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VALUATION ON A RING WITH RESPECT TO A SUBGROUP OF ITS GROUP OF UNITS

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Abstract: Given an integral domain R, K its quotient field, K^* the multiplicative group of K, R^* the semi-group of non-zero elements of R and $U(R)$ the multiplicative group of units of R, the canonical map of K^* onto $K^*/U(R)$ is the well known semi-valuation. In this paper we prove that if – instead of $U(R)$ – we consider an adequate subgroup of $U(R)$, we may define another kind of valuation, the G-valuation, whose the value group has torsion and the triangle property differs slightly from the one of the semi-valuation. We prove that both of these triangle properties may be presented by two completions of an ordered space. Apart of some direct algebraic and topological consequences of the new definition we construct a G-valuated field with value group a given splitting Abelian ordered group.

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1. Introduction

(1) The well known construction of a semi-valuation begins with an integral domain R, K its quotient field, K^* the multiplicative group of K, R^* the multiplicative semi-group of non-zero elements of R and $U(R)$ the multiplicative group of units of R. Then a preorder is defined on K^* by taking $(K^*)^+ = R^*$ and the canonical map

$$w : K^* \to K^*/U(R) \tag{1}$$

of K^* onto the associated ordered group $K^*/U(R)$, written additively, is a *semi-valuation.*

More generally, (cf. Ohm [15, p. 578], Močkoř [13, p. 13]) a map w from a field K into an additive Abelian po-group G is called *semi-valuation* provided that for every x, y in K the following hold:

(i) $w(xy) = w(x) + w(y)$;

(ii) $w(x + y) \geq \inf_{\overline{G}}\{w(x), w(y)\}$;

(iii) $w(-1) = 0$;

(iv) $w(x) = \infty$ iff $x = 0$,

where ∞ is an element attached to G, greater than all $a \in G$.

The crucial property is the triangle one, that is the above (ii) since G, in the history of valuations, was firstly the multiplicative group R^*_+ of real numbers, then a totally ordered group and in the semi-valuation case a torsion free (partially) ordered (po) group. Ohm in [15] has put the expression (ii) explicitly saying that it is simply a symbolism which we put instead of the relation:

$$\text{if } w(x) \geq \gamma \text{ and } w(y) \geq \gamma, \text{ then } w(x + y) \geq \gamma. \tag{ii)'}$$

It is a surprising result that (ii) and $(ii)'$ are equivalent provided that one considers the infimum in the right hand side of (ii) as the infimum of $w(x)$ and $w(y)$ in the MacNeille's completion of G.

It is our main aim in this paper to study what is changed in the case where $(ii)'$ is substituted by the relation:

$$\text{if } w(x) > \gamma \text{ and } w(y) > \gamma, \text{ then } w(x + y) > \gamma. \tag{$\overline{ii}$)'}$$

Automatically $(\overline{ii})'$ takes an equivalent expression

$$w(x + y) \geq \inf_{\widetilde{G}}\{w(x), w(y)\}\,, \tag{$\overline{ii}$}$$

where in the right hand side of $(\overline{ii})$ we consider the infimum of $w(x), w(y)$ in another completion of G, very relative to the MacNeille's one, the so called

Kurepa's completion (Dokas [4] and Hungerford [8]). The natural question which arises is whether a homomorphism of a commutative field satisfying $(\overline{ii})$ exists, that is if a kind of valuation exists, where the triangle property has $>$ instead of $\geq$ in $(\overline{ii})'$. The answer is positive provided that the kernel in the new completion is not the whole group $U(R)$ of units, but a subgroup, say $U_1^*(R)$, of it. The rest cosets of $U(R)/U_1^*$ represent, in the quotient group K^*/U_1^*, points non comparable to the neutral element of K^*/U_1^*.

Then, a second natural question is generated: which are the proper groups U_1^* which fit to be these factor groups? An answer may be for one to make use of the R-Prüfer rings of the given ring R. In fact in an R-Prüfer ring, say A, of R, a valuation pair (A, P) is formed (cf. Močkoř et al [14, pp. 66-67]), where the subring A is a valuation ring and P is its maximal ideal. So, the group of units of A may be the required U_1^*, the new valuation w takes 0 to the points of U_1^*, and the rest points of R have w-images parallel to 0 (see the Section 3.1 below). This canonical epimorphism satisfies the triangle property under the form $(\overline{ii})'$. Another answer may be that, although we exclusively refer to the commutative case, in a skew field – for which the new valuation has a meaning – there is a subgroup of units which must take the zero value; it is the derived group K^c of K^*, generated by all the commutators of K^*.

(2) We call this new valuation *G-valuation* (from the word *generalization*, or from the value group G to which such a valuation ranges). It is an easy consequence that the classical exponential valuation or the Krull valuation may be G-valuations. On the other side, we can easily see the case, where a semi-valuation and a G-valuation coincide. Furthermore, the uniformity and proximity theory which holds for a semi-valuated field, is valid for a G-valuated field as well.

With the notation of Section 1 we remark that, if υ is a semi-valuation on the quotient field K of the integral domain R and w a G-valuation and if f is a function of K^*/U_1^* onto $K^*/U(R)$ such that to xU_1^* corresponds the element $xU(R)$, then $\upsilon = f \circ w$.

It is well known that for a semi-valuation υ the set

$$A_\upsilon = \{x \in K / \upsilon(x) \geq 0\} \tag{2}$$

is a unitary ring (*the ring of the semi-valuation* υ).

Respectively, for a G-valuation w on K^* the set

$$A_w = \{x \in K / w(x) > \gamma, \gamma \text{ any negative element of } G\} \tag{3}$$

is a ring (*the ring of the* G*-valuation* w).

If A_w is a local ring, then for $w(x) < w(y)$, $w(x+y)$ is, in general, parallel to $w(x)$. Thus, it appears a great difference from what happens in the semi-valuation case (see Section 3.1 below) and further in the "representation" of a G-valuated field (see for the later Stabakis et al [17]). In fact, the more impressing difference of a G-valuation from a semi-valuation is that the value group of a G-valuation w may be with torsion, that is $w(K^*)$ may have torsion, surely a considerable and daring change.

The consequences of such a result are very serious. Since, for instance, $w(U(R))$ is not zero in general, the well known preorder defined in K with respect to the integral domain R does not become an order. The classical aspect is that if $\alpha \mid_R \beta$ (that is there is a $c \in R$ such that $\alpha c = \beta$) then an order is defined by the equivalence $\alpha \mid_R \beta$ and $\beta \mid_R \alpha$ and that if $\alpha \mid_R \beta$, then $v(\alpha) \leq v(\beta)$, where v is the corresponded semi-valuation. A crucial point is that, in the semi-valuation case, if $\gamma \mid_R \alpha$ and $\gamma \mid_R \beta$ then it is possible $\alpha+\beta \mid_R \alpha$ and $\alpha+\beta \mid_R \beta$, which relation is expressed by the fact that $v(\alpha+\beta)$ may be the greatest common divisor of $v(\alpha)$ and $v(\beta)$. In the G-valuation case we may define a "strict" order by:

$$\alpha_s \mid_R \beta \text{ iff } \alpha \mid_R \beta \text{ and } \beta \nmid_R \alpha.$$

In this case $w(\alpha) < w(\beta)$. So the relative classical problem which has a partial solution by Jaffard [9] and Ohm [15] has another meaning and another orientation. In fact, Section 4 is devoted to such a problem: Given a partially ordered Abelian group, mixed in general, find conditions and a G-valuated field whose value group is the given group. The problem is solved in the case of splitting groups and the constructions of such fields constitute generalizations of the constructions in the classical case.

On the other hand, the well known theorem of Lorenzen-Simbireva-Everett permits us to extend a po-Abelian group to a totally ordered one, so to extend the order of G into an order which immediately becomes a total order after an equivalence; at the same time we would transfer the new form of G into the G-valuated field.

Section 2 refers to the MacNeille's and Kurepa's completions. In Section 3 we refer to the introduction of the G-valuation and give some of the basic results of algebraic or topological characters. Section 4 is devoted to the above theorem and to the problem we have already mentioned.

Some of the results of the present paper may be found in some older work of the authors, for instance in Kontolatou et al [10] and Stabakis et al [17]. We must restate some of them mainly because our present orientation is absolutely different from the old one.

2. Prerequisites for the New Valuation

We give some well known definitions and results (cf. Fuchs [5], Fuchs [6], Kopytov et al [12]).

Let $(G, +, \leq)$ be an ordered Abelian group and $G^+ = \{x \in G : 0 \leq x\}$ its positive cone. For A, B subsets of G we will write $B < A$ if for every $a \in A$ and every $b \in B$, $b < a$; in particular we write $B < a$ (resp. $b < A$) if for every $b \in B$, $b < a$ (resp. for every $a \in A$, $b < a$). We symbolize two non comparable elements a and b by $a \parallel b$.

