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APPROXIMATE SCHEMES FOR STOCHASTIC OSCILLATORS AND STOCHASTIC RESONANCE OF NONINVASIVE CONTROL

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Abstract: In this work a new approach that apply approximate schemes to study stochastic resonance of noninvasive control is suggested.

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1. Introduction

1.1. About Stochastic Resonance

20 years ago the name of stochastic resonance (SR) first time appeared in the seminal paper by Benzi and his collaborators for explaining the periodicity of the Earth's ice ages (see [1]). After that, it has become very popular in many fields of natural science and engineering such as physics, chemistry, biology and biomedical engineering and numerous contributions to SR have appeared in a lot of journals. Now, it is safe to say understanding of SR has reached a natural level.

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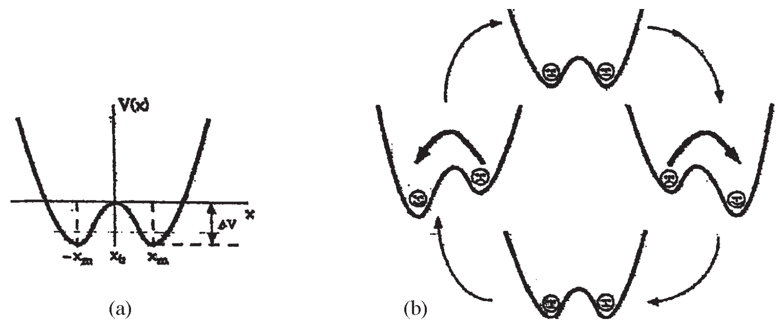


Figure 1.

SR is a phenomenon in non-linear dynamic system which possesses 3 features: (a) a form of threshold; (b) a source of noise; (c) a generally weak input source. In such system, the response of the system occurs a kind of resonance-like behavior due to resonance between the weak deterministic signal and stochastic noise. Hence the name - stochastic resonance. This phenomenon has been observed in a large variety of systems, including bistable ring lasers, semiconductor devices, chemical reactions and mechanoreceptor cells in the tail fan of a crayfish.

1.2. Characterization of Stochastic Resonance

Although in the recent literature the notion of SR gained broader significance, the archetype of SR models is simple and its mechanism is easy to explain. We use a typical example to illustrate its mechanism and characters.

The canonical example of SR involves a sinusoidally driven overdamped noisy bistable oscillator. It can be described by equation below:

$$m\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) = \frac{d}{dx}V(x) + F_N(t) + F_s(t)\,,$$

where $V(x) = \frac{1}{4}bx^4 - \frac{1}{2}ax^2$, with $a, b > 0$ is a bistable potential. Its minima are located $at \pm x_m$ with $x_m = \sqrt{\frac{a}{b}}$, these are separated by a potential barrier with height (threshold) $Dv = \frac{a^2}{4b}$. The barrier top is located at $x_b = 0$.

Sometimes, it is called as double well potential, also (see Figure 1a).

$F_s(t) = A_s \cos(Wt)$ represents periodic driving which changes the potential to $V(x,t) = V(x) - xA_s \cos(Wt)$ which is tilted back and forth, thereby raising and lowering successively the potential barrier of the right and left well, respectively, in an antisymmetric manner (see Figure 1b).

$F_N(t) = DN(t)$ represents the stochastic force produced by oscillator's internal or background noise; D is noise strength and $N(t)$ is Gaussian white noise with zero mean and unit variance. This stochastic force is fluctuational and causes transitions (hopping) between the neighboring potential wells with a rate given by famous Kramers rate

$$r_k = \frac{\omega_0 \omega_b}{2\pi\gamma} \exp\left(\frac{-\Delta v}{D}\right),$$

where $(\omega_0)^2 = \frac{V''(x_m)}{m}$ and $(\omega_b)^2 = \left|\frac{V''(x_b)}{m}\right|$.

Usually, the periodic driving is too weak to let the particle roll periodically from one potential well into the another one. But, the noise-induced hopping between the potential wells can become synchronized with the periodic forcing. This statistical synchronization takes place when the average waiting time $T_k(D) = \frac{1}{r_k}$, where r_k is Kramers rate, between two noise-induced interwell transitions is comparable with half the period T_Ω of periodic forcing. This yield time-scale matching condition for stochastic resonance, i.e. $2T_k(D) = T_\Omega$.

In short, SR in a bistable system occurs by a synchronization of activated hopping events between the potential minima with the weak periodic forcing. For a given T_Ω this can be realized by tuning the noise level D to the value determined by time-scale matching condition.

