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LINEAR SYSTEMS ON THE FINITE PROJECTIVE PLANE: INTERPOLATION AND BASE POINTS

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Abstract: Here we want to discuss a few interpolation problems for linear systems on a projective space over a finite field. For instance, we prove the following result. Fix an integer $d \geq 3$, a field K containing at least $d+1$ points and $P \in \mathbf{P}^2_K$. Set $m_1 := d-2$, $m_j := 2$ for $2 \leq j \leq d-1$ and $m_i := 1$ for $d \leq i \leq d+2$. Then there are points $P_i \in \mathbf{P}^2_K$, $1 \leq i \leq d+2$ such that $h^0(\mathbf{P}^2_K, \mathcal{I}_{m_1P_1 \cup \cdots \cup m_{d+2}P_{d+2}}(d)) = 2$ and the linear system $H^0(\mathbf{P}^2_K, \mathcal{I}_{m_1P_1 \cup \cdots \cup m_{d+2}}(d))$ has P as the only base point outside $\{P_1, \dots, P_{d+2}\}$.

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Here we want to discuss a few interpolation problems for linear systems on a projective space over a finite field. For all prime powers q and positive integers n, d let $V(n,q,d)$ denote the vector space over $GF(q)$ of all homogeneous polynomials of degree d in $n+1$ variables over $GF(q)$. Hence $\dim(V(n,q,d)) = \binom{n+d}{n}$. For any $S \subseteq PG(n,q)$ set $V(n,q,d)(S) := \{F \in V(n,q,d) : F(P) = 0$ for every $P \in S\}$. Hence $\dim(V(n,q,d)) \geq \binom{n+d}{n} - \operatorname{card}(S)$ and we have equality if and only if S imposes independent conditions to $V(n,q,d)$. Set

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$BV(n,q,d,S) := \{P \in PG(n,q)\backslash S : V(n,d,q)(-(S \cup \{P\}) \neq V(n,d,q)(-S)\}$ (i.e. the set of all base points outside S of the linear system $V(n,q,d)(-S)$). Let $W(n,q,d)$ be the vector space over $GF(q)$ of all polynomials of degree at most d in n variables over $GF(q)$. For any $S \subseteq \mathbb{A}(q)^n$ set $W(n,q,d)(-S) := \{f \in W(n,q,d) : F(P) = 0$ for every $P \in S\}$. Set $BW(n,q,d,S) := \{P \in \mathbb{A}(q)^n\backslash S : W(n,d,q)(-(S \cup \{P\}) \neq W(n,d,q)(-S)\}$

Question 1. Fix a prime power q and positive integers n,d,s such that $s < \binom{n+d}{n}$ and $d \leq q$. Let $m(n,q,d,s)$ be the maximal integer t such that there are $S,T \subset PG(n,q)$ with $\mathrm{card}(S) = s$, $\mathrm{card}(T) = t$, $S \cap T = \emptyset$, the vector space $V(n,q,d)(-S)$ has dimension $\binom{n+d}{n} - s$ and $V(q,n,d)(-(S \cup T)) = V(q,n,d)(-S)$. Compute $m(n,q,d,s)$. Is it possible to classify all extremal pairs (S,T) as above? Similarly, call $m'(n,q,d)$ the corresponding integer for the affine space $\mathbb{A}(q)^n$ and for $W(n,q,d)$.

Remark 1. Take S,T as in Question 1 with $\mathrm{card}(T) = m(n,q,d,s)$. By the maximality of the integer $m(n,q,d,s)$ the set T is exactly the base locus of $V(n,q,d)(-S)$.

Question 2. What are the integers t such that there are $S,T \subset PG(n,q)$ with $\mathrm{card}(S) = s$, $\mathrm{card}(T) = t$, $S \cap T = \emptyset$, the vector space $V(n,q,d)(-S)$ has dimension $\binom{n+d}{n} - s$, $V(q,n,d)(-(S \cup T)) = V(q,n,d)(-S)$ and $V(-(S \cup T))$ has no base point in $PG(n,q)\backslash(S \cup T)$? In particular, are there gaps in the possible such integers just near the maximal value $m(n,d,d,s)$? For what integers n,q,d,s are there S,T as above with $\mathrm{card}(T) = 1$, i.e. such that the set S gives independent conditions to $V(n,q,d)$ and such that $V(n,q,d)(-S)$ has exactly one point as a base locus in $PG(n,q)\backslash S$? Answer the same questions for the affine space and $W(n,q,d)$.