We recall that a subset A of an ordered set is called *join semilattice* if every finite non void subset of A has a supremum in A. Dually it is defined the *meet semilattice*. It is called *complete join (or meet) semilattice* if any subset of A has a supremum (or respectively an infimum) in A. We call *lattice* (resp. *complete lattice*) a join and meet semilattice (resp. a complete semilattice). It is evident that a join (or meet) semilattice group is a lattice group.

It is well known (cf. Kopytov et al [12, p. 16]) that every lattice group is torsion free.

2.1. The MacNeille Completion

(1) The *MacNeille completion* of an ordered set is the least complete lattice (join or meet) in which the set is embedded (cf. Birkoff [1], p. 126).

Let $(E, \leq)$ be an ordered structure. For every $A \subseteq E, A \neq \emptyset$ we put

$$A^- = \{x \in E/x < A \text{ or } x = \text{minimum of } A\}.$$

Dually we symbolize by

$$A^+ = \{x \in E/x > A \text{ or } x = \text{maximum of } A\}.$$

We call *MacNeille's cut* the couple $(A^-, (A^-)^+)$ for a subset A.

The set of all MacNeille's cuts of E is called *the MacNeille completion of* E and we symbolize it by E_M. The subsets $A^-, (A^-)^+$ are called, *lower and upper class of the cut*, respectively. We may identify every $x \in E$ with the cut $(\{x\}^-, (\{x\}^-)^+)$, thus we may consider that $E \subseteq E_M$. Moreover, we may induce an ordering $\leq^*$ in E_M as follows (for two cuts $x^* = (A_1, B_1)$ and $y^* = (A_2, B_2)$): $x^* \leq^* y^*$ iff (i) x^*, y^* are identified with elements of E and $x^* \leq y^*$, (ii) $x^* \in E$, $y^* \notin E$ and $x^* \in A_2$, (iii) $x^* \notin E$, $y^* \in E$ and $y^* \in B_1$, (iv) $x^* \notin E$, $y^* \notin E$ and $A_1 \subseteq A_2$. We shall write in the sequel $\leq$ instead of $\leq^*$.

It is easily proved that this E_M is a meet lattice. If an $A \subseteq E$ has minimum, then the cut which assigns to this point is the infimum of A in E_M. In any case, every $A \subseteq E$ has infimum in E_M and we denote it by $\inf_{E_M} A$.

(2) By analogy, we would define a dual completion of E taking for a subset $A \subseteq E$ firstly the set A^+ and then the $(A^+)^-$, so we define the MacNeille's cut $((A^+)^-, A^+)$ and a completion which contains the suprema of all the subsets of E. We shall use only infima, for which case we do not give any more notation or comment for this completion.

(3) According to J. Ohm in [15], in the axiom (ii) of the definition of a semi-valuation with value group G (see in the introduction) we write:

$$a_0 \geq \inf_G\{a_1, ..., a_n\}, \text{ iff } a_0 \geq a \text{ for all } a, \quad a \leq a_1, ..., a_n. \tag{1}$$

We observe that the relation (1) is equivalent to the relation:

$$a_0 \geq \inf{}_{G_M}\{a_1, ..., a_n\}. \tag{M}$$

In fact; all the elements a which are smaller than or equal to $a_1, ..., a_n$ constitute the set $\{a_1, ..., a_n\}^-$ and next we consider the set $(\{a_1, ..., a_n\}^-)^+$ which is the infimum of $\{a_1, ..., a_n\}$ into the completion G_M. The relation (M) means that a_0 is greater than or equal to this infimum. We observe that if $\inf\{a_1, ..., a_n\} \in G$, then a_0 may become equal to this infimum. Thus, the property $(ii)'$ of the definition of a semi-valuation (Section 1(1)) changes into the statement:

$$w(x+y) \geq \inf{}_{(w(K^*))_M}\{w(x), w(y)\},$$

where $(w(K^*))_M$ is the MacNeille completion of the value group $w(K^*)$.

(4) It has been proved that $(E_M)_M = E_M$.

2.2. The Kurepa Completion

(1) The *Kurepa completion* of an ordered set E (cf. Dokas [4] and Varouchakis [18]) is the join of E itself and of a set of classes (only the lower or only the upper) of a new kind of "cuts", which is a semi-lattice for a suitable ordering.

We consider an ordered structure $(E, \leq)$ and a subset $A \subseteq E$. We put

$$A^-_* = \{x \in E / x < A\}$$

(in distinguish to the symbol A^- in the MacNeille's case) and next

$$(A^-_*)^+_* = \{x \in E / x > A^-_*\}.$$

The couple $(A^-_*, (A^-_*)^+_*)$ is said to be a *Kurepa's cut* and the upper class $(A^-_*)^+_*$ is *the class the corresponded to* A. We symbolize by $C^+(E)$ the set of all the upper classes we form in the above procedure for all the subsets of E, which subsets are non-void and correspond to non-void classes. A singleton $\{x\}$ corresponds to the class $(x_*^-)_*^+$ which, in general, is different of $\{x\}$. If $A \subseteq E$, the join of the classes which correspond to the elements of A, constitutes a new class – element of $C^+(E)$ – which corresponds to A.

We order the set $E \cup C^+(E)$ by an ordering $\leq^*$ as follows: For x, y in E, $x \leq^* y \Leftrightarrow x \leq y$. For B_1, B_2 in $C^+(E)$, $B_1 \leq^* B_2 \Leftrightarrow B_1 \supseteq B_2$. For $x \in E$, $B \in C^+(E)$, $B \leq^* x \Leftrightarrow x \in B$ (we will omit the star of $\leq^*$ in the sequel). It is easily proved that the set

$$E_{Ku} = E \cup C^+(E)\,,$$

with the above ordering is a meet semilattice and we call it *Kurepa completion.* For $A \subseteq E$ we denote the infimum of A into the Kurepa completion by $\inf_{E_{Ku}} A$.

(2) Dually, we define, for an $A \subseteq E$, the cut $((A_*^+)_*^-, A_*^+)$, with evident the meaning of the symbols, and for all the cuts of this type taking all the subsets (also non-void and which correspond to non-void classes) we form from the lower classes the set $C^-(E)$ which is ordered by the inclusion and so the set $E \cup C^-(E)$ becomes a join semilattice.

In this paper we will make exclusively use of the $E \cup C^+(E)$ completion and saying *Kurepa completion* we will mean this one.

(3) If (A, B) is a MacNeille's cut and neither A nor B have an end (maximum of A or minimum of B), then B is an element of $C^+(E)$. On the other hand, if there is $x = \max A = \min B$, then in the Kurepa completion the class $(x_*^-)_*^+$ does not generally equal B, thus we have not obligatory $\inf_{E_{Ku}}\{x\} = x$ (see Bruns [3]). However, it is possible $\inf_{E_{Ku}}\{x\} = x$; moreover there is an immersion even of a complete lattice into the Kurepa completion. Thus, in the complete lattice R, R the set of real numbers, $\inf_{R_{Ku}}\{5\} = 5$, but $\inf_{R_{Ku}}\{x \in R \diagup x > 5\}$ is an element of Kurepa completion greater that 5, say 5^+.

(4) By analogy of the above 2.1 (3), let us consider in E the following implication:

$$a_0 \geq \inf{}_A\{a_1, ..., a_n\} \text{ iff } a_0 > a \text{ for all } a < a_1, ..., a_n,$$

where A is a suitable symbol.

Taking all a which are smaller than $a_1, ..., a_n$, we form the class $\{a_1, ..., a_n\}_*^-$ and then taking all the greater elements than the elements of this class, we form the class $(\{a_1, ..., a_n\}_*^-)_*^+$ which is a point of $C^+(E)$. Hence, the above relation is equivalent to the relation

$$a_0 \geq \inf{}_{E_{Ku}}\{a_1, ..., a_n\}. \tag{K}$$

(5) It has been proved that $(E_{ku})_{ku} = E_{ku}$.

2.3. On the Semi-Valuation

We make some remarks on the semi-valuation whose the definition we have put in the introduction.

Let, again, R be an integral domain, K its quotient field (K^*, R^* as above), $U^*(R)$ the torsion group of K^* and $U(R)$ the group of units of R.

We have the following:

(1) The canonical map $w : K^* \to K^*/U(R)$ is a semi-valuation.

(2) We have already said in 2.1 (3) that the property (ii) is equivalent to

$$w(x+y) \geq \inf{}_{(w(K^*))_M}\{w(x), w(y)\},$$

where $(w(K^*))_M$ is the MacNeille completion of $w(K^*)$.