1.3. Approximate Schemes for Stochastic Oscillators

Approximate scheme of order 2 and 3 for stochastic oscillators (SO):

For SO defined on $[0, T]$,

$$x = x_0 + \int_0^t y(s)\,ds \text{ and } y = y_0 + \int_0^t \left(K(x(s), y(s), s)\,ds + \sigma\,dw_x\right).$$

Its approximate solution $\{x_i, y_i\}$ produced by approximate scheme of order 2 is

$$x_{i+1} = x_i + y_i h + \frac{2}{3}\sigma h \Delta w_i + \frac{1}{2} K_i h^2, \quad i = 0, 1, \ldots, n-1,$$

$$y_{i+1} = y_i + sDw_i + k_i h + \frac{\sigma^2}{4}\frac{\partial^2}{\partial y^2} K_i h (\Delta w_i)^2 + \frac{1}{2}\left(\frac{\partial}{\partial t}k + y\frac{\partial}{\partial x}k + k\frac{\partial}{\partial y}k\right)_i h^2. \quad (1.3.1)$$

Its approximate solution $\{x_i, y_i\}$ produced by approximate scheme of order 3 is:

$$x_{i+1} = x_i + y_i h + \frac{2}{3}\sigma h \Delta w_i + \frac{1}{2}k_i h^2 + \frac{4\sigma}{15}\left(\frac{\partial}{\partial y}k\right)_i h^2 \Delta w_i$$
$$+ \frac{1}{6}\left(\frac{\partial}{\partial t}k + y\frac{\partial}{\partial x}k + k\frac{\partial}{\partial y}k\right)_i h^3 + \frac{\sigma^2}{12}\left(\frac{\partial^2}{\partial y^2}k\right)_i h^2 (\Delta w_i)^2 , \quad (1.3.2)$$

$$y_{i+1} = y_i + \sigma \Delta w_i + k_i h + \frac{\sigma^2}{4}\left(\frac{\partial^2}{\partial y^2}k\right)_i h (\Delta w_i)^2$$
$$+ \frac{1}{2}\left(\frac{\partial}{\partial t}k + y\frac{\partial}{\partial x}k + k\frac{\partial}{\partial y}k\right)_i h^2$$
$$+ \left[\frac{4\sigma}{15}\frac{\partial}{\partial t}k + \frac{4\sigma}{15}\left(\frac{\partial^2}{\partial y^2}k\right)^2 + \frac{2\sigma}{5}\frac{\partial^2}{\partial y^2}k + \frac{2\sigma}{5}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}k\right)\right]_i h^2 \Delta w_i$$
$$+ \sigma\left[\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial y^2}k\right) + k\frac{\partial^3}{\partial y^3}k + \sigma y \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial y^2}k\right)\right] h^2 (\Delta w_i)^2$$
$$+ \frac{1}{6}\left[k\frac{\partial}{\partial t}k + \left(\frac{\partial}{\partial t}k\right)\left(\frac{\partial}{\partial x}k\right) + y\left(\frac{\partial}{\partial x}k\right)\left(\frac{\partial}{\partial y}k\right) + y\left(\frac{\partial}{\partial x}k\right)^2 + \frac{\partial^2}{\partial t^2}k\right.$$
$$+ y^2\frac{\partial^2}{\partial x^2}k + k^2\frac{\partial^2}{\partial y^2}k + 2y\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}k\right) + 2y\frac{\partial}{\partial t}\left(\frac{\partial}{\partial y}k\right)$$
$$\left. + 2ky\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}k\right)\right]_i h^3 ,$$

where $k(t_i, x_i, y_i)$, $\frac{d}{dt}k(t_i, x_i, y_i)$, $\frac{\partial^2}{\partial y^2}k(t_i, x_i, y_i)$, ... are replaced by k_i, $\left(\frac{\partial}{\partial t}k\right)_i$, and $\left(\frac{\partial^2}{\partial y^2}k\right)_i$, ...

The following theorem describes the convergence of schemes above (see [4]).

Theorem. *Let the following assumptions hold for stochastic oscillator:*

(1) $E\left((|x_0|)^4 + (|y_0|)^4\right) < \infty$;

(2) $K(t, x, y) \in C^2(R^4 \times R^2)$ *or* $C^3(R^4 \times R^2)$ *and* K *and its partial derivatives up to order 2 or 3 satisfy Lipschitz condition in* x, y.