The last part of Question 2 is the recognition problem of the point T starting with a set S giving independent conditions to $V(n,q,d)$. If we drop the independence condition the corresponding problem is trivial: it is sufficient to find a set S' such that $V(n,q,d)(-S')$ has at least another base point and then take as S the union of S' and all exactly one of the base points of $V(n,q,d)(-S')$ not on S'.

Question 3. Fix a prime power q and integers $n \geq 2$, $d < q$. What is the maximal integer t such that for every $S \subset PG(n,q)$ we have $V(n,q,d)(-S) = \{0\}$? Call $(q-1)^{n+1}/(q-1) - \alpha(q,n,d) + 1$ this integer. Notice that $\alpha(q,n,d)$ is the maximal number of points with $GF(q)$ as residue field of a degree d hypersurface defined over $GF(q)$. We conjecture that $\alpha(n,q,d) = (q-1)^{n-1}/(q-1) + dq^{n-1}$ and that we have the equality exactly for the hypersurfaces union of

d hyperplanes of $PG(n, q)$, all of them passing through a common codimenaion 2 linear subspace. Now assume $d \geq 2$ and call $\alpha_2(n, q, d)$ the maximal integer $t < \alpha(n, q, d)$ such that there is a degree d hypersurface defined over $GF(q)$ and with t points with with $GF(q)$ as residue field. The same problems may be studied for the affine space $\mathbb{A}(q)^n$.

Remark 2. Fix a prime power q and integers $n \geq 2$, $d < q$. Fix a codimension two linear subspace A of $PG(n, q)$. Let $S \subset PG(n, q)$ be the union of d distinct hyperplanes containing A. Hence $\text{card}(S) = (q-1)^{n-1}/(q-1) + dq^{n-1}$. Let $T \subset PG(n, q)$ be the union of d planes. Then $\text{card}(T) \leq (q-1)^{n-1}/(q-1) + dq^{n-1}$ and we have equality if and only if T is projectively equivalent to S.

Proposition 1. *Fix a prime power q and an integer $d < q$. Then $\alpha(q, 2, d) = 1 + dq$ and the only degree d curves of $PG(2, q)$ defined over $GF(q)$ and with $1 + dq$ points with $GF(q)$ as a residue field are the union of d lines of $PG(2, q)$ through a common $P \in PG(2, q)$, i.e. the first conjecture raised in Question 3 is true for $n = 2$.*

Proof. Since the case $d = 1$ is trivial, we may assume $d \geq 2$. Take $S \subseteq PG(2, q)$ with $\text{card}(S) \geq 1 + qd$ and S contained in the zero-locus of a degree d homogeneous polynomial. If S is a union of lines, then use Remark 2. If S contains at least one line, then we easily conclude by induction on the integer d. Now assume that S contains no line. Fix $Q \in S$. There are $q + 1$ lines of $PG(2, q)$ through Q and each of them contains at most d point of S because S contains no line. Hence $\text{card}(S) \leq 1 + (q+1)(d-1) = qd + d - q < 1 + qd$. □

Take $S \subset PG(n, q)$, $n > 2$. The proof of Proposition 2 works verbatim and gives $\text{card}(S) \leq (q-1)^{n-1}/(q-1) + (q+1)(\alpha(n-1, q, d) - (q-1)^{n-1}/(q-1))$ if S contains a codimension two linear subspace, but no hyperplane of $PG(n, q)$. More generally, the same proof gives the following result.

Proposition 2. *Fix a prime power q and integers $n \geq 3$, $d < q$. Take $S \subset PG(n, q)$ which is contained in the zero-locus of a non-zero degree d polynomial over $GF(q)$. Assume h that S contains no hyperplane of $PG(n, q)$ and call β the maximal number of points of S contained in a codimension two linear subspace of $PG(n, q)$. Then* $\text{card}(S) \leq \beta + (q+1)(\alpha(n-1, d, d) - \beta)$.

Remark 3. For every positive integer $s \leq d + 1$ and every $S \subset PG(n, q)$ with $\text{card}(S) = s$ we have $\dim(V(n, q, d)(-S)) = \binom{n+d}{n} - s$. Hence for every $A \subset PG(n, q)$ such that $\text{card}(A) \leq d$ we have $BV(n, q, d, a) = \emptyset$. Now assume $d \leq q$. For every integer s such that $d + 1 \leq s < ((q+1)^{n+1} - 1)/(q-1)$ there is $S \subset PG(n,q)$ such that $\mathrm{card}(S) = s$ and $BV(n,q,d,S) \neq \emptyset$): just take S containing at least $d+1$ collinear points, but not all points of the corresponding line.