(3) We have defined $(K^*/U(R))^+ = \{w(x)/x \in R^*\}$. In particular, $w(x) > 0 \Leftrightarrow x \in R^* - U(R)$.

(4) If w is a semi-valuation of a field K and for x, y in K, $w(x) \parallel w(y)$, then $w(x+y) \geq \inf{}_{(w(K^*))_M}\{w(x), w(y)\}$ and since the MacNeille completion is a meet lattice it is possible $w(x+y)$ equals $\inf{}_{(w(K^*))_M}\{w(x),\ w(y)\}$. For instance, if $w(x) < w(y)$ and $w(x+y)$ is comparable with $w(x)$ and $w(y)$, then $w(x+y)$ equals $w(x)$.

Generally, if $(E, \leq)$ is an ordered structure, the relation

$$a_0 \geq \inf{}_{E_M}\{a_1, ..., a_n\}, \quad a_0, a_1, ..., a_n \text{ in } E$$

means that it is possible the element a_0 to be equal to this infimum. From this remark we conclude that the value group is torsion free. In addition, the above remark (3) and the fact that every element of K^* with torsion is a unit (c.f. Section 3.2 below) lead to the same conclusion.

(5) Given a semi-valuation w on a commutative field K, there is – as we have already said – a ring (whose the quotient field is K) such that the canonical application of K^* onto $K^*/U(R)$ is equivalent to w. In fact, it is enough to take as ring the set $A_w = \{x \in K^*/w(x) \geq 0\}$. In this case

$$U(R) = \{x \in K^*/w(x) = 0\}.$$

In the case where the ring A_w is a quasi-local ring, that is in the case where it has exactly one maximal ideal, this ideal is the set

$$M_w = \{x \in K / w(x) > 0\}$$

and if $w(x) < w(y)$, then $w(x+y) = w(x)$.

3. The *G*-Valuation

We come now to the introduction of the new valuation. We begin again with an integral domain and we suppose that all the subrings of the given one contain unity.

3.1. The Typical Definition

Definition 3.1. We call G-valuation a map w of a field K onto an additive ordered group G which fulfils the following:

$(\bar{i})$ $w(xy) = w(x) + w(y)$;

$(\overline{ii})$ $w(x+y) \geq inf_{G_{ku}}\{w(x), w(y)\}$, where $inf_{G_{ku}}$ is the infimum into the Kurepa completion;

$(\overline{iii})$ $w(-1) = 0$;

$(\overline{iv})$ $w(x) = \inf ty$ iff $x = 0$, where $\inf ty$ is an element adjoint to G and greatest of all $a \in G$, such that $\inf ty + a = \inf ty + \inf ty = \inf ty$.

It is not difficult for one to understand that the above condition $(\overline{ii})$ coincides with the statement:

$$\text{if } w(x) > \gamma \text{ and } w(y) > \gamma, \text{ then } w(x+y) > \gamma, \text{ for any } \gamma \in G, \qquad (\overline{ii})\text{'}$$

In fact, the set $\{\gamma \in w(K^*) : w(x) > \gamma, w(y) > \gamma\}$ is the lower class in the Kurepa's process for the set $\{w(x), w(y)\}$ and then, since $w(x+y)$ is greater than all the elements of this lower class, it is greater than the infimum of the upper class in the same process.

The following are easy consequences:

Proposition 3.2. *Let w be a G-valuation $w : K \to \hat{G} = G \cup \{\inf ty\}$.*

(a) If 1 is the unit of K, then $w(1) = 0$.

(b) $w(x-y) \geq inf_{G_{Ku}}\{w(x), w(y)\}$.

(c) If the field is finite and G is torsion free, then the only possible G-valuation is the trivial one (that is, the one which fulfils: $w(0) = \inf ty$ and $w(x) = 0$ for all $x \in K^*$).

(d) The set $\{w(x) : x \in K^*\}$ is a group (the value group of w).

Remarks 3.3. 1. The MacNeille completion of a group is a lattice group. On the other hand, a torsion group is not a lattice group. It is remarkable that the definition of the semi-valuation is given in such a way that the semi-value group is embeddable to its MacNeille completion, so a group with torsion is not a semi-value group.

In the case of a G-valuation the element $w(x+y)$ belongs to the upper class B of the Kurepa's cut (A, B), where A is the set of elements which are smaller than $w(x)$ and $w(y)$. In particular, if $w(x) \parallel w(y)$ and there is the $infimum\{w(x), w(y)\}$, then $w(x+y)$ (in the contrary to what happens in the semi-valuation case) is never equal to this infimum. That is, the $inf_{G_{Ku}}\{w(x), w(y)\}$ is not obligatory an element of the G-value group. This phenomenon permits the G-value group to be a torsion group.

2. The so called "G-valuation" in Kontolatou et al [10] and Stabakis et al [17] is exactly the G-valuation defined in this section.

Proposition 3.4. *In every field K, G-valuated by a G-valuation $w : K \longrightarrow \hat{G}$, the set*

$$A_w = \{x \in K : w(x) \geq 0\} \cup I\,,$$

where $I = \{x \in K : w(x) \parallel 0$ and $w(x) > \gamma,$ for every $\gamma < 0\}$, is a ring.

Proof. If $w(x) > \gamma$, $w(y) > \gamma$ for every $\gamma < 0$, then $w(x+y) > \gamma$ and A_w is a group. On the other hand, A_w is multiplicatively closed. In fact: if $x \in I$, $w(y) > 0$ and $w(x)+w(y) \parallel \gamma$ for a $\gamma < 0$, then $w(x) \parallel \gamma - w(y) < 0$, an absurd. Similarly, if x, y in I, then $w(x) + w(y) \parallel \gamma$ implies $w(x) \parallel \gamma - w(y) < 0$, an absurd as well. The rest is obvious. □

Proposition 3.5. *With the notation of the former Proposition the subset*

$$I_0 = \{x \in A_w / w(x) \parallel 0,\ w(x) > \gamma,\ -w(x) > \vartheta,\ \text{for any } \gamma < 0 \text{ and } \vartheta > 0\}$$

of A_w and the set $\{x \in K/w(x) = 0\}$ constitute the set of units of A_w.

Proof. It is obvious that for any $x \in I_0$,$x^{-1} \in I_0$. On the other hand, every unit $x \in A_w$ must have $w(x)$ and $-w(x)$ into A_w, hence $x \in I_0$. If for a unit x there holds $w(x) \parallel \vartheta$, ϑ a positive element, then $-w(x) \parallel -\vartheta = \gamma < 0$, an absurd. □

Proposition 3.6. *With the notation of the Proposition 3.4, if A_w is a quasi-local ring, then the set $M_w = M_0 \cup I_1$, where $M_0 = \{x \in K/w(x) > 0\}$ and $I_1 = \{x \in K/w(x) \parallel 0, w(x) > \gamma\ , -w(x) > \gamma$ for any $\gamma < 0\}$ is its maximal ideal. Moreover for x, y in K if $w(x) < w(y)$, then $w(x+y) = w(x)$ or $w(x+y) \parallel w(x)$.*

Proof. We have already demonstrated that the units of A_w are the elements of I_0 and of the set $\{x \in K/w(x) = 0\}$. Besides, if $x \in M_w$, then xA_w is a proper ideal. In fact, if u a unit with $w(u) = 0$ and $y \in A_w$, then $xy = u$, hence $y \notin U(A_w)$, ($U(A_w)$ is the set of units of A_w) and $w(x) + w(y) = 0$, or $w(x) = w(\frac{1}{y})$, that is y is a unit, an absurd. Hence, M_w is the maximal ideal.

On the other hand, if $w(x) < w(y)$, $w(x+y)$ equals $w(x)$ or it is parallel to it, since $w(x)$ must be greater than or equal to $\inf_{w(K)_{Ku}}\{w(x+y), w(y)\}$, hence $w(x+y) = w(x)$ or $w(x+y) \parallel w(x)$. The proof is direct. □

Remarks 3.7. One of the first statements on the semi-valuation theory is that if an integral domain A is a GCD-domain, then the group of divisibility of A is a lattice. Unfortunately such a thing is not true for a G-valuation: if x, y are elements of A and a GCD of them is z, then every t which divides x and y with respect to the A, divides z as well. If zt^{-1} is a unit of A, then for a semi-valuation v it is $v(z) = v(t)$, while for a G-valuation w may be $w(z) \parallel w(t)$.

