wedge^i U_1 \otimes \wedge^j U_2 \otimes \wedge^k U_3.$
- *As U_1 and U_2 are isomorphic to the standard representation of $\mathrm{SO}(n-2q)$, the decomposition of $\wedge^i U_1 \otimes \wedge^j U_2$ is determined by applying the Proposition 12.*
- *We decompose $\wedge^k U_3$ recursively.*

(iii) To obtain the decomposition of $\wedge^p \mathfrak{m}_{\mathbb{C}}^*$ as $\mathrm{SO}(2q) \times \mathrm{SO}(n-2q)$-module, we regroup the irreducible $\mathrm{SO}(2) \times \mathrm{SO}(2q-2)$-modules occurring in the decomposition into irreducible $\mathrm{SO}(2q)$-modules.

5. Conclusion

The proposed method gives for every positive integers n and q, the eigenvalues of the Laplace operator acting on the forms for Grassmann manifolds $\mathrm{SO}(n+2q)/\mathrm{SO}(2q) \times \mathrm{SO}(n)$. We can find examples in my Ph.D. Thesis [4].

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