Then:

(a) $E|x(T) - x_n| < O(h^2)$ *for* $\{x_n\}$ *produced by (1.3.1) in strong sense;*

(b) $E|x(T) - x_n| < O(h^3)$ for $\{x_n\}$ produced by (1.3.2) in strong sense.

2. Noninvasive Control of Stochastic Resonance and Approach

In this work we shall investigate the SR of sinusoidally driven overdamped noisy bistable oscillator and control its SR by adding external feedback into system and estimate the strength of SR by mean square of its approximate solution.

2.1. Sinusoidally Driven Overdamped Noisy Bistable Oscillator

In this work we shall deal with the SR of sinusoidally driven overdamped noisy bistable oscillator mentioned in Subsection 1.2 with equation

$$m\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) = -\frac{d}{dx}V(x) + F_s(t) + F_N(t)\,. \tag{2.1.1}$$

In studying SR of bistable system we note it can be controlled, either to suppress or enhance the output by sinusoidally moduling the barrier height between the both of a bistable system. But, in many system of interest, such as neuron, it is difficult or impossible to modulate the relevant barrier. Therefore, we must have other ways. One different way is to add external feedback into system to enhance the response of the system to periodic driving and significantly magnify its natural SR. Since this kind of control is based on the output of the system, it is called noninvasive control.

Thus, we shall deal with the controlled sinusoidally driven overdamped noisy bistable oscillator by adding a controller $F_c(x)$ into the system. Its equation is

$$m\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) = -\frac{d}{dx}V(x) + F_s(x) + F_N(t) + F_c(x)\,, \tag{2.1.2}$$

where controller $F_c(x)$ is the external feedback which is implicitly dependent on t. We can choose $F_c(x)$ to lower the hight of barrier of potential so that the likelihood of switching between states increases.

2.2. Suggested Approach and Criterion

In this work we shall adopt a different criterion to determine the strength of SR of a oscillator. That is we use the max value of mean square of responce of the system to estimate the strength of SR. Thus, we can directly use its approximate solution to get approximate value of mean square and estimate the strength of SR. So, the approach we shall adopt in this work is to apply the approximate schemes for stochastic oscillators to the controlled bistable system to get approximate solution and mean square, then estimate the strengthof a SR. In next two sections, we shall the follow approach above to investigate the controlled system with external feedback of binary pulse.

3. Controlled Noisy Bistable Oscillator with Feedback of Binary Pulse

In this section, we shall assume the external feedback is binary pulse:

$$F_c(x) = -A\frac{x}{|x|}\,, \quad A \text{ — pulse amplitude.}$$

For simplicity, let $m = 1$, $g = 10$, $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$, $F_N(t) = sN(t)$ with $s = 0.1$, $F_s(t) = A_s \sin(wt)$ with $A_s = 0.04$ and $w = 10p2^{-9}$, and the pulse produces a force $F_c(x > 0) = -A$ and $F_c(x < 0) = A$, or changes potential to $V = \frac{1}{4}x^4 - \frac{1}{2}x^2 - F_c x$, then equation (2.1.2) becomes

$$\frac{d^2}{dt^2}x(t) + 10\frac{d}{dt}x(t) = x^3 - x + 0.1N(t) + 0.04\sin(\omega t) - A\,.$$

In order to use approximate scheme in Section 1.3, let $\frac{d}{dt}x(t) = y$, then $\frac{d}{dt}y = x^3 - x - 10y - A + 0.04\sin(\omega t) + 0.1N(t)$, where $k(t, x, y) = x^3 - x - 10y - A + 0.04\sin(\omega t)$ and $s = 0.1$.

Thus, its approximate solution produced by approximate scheme of order 3 in Section 1.3:

$$x_{i+1} = x_i + y_i h + \frac{0.2}{3}h\Delta w_i + \frac{1}{2}k_i h^2 - \frac{4}{15}h^2\Delta w_i + \frac{1}{6}\left[0.04\omega\sin(\omega t_i) + y_i\left[3(x_i^2) - 1\right] - 10k_i\right]h^3\,,$$

$$\begin{aligned} y_{i+1} = {} & y_i + 0.1\Delta w_i + k_i h + \frac{1}{2}\left[0.04\omega\sin(\omega t_i) + y_i\left[3(x_i)^2 - 1\right] - 10k_i\right]h^2 \\ & + \left(\frac{0.016}{15}\omega\cos(\omega t_i) + \frac{8}{3}\right)h^2\Delta w_i \\ & + \frac{1}{6}\Big[\left[3(x_i)^2 - 1\right]k_i + 0.04\left[3(x_i)^2 - 1\right]\omega\cos(\omega t_i) - 10y_i\left[3(x_i)^2 - 1\right] \\ & + k_i\left[3(x_i)^2 - 1\right]^2 - 0.04\omega^2\sin(\omega t_i) + 6x_i(y_i)^2\Big]h^3\,. \end{aligned}$$

In expression above $\{x_n\}$ is our desired solution which is the approximate stochastic process over time interval $[0, T]$.