Proposition 3.8 *If the ring A_w is a quasi-local ring, the factor group A_w/M_w is a field.*

3.2. The G-Valuation as a Canonical Epimorphism

We preserve the notation: R is an integral domain, K its quotient field, $U(R)$ the multiplicative group of units of R and let $U^*(R)$ be the torsion subgroup of the multiplicative group $K^* = K \backslash \{\emptyset\}$. We recall that a natural semi-valuation is defined, that $K^*/U(R)$ may be considered as its semi-value group and that the elements of $U(R)$ are "equivalent" under the division $|_R$ in R, that is x, y in $U(R)$ if $x \,|_R\, y$ and $y \,|_R\, x$. If for a $x \in K, \overline{x}$ is its class under this equivalence, an order $<$ is defined in $K^*/U(R)$ and if $x \,|_R\, y$, $y \nmid_R x$, then $\overline{x} < \overline{y}$; zero is the $U(R)$.

We seek, here, for a subgroup U_1^* of $U(R)$ such that K^*/U_1^* is again a po group, by an order $<$, U_1^* determines an equivalence, if $\overline{x}$ is the class of $x \in K$ the division $|_A$ gives $\overline{x} < \overline{y}$ if $x \,|_A\, y$ and $y \nmid_A x$, for a subring A of R with U_1^* the set of its units, and the canonical epimorphism.

$$w : K^* \to K^*/U_1^*$$

fulfils the relation:

$$\text{if } w(x) > a \text{ and } w(y) > a, \text{ then } w(x+y) > a. \qquad (\overline{ii})$$

For reasons we will give below, we firstly put the requirement:

$$U^*(R) \subseteq U_1^* \subseteq U(R)\,. \qquad (G1)$$

We may have the resolution if we have a subring A of R, U_1^* is the set of its units and A is a local ring. We will see below some such cases. Thus, we may put as a second requirement the following:

A is a subring of R, U_1^* is the group of units of $A$and $A \setminus U_1^*$ is the maximal ideal of A. G2

We order the group K^*/U_1^* considering as its positive cone the w-images of the elements of A. Moreover we define that the classes of the elements $x \in U(R)\backslash U_1^*$ are parallel to 0, greater than any negative element and smaller than any positive one.

Remarks 3.9. We also have:

(α) The relation $(\overline{\overline{ii}})'$ in 3.1 does not hold in general in the case of a semi-valuation v. In fact, for a semi-valuation v, the relations $v(x) > 0$ and $v(y) > 0$, sometimes give $v(x + y) = 0$. The relation $(\overline{\overline{ii}})'$ is valid for a semi-valuation v, if the semi-value ring $A_v = \{x \in K/v(x) \geq 0\}$ of v, is a local ring, in which case the set $M_v = \{x \in K/v(x) > 0\}$ is the only maximal ideal of A_v.

(β) Since A is a local ring, if x_1, x_2 belong to $A\backslash U_1^*$, $x_1 + x_2 \in A\backslash U_1^*$ and $x_1x_2 \in A\backslash U_1^*$. Similarly if $u \in U_1^*$ and $x \in A\backslash U_1^*$, $x+u$ and xu belong to $A\backslash U_1^*$.

(γ) If $u \in U(R)$ and $x \notin U(R)$, then $xu \notin U(R)$

(δ) If x^* and y^* are equivalent to x and y respectively and $x + y \notin U(R)$ so $x^* + y^*$ does.

(ϵ) If $x \notin U(R)$ and $y \notin U(R)$, then $xy \notin U(R)$. In fact, if $xy = u \in U(R)$, then $u^{-1}xy = 1$ and $y \in U(R)$. On the other hand, from (γ) we conclude that $x + y \notin U(R)$, since for $x^* \in A\backslash U_1^*$, $y^* \in A\backslash U_1^*$ where x^*, y^* are equivalent to x, y, respectively, modulo $U(R)$, we have $x^*+y^* \in A\backslash U_1^*$, that is $x^*+y^* \notin U(R)$.

From these statements we conclude the following:

($\sigma\tau$) The elements of $R\backslash U(R)$ have w-images parallel to 0 and parallel to some positive elements.

(ζ) The order of K^*/U_1^* is compatible with the operation of the group.

(η) The canonical epimorphism $w : K^* \to K^*/U_1^*$ fulfils the property $(\overline{\overline{ii}})$.

Remarks 3.10. We make some remarks on (G1). We remark that if $U(R) \subsetneq U_1^*$, then there are in general, elements x, y in U_1^* such that $w(x) = w(y) = 0$, while $w(x) + w(y) > 0$, a conclusion which contradicts to the compatibility of the addition of the group K^*/U_1^* and of its order. Besides, the theory of the Section 4 is based in splitting groups and in torsion free quotient groups. So, it is a natural demand the group K^*/U_1^* to be a torsion free group.

The following are rather easy conclusions:

(α) If $U^*(R)$ is the torsion part of K^* and $U_1^* \subsetneq U^*(R)$, there would be elements of K^*/U_1^* with torsion.

(β) The quotient groups $K^*/U^*(R)$ and $K^*/U(R)$ are torsion free. In fact, for the latter assertion, if $x \in K^*$, $x^k = y \in U(R)$ and $yy' = 1$, the element $y'x^{k-1}$ is the inverse element of x. The rest are trivial.

(γ) If $U^*(R) \subseteq U_1^* \subseteq U(R)$ and the factor group $U(R)/U_1^*$ is torsion free, then the group K^*/U_1^* is torsion free too. In fact: let $x \in K^*$, $x \notin U(R)$ and $x^n = \tau \in U_1^*$. Then $\tau \in U(R)$, a contradiction, since $K^*/U(R)$ is torsion free.

After the remarks above it seems that the condition (G1) refering to the U_1^* is convenient and we conclude that the G-valuation may be defined under the above conditions (G1) and (G2).

Remark 3.11. There is at least one case in commutative algebra where we have a subring A of R and $A \setminus U(A), U(A)$ the set of units of A, is its maximal ideal. In fact, we have the following:

The large quotient field $A^R_{[S]}$ of A in R with respect to S, where A is a subring of R and S a multiplicatively closed subset of A, is the set $\{x \in R/(\exists s \in S)[xs \in A]\}$ (cf. [14, p. 51]). Furthermore, if I is an arbitary subset of R, *the large extension of* I is defined by $[I]A^R_{[S]} = \{x \in R/(\exists s \in S)[xs \in I]\}$.

It is also well known (cf. Močkoř et al [14, p. 66]) that if the subring A of R is an R-Prüfer ring, then the couple $(A^R_{[M]}, [M]A^R_{[M]})$ is *a valuation pair* of R, where $A^R_{[M]}$ is the *localization of A with respect to the multiplicative closed system M* and M is a maximal ideal of A. Consequently, if A is an R-Prüfer ring of R and P is a prime ideal of A, then $(A^R_{[P]}, [P]A^R_{[P]})$ is a valuation pair of R. The final conclusion (cf. Močkoř et al [14, p. 69]) is that if (A, P) is a non trivial valuation pair of the ring R and A is an R-Prüfer ring, then the ideal P is a maximal one in A.

3.3. The Uniformity and the Proximity of a G-Valuated Field

We consider again a G-valuation w of a field K

$$w : K \to \hat{G} = G \cup \{\inf ty\},$$

G an ordered Abelian group $(G, +, \leq)$; we suppose that w is onto and that the positive cone G_+ of G is $\neq \{0\}$.

We consider the sets

$$U_\gamma = \{(x,y) \in K \times K : w(x-y) > \gamma\}\,, \tag{1}'$$

for every $\gamma \in G_+$. Then, the family of the sets

$$U = \{U_\gamma : \gamma \in G_+\}\,, \tag{2}'$$

constitutes a uniformity on K.

From this result we have the following theorem.

Theorem 3.12. *Every $G-$ (or semi-) valuated field is a uniform space. Hence, it is a completely regular space.*

In fact, it is a Tychonoff space.

For every $\gamma \in G_+$ and every $x_0 \in K^*$ we put:

$$V_\gamma(x_0) : \{x \in K : w(x - x_0) > \gamma\} \tag{3}'$$

(sometimes we call $V_\gamma(x_0)$ *sphere with center* x_0 *and radius* γ, cf. Varouchakis [19]).

There following proposition holds.

Proposition 3.13. $\bigcap\{V_\gamma(x_0) : \gamma \in G_+\} = \{x_0\}$. *Hence, the space is* T_1.

Proof. Let $y \in \bigcap\{V_\gamma(x_0) : \gamma \in G_+\}, y \neq x_0$. Then, for any $\gamma \in G_+, w(x_0 - y) > \gamma$. Thus, if $w(x_0 - y) = \gamma_1$, since G_+ is a directed group, there is a γ, $\gamma > \gamma_1$ and $w(x_0 - y) > \gamma$, an absurd. So, $w(x_0 - y) = \inf ty$. □

We also have the following proposition.

Proposition 3.14. $V_\gamma(x_0) = V_\gamma(0) + \{x_0\}$.