Now we consider the same base locus recognition problem allowing also derivatives as interpolation data. In this way we will be able to give base point sets for some interpolation problem consisting of a unique point. We fix an integer $d \geq 3$ and we work over an arbitrary field with the only restriction that it has at least $d+1$ elements. To simplify the numerology we consider only the plane, leaving to the interested reader the case of a projective space of dimension at least three. It is very easy to state and prove similar in the affine case. The key of our construction is to use interpolations problems involving Taylor's expansions, i.e. higer order derivatives. We recall that in positive characteristic one uses the Hasse derivatives, not the ordinary ones, to get the Taylor expansions of polynomials. If P_i, $1 \leq i \leq k$, are distinct points of $\mathbf{P}^n_K$ and m_i, $1 \leq i \leq k$, are positive integers, $H^0(\mathbf{P}^n_K, \mathcal{I}_{m_1P_1 \cup \cdots \cup m_kP_k}(d))$ denotes the vector space of all homogeneous degree d polynomials in $n+1$ variables vanishing at each P_i with multiplicity at least m_i. $H^0(\mathbf{P}^n_K, \mathcal{I}_{m_1P_1 \cup \cdots \cup m_kP_k}(d))$ is a K-vector space of dimension at least $\binom{n+d}{n} - \sum_{i=1}^k \binom{n+m_i-1}{n}$

Theorem 1. *Fix an integer $d \geq 3$, a field K contains at least $d+1$ points and $P \in \mathbf{P}^2_K$. Set $m_1 := d-2$, $m_j := 2$ for $2 \leq j \leq d-1$ and $m_i := 1$ for $d \leq i \leq d+2$. Then there are points $P_i \in \mathbf{P}^2_K$, $1 \leq i \leq d+2$ such that $h^0(\mathbf{P}^2_K, \mathcal{I}_{m_1P_1 \cup \cdots \cup m_{d+2}P_{d+2}}(d)) = 2$ and the linear system $H^0(\mathbf{P}^2_K, \mathcal{I}_{m_1P_1 \cup \cdots \cup m_{d+2}P_{d+2}}(d))$ has P as the only base point outside $\{P_1, \ldots, P_{d+2}\}$. Furthermore, in this linear system each point P_i is a base point with multiplicity exactly m_i and its general member has as zero-locus a curve C which has each P_i an ordinary multiple point with multiplicity m_i with m_i different tangents; for general C, C' there intersection multiplicity at P_i is exactly m_i^2, i.e. they have different tangents at P_i. Furthermore, if K is infinite, this is true for "sufficiently general" points $P_1, \ldots, P_{d+2} \in \mathbf{P}^2_K$.*

Proof. Fix any two degree d plane curves C, C' without common components. Their total intersection number is d^2. If C, C' have P_i as an ordinary multiple points with multiplicity m_i and no common tangent at P_i, then their intersection multiplicity at P_i is exactly m_i^2. We choosed the data so that $d^2 - \sum_{i=1}^{d+2} m_i^2 = 1$. Hence by Bezout Theorem the linear system generated by C and C' has (over the algebraic closure $\mathbb{K}$ of K) exactly one base point, Q. The uniqueness of Q gives that it is defined over K if C, C' and all P_i, $1 \leq i \leq d+2$, are defined over K. Since $\mathbf{P}^2_K$ is homogeneous, to check the first two parts it is sufficient to find distinct points $P_i \in \mathbf{P}^2_K$, $1 \leq i \leq d+2$ such that

$h^0(\mathbf{P}^2_K, \mathcal{I}_{m_1P_1\cup\cdots\cup m_{d+2}}(d)) = 2$ and the linear system $H^0(\mathbf{P}^2_K, \mathcal{I}_{m_1P_1\cup\cdots\cup m_{d+2}}(d))$ has each P_i as a base point with multiplicity exactly m_i and different tangents and (for general different curves as above, different tangents at each P_i) and hence (by the first part) with exactly one base point. Fix $P_1 \in \mathbf{P}^2_K$ and call $D \subset \mathbf{P}^2_K$ the union of $d-2$ general lines $D_2, \dots, D_{d-1}$ through P_1 and two general lines D, R. Set $P_i := D_i \cap D$ for $2 \le j \le [(d-1)/2]+1$, $P_i : D_i \cap R$ for $[(d-1/2]+2 \le i \le d-1$ and take as P_j, $d \le j \le d-2$, two general points of R and one general point of D. It is easy to check that all conditions are satisfied. the last statement of Theorem 1 follows from the semicontinuity theorem for cohomology and the openess of smoothness. □

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