Now, we use below PC program to perform the approximate scheme to get its approximate solution over interval $[0, T]$ with n steps and step size h. This program is produced by *Mathcad 2001i*. In the program we replace Dw with $\sqrt{h}N(t)$ and use routine "*rnorm*" to produce $N(t)$. Also,we assume the initial conditions: $x_0 = a = 0$ and $y_0 = b = 1$.

We proceed to our investigation with the cases of $A = 0, 0.2, 0.4, 0.5$. Their solutions are expressed by matrix $B0$, $B1$, $B2$, $B3$ with size 100×11 which are produced by programs $Bp(a, b, T, m, n, A)$.

$$
\begin{aligned}
\mathrm{Bp}(a,b,T,n,m,A) := \;& x_0 \leftarrow a \\
& y_0 \leftarrow b \\
& h \leftarrow \frac{T}{n} \\
& t_0 \leftarrow 0 \\
& \omega \leftarrow 10\cdot\pi\cdot 2^{-9} \\
& \text{for } i \in 0..n-1 \\
& \quad t_{i+1} \leftarrow t_i + h \\
& \quad x_{i+1} \leftarrow x_i + y_i\cdot h + \frac{0.2}{3}\cdot h\cdot \mathrm{rnorm}(m,0,1)\cdot\sqrt{h} - \frac{4}{15}\cdot h^2\cdot\sqrt{h}\cdot\mathrm{rnorm}(m,0,1)\;\ldots \\
& \qquad + \frac{1}{2}\cdot\left[(x_i)^3 - x_i - 10\cdot y_i - A + 0.04\cdot\sin(\omega\cdot t_i)\right]\cdot h^2\;\ldots \\
& \qquad + \frac{1}{6}\cdot\left[\begin{array}{l} 0.04\cdot\omega\cdot\cos(\omega\cdot t_i) + y_i\cdot\left[3\cdot(x_i)^2 - 1\right]\;\ldots \\ +(-10)\cdot\left[(x_i)^3 - x_i - 10\cdot y_i - A + 0.04\cdot\sin(\omega\cdot t_i)\right] \end{array}\right]\cdot h^3 \\
& \quad y_{i+1} \leftarrow y_i + \frac{\mathrm{rnorm}(m,0,1)\cdot\sqrt{h}}{10} + \left[(x_i)^3 - x_i - 10\cdot y_i - A + 0.04\cdot\sin(\omega\cdot t_i)\right]\cdot h\;\ldots \\
& \qquad + \frac{1}{2}\cdot\left[\begin{array}{l} 0.04\cdot\omega\cdot\cos(\omega\cdot t_i) + y_i\cdot\left[3\cdot(x_i)^2 - 1\right]\;\ldots \\ +(-10)\cdot\left[(x_i)^3 - x_i - 10\cdot y_i - A + 0.04\cdot\sin(\omega\cdot t_i)\right] \end{array}\right]\cdot h^2\;\ldots \\
& \qquad + \left(\frac{0.016}{15}\cdot\omega\cdot\cos(\omega\cdot t_i) + \frac{8}{3}\right)\cdot h^2\cdot\sqrt{h}\cdot\mathrm{rnorm}(m,0,1)\;\ldots \\
& \qquad + \frac{1}{6}\cdot\left[\begin{array}{l} \left[3\cdot(x_i)^2 - 1\right]\cdot\left[\left[\begin{array}{l} (x_i)^3 - x_i\;\ldots \\ +(-20\cdot y_i)\;\ldots \\ +(-A)\;\ldots \\ +0.04\cdot\sin(\omega\cdot t_i) \end{array}\right] + 0.04\cdot\omega\cdot\cos(\omega\cdot t_i)\right]\;\ldots \\ + y_i\cdot\left[3\cdot(x_i)^2 - 1\right]^2 - 0.04\cdot\omega^2\cdot\sin(\omega\cdot t_i) + 6\cdot x_i\cdot(y_i)^2 \end{array}\right]\cdot h^3 \\
& x \leftarrow \mathrm{augment}(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \\
& x
\end{aligned}
$$

Define parameters: $n := 10$, $m := 100$, $T := 1$, $h := \frac{T}{n}$.
Define initial condition: $j := 1..m-1$, $a_{j,0} := 0$, $b_{j,0} := 1$.
For the case of $A = 0$ (without external feedback): $A := A$, $A := 0$.
Call program and define approximate solution as matrix $B0$: $B := Bp(a, b, T, n, m, A)$.