Proof. Let $y \in (V_\gamma(0) + \{x_0\})$, that is $y = x + x_0$ and $w(x) > \gamma_0$. Then, $w(x_0 - y) = w(x) > \gamma$ and $y \in V_\gamma(x_0)$.

Inversely: let $y \in V_\gamma(x_0)$, hence $w(x_0 - y) > \gamma$. We put $y - x_0 = x$. Then, $w(x) > \gamma$ and $x \in V_\gamma(0)$, hence $y \in V_\gamma(0) + \{x_0\}$. □

Proposition 3.15. $V_\gamma(x) + V_\gamma(y) = V_\gamma(x + y)$.

Remark. We consider in K the topology with subbase the upper open intervals, that is the sets of the form $]a, \rightarrow, [, a \in G$, and the initial topology in G generated by w. This topology is coarser than the topology which the uniformity induces, that is every set of the form $\Gamma = w^{-1}(]\gamma, \rightarrow [), \gamma \in G$ is open for the topology of the uniformity. In fact; let $x \in K$. Then, for every $y \in V_\gamma(x)$, that is for every y such that $w(x - y) > \gamma$, there holds $w(y) > \gamma$, since $w(x) > \gamma$ as well. Hence, $V_\gamma(x) \subseteq \Gamma$. On the other hand, the initial topology generated by a function is the coarsest one for which this function is continuous.

So, we conclude.

Proposition 3.16. *For the topology of the upper open intervals in G and the topology of the uniformity in K, the G-valuation is continuous.*

As usual, from the uniformity we can define a proximity, but here we have an immediate way of defining proximity.

In fact, in the set $P(K)$ of all the subsets of a field K G-valued by a G-valuation w, we define a relation δ:

$$\text{“}A\delta B \text{ iff the set } \{w(x-y) : x \in A, y \in B\} \text{ is cofinal with } w(K^*) \text{ or contains the element } \inf ty\text{.''} \quad (4)\text{'}$$

We symbolize by $A\overline{\delta}B$ the case where A and B do not fulfil δ. It is not difficult for one to prove the following theorem.

Theorem 3.17. *Every G-valuated field is a proximity space with respect to the above relation δ.*

We can define a topology on the G-valuated field by this relation δ, in the usual way: if K is the field and $A \subseteq K^*$, then

$$\operatorname{cl} A = \{x \in K^* : w(x-A) \text{ cofinal with } G_+ \cup \{\inf ty\}\}, \qquad (5)'$$

where cl$A$ is the closure of A for this topology. We have the following result.

Theorem 3.18. *In a G-valuated field the topology the induced by the above proximity δ coincides with the topology the induced by the above uniformity U.*

Proof. With the notation we have in use, let $x \in clA$. Then, $x\delta A$ and the set $w(x-A)$ is cofinal with G_+ or contains $\inf ty$. Hence, for every $\gamma \in G_+$, there is an $a \in A$ with $w(x-a) > \gamma$, so $a \in A \cap V_\gamma(x)$.

Inversely: let $x \in \overline{A}$, $\overline{A}$ the closure of A for the topology of the uniformity. Then, for every $\gamma \in G_+, V_\gamma(x) \cap A \neq \emptyset$, that is there is an $a \in A$ with $w(x-a) > \gamma$, hence $w(x-a)$ is cofinal with G_+ or contains $\inf ty$, thus $x\delta A$ or $x \in clA$. □

Remarks 3.19. (1) The above theorem means that a G-valuated field, endowed with the topology induced by δ, is a T_1-space. Moreover the relation $x\delta y$ means that $x \leq_s y$, where $\leq_s$ is the specialization ordering of the space. Thus, the theorem says that the specialization ordering is the trivial ordering.

(2) Another conclusion from the previous Theorem is that the above uniformity is one of these uniformities which induce the proximity δ. It is well

known that if a uniformity U induces a proximity δ, then for every couple A, B of subsets, we have that

$$\text{if } A\delta B, \text{ then } u \cap (A \times B) \neq \emptyset \text{ for every } u \in U. \qquad (6)'$$

We also know that for all uniformities which induce in the space the same proximity and (of course) the same topology, there is one which is the "coarser", "totally bounded" and "transitive". The uniformity we have given is a transitive uniformity.

We point out a last thing on the topological properties of a G-valuation: If the G-valuation is an additive one, then we can prove that every sphere is open and as well as a closed one, hence the space is *extremely disconnected.*

4. *G*-Valuated Fields Ranged onto Splitting Groups

This last section is devoted to the construction of G-valuated fields ranged onto given groups in particular onto splitting groups. It is well known that given the integral domain R, its quotient field K and the induced valuation v, there holds:

$$x \mid_R y \text{ if and only if } v(x) \leq v(y),$$

where $x \mid_R y$ means that there exists $\alpha \in R$ such that $y = \alpha x$. The relation $\mid_R$ is a preorder which in the next step, because of the coincidence of all the units, becomes an order. This order in K is in correspondence with the relation $\leq$ in the group $v(K)$. After this consideration, with the term *group of divisibility* we mean the value group of a semi-valuation or more precisely a group isomorphic to such a value group. The natural question, which has been arisen, is which are the *groups of divisibility.* J. Ohm in [15] proved that every lattice group is a group of divisibility, while Jaffard in [9] gave an example of a directed group which is not such a group.

When we have a G-valuation, it is defined a transitive relation ${}_s\mid_R$ in K^* with respect to R:

$$x_s \mid_R y \text{ if and only if } x \text{ divides } y \text{ and } y \text{ does not divide } x.$$

The division is considered with respect to the ring R and we say that x *strictly divides* y *with respect to* R.

If w is a corresponded G-valuation there holds:

$$\text{if } x_s \mid_R y, \text{ then } w(x) < w(y).$$

Since we do not identify the units in the case of a G-valuation the relation is not an order. However, the problems with the value groups are similar for the two kinds of valuation and this study is the object of the rest section.

Before we begin the study of the new relation ${}_s\,|_R$, we refer to a useful construction referring to the group G/G^*, where $(G, +, \leq)$ is a partially ordered Abelian group, mixed in general, and G^* its torsion subgroup. We will concern, in some cases below, with the ordering we can equip the group G/G^*.

For every $a \in G$, let $\overline{a}$, be the class of a mod G^*. The following are easily demonstrated.

(α) *The group G/G^* is torsion free.*

In fact; if $x \in G$ and $\overline{x}$ has torsion, then there is a natural number n such that $nx = r \in G^*$, where r has *rank* m (i.e. $mr = 0$). Then $mnx = 0$ and $x \in G^*$.

(β) *The elements of G^* are non comparable. Hence, the elements of every class mod G^* are non comparable.*

In fact; if $a, b \in G^*$ with rank m and n respectively and $a - b > 0$, then $mn(a - b) = 0$, an absurd. In particular, if $a \in G^*$, it is $a \parallel 0$, because if, say, $a > 0$ and a has rank n, then $na > 0$, an absurd. On the other hand, if a and b belong to the same class mod G^*, then $a - b \in G^*$ or $a \parallel b$.

(γ) *If $a < b$, $a, b \in G$ and $\alpha' \in \overline{a}, \beta' \in \overline{b}$, then $\alpha' < \beta'$ or $\alpha' \parallel \beta'$.*

In fact; it is $a - \alpha' \in G$ with rank, say n, and $b - \beta' \in G^*$ with rank m, that is $n\alpha' = na$ and $m\beta' = mb$. If $\beta' - \alpha' < 0$, then $mn(\beta' - \alpha') = mn(b - a) > 0$, an absurd.

In the group G/G^* we define an ordering (we symbolize it, for simplicity, with the same $\leq$) by:

$$\overline{a} \leq \overline{b} \;\text{ iff }\; (\exists \alpha' \in \overline{a})(\exists \beta' \in \overline{b})\ [\alpha' \leq \beta'].$$

Since we may consider G^* as a convex subgroup of G, the definition is meaningful and in valid. Besides, the above proposition (γ) assures the compatibility of the ordering with the group-operation.

Since G/G^* is torsion free we can – according to Lorenzen – Simbireva-Everett Theorem (cf. Fuchs [5, p. 39]) – extend $\leq$ into a linear ordering, say $\widetilde{\leq}$. In this ordering, some classes among these whose the elements are "parallel" to all the elements of G^*, will become positive (by $\widetilde{\leq}$) and the rest, negative.

On the other hand, we can come back and define a new ordering in G, an extension of the initial $\leq$, say $\leq_1$:

$$a \leq_1 b \;\text{ iff }\; (a = b) \text{ or } (\overline{a} \leq \overline{b}).$$

Under this ordering $\leq_1$ all the elements of G are parallel only in the case they belong to the same class modG^*; in all the other cases they are comparable.

We come, now, to our main result of this section. We shall firstly give some lemmas and make some remarks (some of these statements may be found as partial results in Theorem 3.4 of Kontolatou et al [10]).

Lemma 4.1. *Every Abelian group is a homomorphic image of a free Abelian group.*

For the proof see Hungerford [8, p. 73].

Lemma 4.2. *In every ordered group* $(G, +, \leq)$, *for any a,b, c, elements of G, there holds:*

$$\inf{}_{G_{Ku}}\{a, b\} - c = \inf{}_{G_{Ku}}\{a - c, b - c\},$$

where $inf_{G_{Ku}}$ *symbolizes the infimum of the underlying set into the Kurepa completion of* G.

Proof. Let $\inf_{G_{Ku}}\{a, b\} = d$. Then $d \leq a$, $d \leq b$, hence $d - c \leq a - c$, $d - c \leq b - c$ and $d - c \leq \inf_{G_{Ku}}\{a - c, b - c\}$. If d_1 is the latter infimum, we also have: $d_1 \leq a - c$, $d_1 \leq b - c$, or $d_1 + c \leq a$, $d_1 + c \leq b$, hence $d_1 + c \leq d$ or $d_1 \leq d - c$. Hence, we have the result. □

Lemma 4.3. *Any torsion free directed Abelian group is the image, by a G-valuation, of a commutative field.*

Proof. We basically follow the well known process for the construction of valuated fields in general, as it is described – for instance – in Ribenboim [16, p. 32], after an introductory formation.

Let $(G, +, \leq)$ be the given structure. We extend the order $\leq$ into the order $\leq_1$ mentioned above. We consider an arbitrary commutative field k and next the ring $k[x]^{G_1^+}$ of all the polynomials with one variable x, where G_1 is the group G endowed by the ordering $\leq_1$, with coefficients from k and exponents from the positive cone G_1^+ of G_1. The operations of addition and multiplication into $k[x]^{G_1^+}$ are as the ones of the usual polynomials. Thus $k[x]^{G_1^+}$ is a commutative ring with identity.

We define $w : k[x]^{G_1^+} \to G$ with $w(0) = \inf ty$, $w(a_{\gamma_i}) = 0$ for every $a_{\gamma_i} \in k^*$ and $w(s) = r_0$, where $s = a_{r_0}x^{r_0} + a_{r_1}x^{r_1} + ... + a_{r_m}x^{r_m}, a_{r_0} \neq 0$, $0 \leq_1 r_0 \leq_1 r_1 \leq_1 ... \leq_1 r_m$.

It immediately results that $w(s) = \inf ty$ if and only if $s = 0$, $w(st) = w(s) + w(t)$ and $w(s + t) \geq \min\{w(s), w(t)\}$, where s and t in $k[x]^{G_1^+}$.

Lastly, $st = 0$ implies $s = 0$ or $t = 0$. Hence, $k[x]^{G_1^+}$ is an integral domain.

Let K_1 be the quotient field of $k[x]^{G_1^+}$. We remark that if s_1, t_1, s_2, t_2 are elements of $k[x]^{G_1^+}$ and $s_1 t_2 = s_2 t_1$, then $w(s_1) + w(t_2) = w(s_2) + w(t_1)$. Thus, if $s_1/t_1 \in K_1$ and define

$$w_1 : K_1 \to G_1, \quad w_1(s_1/t_1) = w(s_1) - w(t_1) \tag{1}$$

the map w_1 is well defined and it is an extension of w from $k[x]^{G_1^+}$ to K_1.

The map w_1 is a G - valuation:

(i) $w_1\left(\frac{s}{t}\right) = \inf ty \Leftrightarrow \frac{s}{t} = 0 \Leftrightarrow s = 0$.

(ii) If $\frac{s_1}{t_1}, \frac{s_2}{t_2}$ in K, then $w_1(\frac{s_1}{t_1} \cdot \frac{s_2}{t_2}) = w(s_1) - w(t_1) + w(s_2) - w(t_2)$.

(iii) Let $s_1 = a_{\gamma_0} x^{\gamma_0}$ +...+ $a_{\gamma_n} x^{\gamma_n}$, $t_1 = a_{\beta_0} x^{\beta_0}$ +...+ $a_{\beta_m} x^{\beta_m}$, $s_2 = a_{\epsilon_0} x^{\epsilon_0}$ +...+ $a_{\epsilon_\kappa} x^{\epsilon_\kappa}$ and $t_2 = a_{\delta_0} x^{\delta_0}$ +... + $a_{\delta_\lambda} x^{\delta_\lambda}$.

There holds:

$$\begin{aligned} w_1(\frac{s_1}{t_1} + \frac{s_2}{t_2}) &= w(s_1 t_2 + s_2 t_1) - w(t_1 t_2) \\ &= w\{(a_{\gamma_0} x^{\gamma_0} a_{\delta_0} x^{\delta_0} + ... + a_{\gamma_n} x^{\gamma_n} a_{\beta_m} x^{\beta_m}) \\ &+ (a_{\epsilon_0} x^{\epsilon_0} a_{\delta_0} x^{\delta_0} + ... + a_{\epsilon_\kappa} x^{\epsilon_\kappa} a_{\delta_\lambda} x^{\delta_\lambda})\} - w(t_1 t_2) \\ &\geq \min\{w(a_{\gamma_0} a_{\delta_0} x^{\gamma_0 + \delta_0} + ...), w(a_{\epsilon_0} a_{\delta_0} x^{\epsilon_0 + \delta_0} + ...)\} - w(t_1 t_2) \end{aligned}$$

and in the general case (i.e. non zero coefficients or exponents):

$$\begin{aligned} & w_1(\frac{s_1}{t_1} + \frac{s_2}{t_2}) \geq \\ & \min\{\gamma_0 + \delta_0, \epsilon_0 + \beta_0\} - (\beta_0 + \delta_0) = \inf{}_{G_{Ku}}\{\gamma_0 + \delta_0, \epsilon_0 + \beta_0\} \\ & \qquad - (\beta_0 + \delta_0) = \inf{}_{G_{Ku}}\{\gamma_0 - \beta_0, \epsilon_0 - \delta_0\}, \end{aligned}$$

because of (4), lemma and of the fact that $w_1(t_1 t_2) = \beta_0 + \delta_0$.

Thus $w_1(\frac{s_1}{t_1} + \frac{s_2}{t_2}) \geq \inf{}_{G_{Ku}}\{w_1(\frac{s_1}{t_1}), w_1(\frac{s_2}{t_2})\}$. □

Lemma 4.4. *For any torsion free directed Abelian group G there is an integral domain R and its quotient field K_1 with the following properties:*

(α) *There is a homomorphism w_1 of the multiplicative group $K_1^*(= K_1 - \{0\})$ onto G such that, if x divides y, but y does not divide x, with respect to R (in the sense that there is an $a \in R$ such that $y = ax$), then $w_1(x) \leq w_1(y)$ or $w_1(x) \parallel w_1(y)$.*

(β) *This homomorphism is a G-valuation if we consider G endowed true cm with a total order, which is an extension of the given order.*

Proof. Let $(G, +, \leq)$ be the given structure and $\leq_1$ an extension of $\leq$ to a total order. We construct the integral domain $R = k[x]^{G_1^+}$ described in (5) Lemma, next the field K_1 of quotients of R and, lastly, the G-valuation w_1 of K_1 onto $(G, \leq_1)$.

As we have already mentioned, the subset

$$R = \{x \in K^*/0 \leq_1 w_1(x)\}$$

of K_1 is an integral domain. Then, x divides y, means that there is an $a \in R^*$ such that $y = ax$, that is $x/y \in R^*$. In this case $w_1(x) <_1 w_1(y)$ (with the additional supposition that y does not divide x).

We suppose, now, that we come back to the order $\leq$ of G from the order $\leq_1$. Then, the elements x, y with $\frac{y}{x} \in R^*$ fulfil the relations: $w_1(x) < w_1(y)$ or $w_1(x) \parallel w_1(y)$. The inverse is not true: two elements parallel in $(G, \leq)$, say, $\overline{x} = w_1(x)$, $\overline{y} = w_1(y)$, w_1-images of x, y respectively, do not necessarily satisfy the property $\frac{y}{x} \in R^*$, because it is not obligatory for the relation $w_1(x) \parallel w_1(y)$ to imply $w_1 <_1 w_1(y)$. □

The Ordering of a Free Abelian Group

Let $\mathcal{F}$ be an arbitrary free Abelian group considered multiplicatively. We may induce a total order on it as follows: Let X be a free system of generators of F. According to the Zorn Lemma, we order it by a total order $\leq^*$ and consider the set $\mathcal{M}$ of monomials of the type

$$M = x_{i_1}^{a_1} ... x_{i_n}^{a_n} \;, \tag{$*$}$$

n finite, a_k integers and $x_{i_1} <^* x_{i_2} <^* ... <^* x_{i_n}$ (we consider these monomials in their "reduced" form, that is, if $a_k = 0$ the corresponding factor is omitted except if all the exponents are zero, in which case we put $M = 1$).

We define in $\mathcal{M}$ the multiplication of two monomials and the inverse of one of them, in the usual way and we write all the elements of $\mathcal{M}$ in a fixed form. So the structures $(\mathcal{M}, \cdot)$ and $(\mathcal{F}, \cdot)$ coincide. We give an order in $\mathcal{M}$ (M is written as in (*)), by:

$$1 <^* M_1, \text{ if } a_1 > 0 \;,\; M_1 <^* 1, \text{ if } a_1 < 0 \text{ and } M_2 <^* M_1, \text{ if } 1 \leq^* M_1 M_2^{-1}.$$

The relation $\leq^*$ is compatible with the multiplication in $\mathcal{M}$.

Theorem 4.5. *Given a splitting directed Abelian group $(G, +, \leq)$, there are an extension $\leq_1$ of $\leq$ and a commutative field valuated by a G-valuation which maps the field onto $(\hat{G} = G \cup \{\inf ty\}, +, \leq_1)$.*

Proof. Let $G = G^* \oplus \Gamma \quad (i)$, $\Gamma = (\gamma_i)_{i \in I}$ be a system of representatives, which by supposition is a group and $(\mathcal{F}, \cdot)$ a free Abelian group, totally ordered by the order $\leq^*$ defined above. We also order G according to the ordering $\leq_1$ mentioned above in the beginning of the section.

If the group G was torsion free, then we would find the commutative field K_1, as we have constructed it in the Lemma 4.4. So, we suppose that there is a torsion part G^* of G and there holds the relation $(*)$.

We also consider a homomorphism ϑ of a free Abelian group $\mathcal{F}$ onto the torsion group G^* (according to Lemma 4.1).

Since the group Γ under the order $\leq_1$ is a totally ordered group, we consider the field K_1 as we have constructed it in the above Lemma 4.4; we also consider the G-valuation w_1 defined on K_1. Next we consider the ring of the polynomials $K_1[X]$ of the form

$$s = u_1 M_1 + u_2 M_2 + ... + u_n M_n, \qquad (**)$$

where X is a base of $\mathcal{F}$'s generators, $M_i \in \mathcal{M}, i \in \{1, ..., n\}$ and $u_i \in K_1$. The monomials $M_i, i \in \{1, ..., n\}$ are written in increasing order, i.e. $M_i <^* M_{i+1}$, for every i, according to Section 7.

We define a G-valuation $\widetilde{w} : K_1[X] \to \hat{G} = G \cup \{\inf ty\}$ as follows:

$$\widetilde{w}(s) = \inf ty \in \hat{G} \text{ iff } s = 0,\ \widetilde{w}(1) = 0 \in G \text{ and } \widetilde{w}(s) = w_1(u_k) + \vartheta(M_k),$$

where M_k is the minimum nonzero, according to $\leq^*$, of all $M_i, i \in \{1, ..., n\}$. Thus $w_1(u_k) = \gamma_k \in \Gamma$ and $\vartheta(M_k) = g_k^* \in G^*$, hence $\widetilde{w}(s)$ is equivalent to γ_k modulo G^*.

Let s be as in (ii) and $t = p_1 N_1 + ... + p_m N_m$. Let also $\widetilde{w}(s) = w_1(u_k) + \vartheta(M_k)$ and $\widetilde{w}(t) = w_1(p_\lambda) + \vartheta(M_\lambda)$. The polynomial st contains the summand $u_k p_\lambda M_k N_\lambda$, where $M_k N_\lambda$ is the smallest monomial in the st, because of the compatibility of $\leq^*$. Thus:

$$\begin{aligned}\widetilde{w}(st) = w_1(u_k p_\lambda) + \vartheta(M_k M_\lambda) = w_1(u_k) + \vartheta(M_k) + w_1(p_\lambda) + \vartheta(M_\lambda) \\ = \widetilde{w}(s) + \widetilde{w}(t),\end{aligned}$$

hence $\widetilde{w}$ is a homomorphism.

It remains to be proved the triangle property: firstly, we remark that if γ_1, γ_2 in Γ, $\overline{\gamma_i}$, $i \in \{1, 2\}$ the class of γ_i modulo G^* and $\gamma_2 <_1 \gamma_1$, then for $x \in \overline{\gamma_1}$ and $y \in \overline{\gamma_2}$, $x <_1 y$. So, with the above notation, $\widetilde{w}(s+t)$ equals $w_1(u_k) + \vartheta(M_k)$ or $w_1(p_\lambda) + \vartheta(M_\lambda)$ according to whether which of $\overline{w_1(u_k)}$, $w_1(p_\lambda)$ is smaller. In both of the cases $\widetilde{w}(s+t)$ is an element of the class $\overline{w_1(u_k)}$ or of the class $\overline{w_1(u_\lambda)}$ modulo G^*, which class has the smallest elements or $\widetilde{w}(s+t)$ is an element of a class which has greater elements. In any case $\inf_{G_{Ku}}\{\widetilde{w}(s), \widetilde{w}(t)\} \leq_1 \widetilde{w}(s+t)$, where $\inf_{G_{Ku}}$ has been taken according to $\leq_1$.

Since $\widetilde{w}$ is a G-valuation, we define a new G-valuation $\widetilde{w_1}$ on the field of quotients $\widetilde{K_1}$ of the ring $K_1[X]$ by:

$$\widetilde{w_1}\big(\frac{s}{t}\big) = \widetilde{w}(s) - \widetilde{w}(t).$$

Firstly, if $st' = s't$, then $\widetilde{w}(s) + \widetilde{w}(t') = \widetilde{w}(s') + \widetilde{w}(t)$, thus $\widetilde{w_1}$ is well defined.

Now, let

$$s_1 = u_{a_1}M_{a_1} + ... + u_{a_n}M_{a_n}, \quad s_2 = u_{a'_1}M_{a'_1} + ... + u_{a'_m}M_{a'_m},$$

$$t_1 = u_{\beta_1}M_{\beta_1} + ... + u_{\beta_k}M_{\beta_k}, \quad t_2 = u_{\beta'_1}M_{\beta'_1} + ... + u_{\beta'_\lambda}M_{\beta'_\lambda},$$

with $\widetilde{w}(s_1) = w_1(u_{a_1}) + \vartheta(M_{a_1})$, $\widetilde{w}(s_2) = w_1(u_{a'_1}) + \vartheta(M_{a'_1})$ and $\widetilde{w}(t_1) = w_1(u_{\beta_1}) + \vartheta(M_{\beta_1})$, $\widetilde{w}(t_2) = w_1(u_{\beta'_1}) + \vartheta(M_{\beta'_1})$, respectively; we also put $\vartheta(M_{a_1}) = \gamma_1$, $\vartheta(M_{a'_1}) = \gamma'_1$, $\vartheta(M_{\beta_1}) = \gamma_2$ and $\vartheta(M_{\beta'_1}) = \gamma'_2$.

We have:

$$\widetilde{w}_1\big(\frac{s_1 \cdot s_2}{t_1 \cdot t_2}\big) = \widetilde{w}(s_1 s_2) - \widetilde{w}(t_1 t_2) = \widetilde{w}(u_{a_1}u_{a'_1}M_{a_1}M_{a'_1} + ...) - \widetilde{w}(u_{\beta_1}u_{\beta'_1}M_{\beta_1}M_{\beta'_1} + ...),$$

where the summands we have written are the smallest ones (with respect to $\leq^*$) and ϑ is a homomorphism. Thus: $\widetilde{w}_1\big(\frac{s_1 \cdot s_2}{t_1 \cdot t_2}\big) = \widetilde{w}(u_{a_1}u_{a'_1}) - \widetilde{w}(u_{\beta_1}u_{\beta'_1}) + \gamma_1 + \gamma'_1 - \gamma_2 - \gamma'_2 = w_1(u_{a_1}u_{a'_1}) - w_1(u_{\beta_1}u_{\beta'_1}) + (\gamma_1 - \gamma_2) + (\gamma'_1 - \gamma'_2)$, and since w_1 is a homomorphism, we conclude that

$$\widetilde{w}_1\big(\tfrac{s_1 \cdot s_2}{t_1 \cdot t_2}\big) = \widetilde{w}_1\big(\tfrac{s_1}{t_1}\big) + \widetilde{w}_1\big(\tfrac{s_2}{t_2}\big).$$

The homomorphism $\widetilde{w}_1$ fulfils the triangle property: First of all we remark that if $\overline{\gamma_1}, \overline{\gamma_2}$ are the equivalence classes modulo G^* of the elements γ_1, γ_2 in Γ, then $\gamma_1 \leq_1 \gamma_2$ implies $x_1 \leq_1 x_2$ for every $x_1 \in \overline{\gamma_1}$ and $x_2 \in \overline{\gamma_2}$.

Let, now, $\frac{s_1}{t_1} + \frac{s_2}{t_2} = \frac{(u_{a_1}u_{\beta'_1}M_{a_1}M_{\beta'_1} + ...) + (u_{a'_1}u_{\beta_1}M_{a'_1}M_{\beta_1} + ...)}{u_{\beta_1}u_{\beta'_1}M_{\beta_1}M_{\beta'_1} + ...}$, with the above notation. We name A the nominator of the right hand fraction and we suppose that the monomial $M_{a_1}M_{\beta'_1}$ is the smallest one, with respect to $\leq^*$, from all the monomials of $s_1 t_2$ and similarly $M_{a'_1}M_{\beta_1}$ is the smallest of $s_2 t_1$. Thus, the smallest term with respect to $\leq^*$ in A is one of these two, say, $M_{a_1}M_{\beta'_1}$.

Then:

$$\widetilde{w_1}\big(\frac{s_1}{t_1} + \frac{s_2}{t_2}\big) = \widetilde{w}(A) - \widetilde{w}(t_1 t_2), \quad \widetilde{w}(A) = w_1(u_{a_1}u_{\beta'_1}) + \vartheta(M_{a_1}M_{\beta'_1}).$$

If $w_1(u_{a_1}u_{\beta'_1}) \leq_1 w_1(u_{a'_1}u_{\beta_1})$, then $w_1(u_{a_1}u_{\beta'_1}) \geq \inf_{G_{Ku}}\{w_1(u_{a_1}u_{\beta'_1}),$ $w_1(u_{a'_1}u_{\beta_1})\}$ and we have the same result if $w_1(u_{a'_1}u_{\beta_1}) \leq_1 w_1(u_{a_1}u_{\beta'_1})$. Hence, from (iii):

$$\begin{aligned}\widetilde{w_1}\Big(\frac{s_1}{t_1}+\frac{s_2}{t_2}\Big) &\geq \inf{}_{G_{Ku}}\{w_1(u_{a_1}u_{\beta'_1}), w_1(u_{a'_1}u_{\beta_1})\} - w_1(u_{\beta_1}u_{\beta'_1})\\ &\quad + \vartheta(M_{a_1}M_{\beta'_1}) - \vartheta(M_{\beta_1}M_{\beta'_1}) = \inf{}_{G_{Ku}}\{w_1(u_{a_1}) - w_1(u_{\beta_1}), w_1(u_{a'_1})\\ &\quad - w_1(u_{\beta_1})\} + \vartheta(M_{a_1}) - \vartheta(M_{\beta_1})\\ &= \inf{}_{G_{Ku}}\{w_1\Big(\frac{s_1}{t_1}\Big), w_1\Big(\frac{s_2}{t_2}\Big)\} + \vartheta(M_{a_1}) - \vartheta(M_{\beta_2}). \quad \text{(iii)}\end{aligned}$$

The $\widetilde{w_1}\big(\frac{s_1}{t_1}+\frac{s_2}{t_2}\big)$ is an element of the class of an element of Γ modulo G^* and, as we have already said, the family of all these classes is a totally ordered set with respect to $\leq_1$, in the sense that two elements of two different classes are comparable and all the elements of the one class are smaller (or greater) than the elements of the other. The addition (in the relation $(***)$) of the summand $\vartheta(M_{a_1}) - \vartheta(M_{\beta_1})$ in the terms $w_1\big(\frac{s_1}{t_1}\big)$ and $w_1\big(\frac{s_2}{t_2}\big)$ does not change the class of these latter terms. Hence

$$\begin{aligned}\widetilde{w_1}\Big(\frac{s_1}{t_1}+\frac{s_2}{t_2}\Big) \geq inf_{G_{Ku}}\{w_1\Big(\frac{s_1}{t_1}\Big) + \vartheta(M_{a_1}) - \vartheta(M_{\beta_1}), w_1\Big(\frac{s_2}{t_2}\Big)\\ + \vartheta(M_{a_2}) - \vartheta(M_{\beta_2})\}.\end{aligned}$$

In the last part of the paper we concern with the notion of a "group of divisibility". We state the following theorem.

Theorem 4.6. *For every splitting directed Abelian group $(G, \leq, +)$, there is an integral domain R with quotient field $\widetilde{K_1}$ which fulfils the properties:*

(1) The multiplicative group $\widetilde{K_1^*} = \widetilde{K_1} - \{0\}$ is the direct product of two groups $\mathcal{M} \otimes (\widetilde{K_1^*}/\mathcal{M})$, $\mathcal{M}$ contains the torsion elements of $\widetilde{K_1^*}$ and the quotient group $\widetilde{K_1^*}/\mathcal{M}$ is ordered by an order $\widetilde{\leq}$, with positive cone the set $R^* = R - \{0\}$.

(2) The canonical epimorphism $p : \widetilde{K_1^*} \to \widetilde{K_1^*}/\mathcal{M}$ is such that if x divides y (in the sense that there is an $a \in R^*$ such that $y = ax$), then $p(x) \widetilde{\leq} p(y)$ or $p(x) \parallel p(y)$.

(3) There is an epimorphism $\widetilde{w_1}$ of $\widetilde{K_1^*}$ onto G such that if x divides y, then $\widetilde{w_1}(x) \leq \widetilde{w_1}(y)$ or $\widetilde{w_1}(x) \parallel \widetilde{w_1}(y)$.

(4) There is an extension $\leq_1$ of $\leq$, for which the epimorphism $\widetilde{w_1}$ becomes a G-valuation.

Proof. Firstly, we consider the analysis of G into the direct sum $G^* \oplus \Gamma$, G^* the torsion group of G and the total order $\leq_1$, extension of $\leq$, with respect to which Γ is a totally ordered subgroup of G.

We also consider the fields $K_1[X]$ and $\widetilde{K_1}$, the group $\mathcal{M}$ and the monomorphisms (actually G-valuations) w_1 and $\widetilde{w_1}$ as they have been defined in the Theorem 4.5, too. Thus we have

$$\widetilde{K_1^*} = \mathcal{M} \otimes K_1^* \,,$$

where $\mathcal{M}$ contains the torsion subgroup of $\widetilde{K_1^*}$ (since all its ϑ-images cover the G^*). The elements of $\mathcal{M}$ are monomials M_a, the elements of $K_1[X]$ are of the form $s = a_1x^{\gamma_1} + a_2x^{\gamma_2} + ...a_nx^{\gamma_n}$, with $a_i, i \in \{1, ..., n\}$, belonging to another field k and $\gamma_i \in \Gamma$. This element s has as w_1-image the point $\gamma_1 \in \Gamma$ and the same w_1-image have all the elements of $K_1[X]$ with the same first term $a_1x^{\gamma_1}$, forming so a class of an equivalence. Hence, the points of $\widetilde{K_1}$ are of the form u_1M_1 (where $u_1 = a_1x^{\gamma_1} + ...$ and $M_1 \in \mathcal{M}$) and their mapping by $\widetilde{w_1}$ onto G equals $w_1(u_1) + \vartheta(M_1)$.

We consider now the canonical application p of $\widetilde{K_1^*}$ onto $(\widetilde{K_1^*}/\mathcal{M})$. We give an order in $\widetilde{K_1^*}/\mathcal{M}$ taking as positive cone of the structure the set $K_1[X]^*$. Let $x = u_1M_1 + ...$, $y = v_1N_1 + ...$ be two elements of $\widetilde{K_1^*}$, x divides y in $K_1[X]^*$, that is there is a $\sigma_1 \in K_1[X]^*$ such that $y = \sigma_1 x$, but y does not divide x. Then $p(y) = \overline{v_1}$, $\overline{v_1}$ is the equivalence class of v_1, $p(x) = \overline{u_1}$ and $v_1 = \sigma_1 u_1$. Thus yx^{-1} belongs to the ring $K_1[X] = R$ and $p(y) \widetilde{>} p(x)$. There is an isomorphism between the structures $(G, +, \leq_1)$ and $(\widetilde{K_1}^*/M, \cdot, \widetilde{\leq})$. If we come back in G, in the structure $\leq$, then some of the equivalence classes – elements of $\widetilde{K_1^*}/M$ may become parallel. This is the reason that $p(x)$ and $p(y)$ may become parallel as well.

Lastly, if we consider the order $\leq_1$ for G, the epimorphism $\widetilde{w_1}$ becomes a G-valuation.

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