$B0 =$

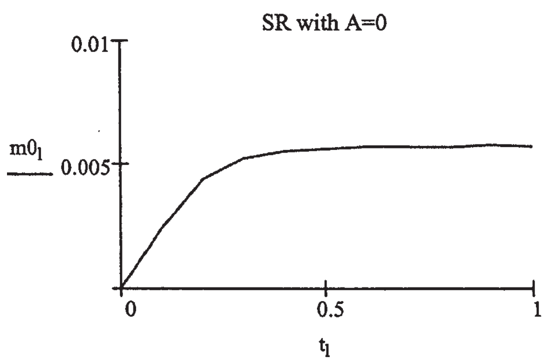
	1	2	3	4	5	6	7	8	9	10
88	0.042	0.063	0.065	0.074	0.076	0.083	0.083	0.092	0.093	0.094
89	0.052	0.069	0.073	0.073	0.072	0.07	0.073	0.072	0.075	0.074
90	0.051	0.068	0.076	0.078	0.08	0.082	0.082	0.082	0.083	0.082
91	0.051	0.066	0.071	0.074	0.078	0.077	0.081	0.085	0.094	0.096
92	0.049	0.069	0.078	0.075	0.078	0.081	0.083	0.082	0.084	0.085
93	0.048	0.064	0.077	0.087	0.088	0.088	0.087	0.086	0.082	0.082
94	0.047	0.064	0.065	0.071	0.072	0.07	0.067	0.068	0.068	0.067
95	0.05	0.067	0.075	0.078	0.079	0.084	0.092	0.09	0.087	0.085
96	0.05	0.064	0.068	0.066	0.062	0.065	0.068	0.069	0.08	0.087
97	0.052	0.071	0.083	0.086	0.092	0.091	0.091	0.091	0.09	0.088
98	0.047	0.063	0.071	0.071	0.067	0.07	0.069	0.07	0.074	0.07
99	0.052	0.069	0.072	0.069	0.063	0.057	0.054	0.047	0.048	0.044

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $m0_1 :=$mean$\left[\left(B0^{\langle 1\rangle}\right)^2\right]$.

$m0^T =$

	4	5	6	7	8	9	10
0	$5.559\,10^{-3}$	$5.651\,10^{-3}$	$5.745\,10^{-3}$	$5.734\,10^{-3}$	$5.718\,10^{-3}$	$5.822\,10^{-3}$	$5.75\,10^{-3}$

Its graph:



Finally get strength of SR: $SH := \max(m0)$, $SH = 5.822 \times 10^{-3}$.

For the case of $A = 0.2$: $A := A$, $A := 0.2$.

Call program and define approximate solution as matrix $B1$: $B1 := Bp(a, b, T, n, m, A)$.

$B1 =$

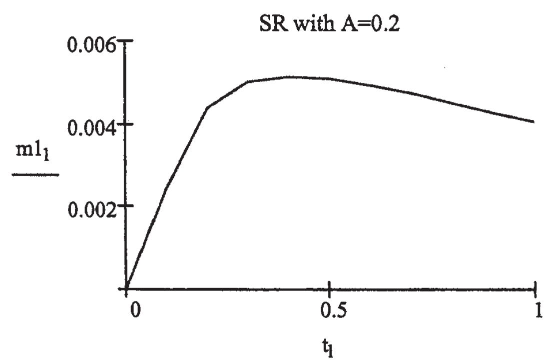
	1	2	3	4	5	6	7	8	9	10
88	0.051	0.068	0.073	0.078	0.076	0.072	0.072	0.067	0.066	0.066
89	0.046	0.066	0.068	0.065	0.065	0.067	0.068	0.067	0.074	0.079
90	0.049	0.067	0.074	0.079	0.085	0.088	0.085	0.083	0.088	0.087
91	0.049	0.071	0.079	0.079	0.083	0.089	0.088	0.089	0.092	0.091
92	0.049	0.069	0.074	0.074	0.07	0.067	0.068	0.07	0.071	0.069
93	0.052	0.073	0.079	0.075	0.074	0.063	0.06	0.059	0.058	0.058
94	0.051	0.069	0.075	0.082	0.081	0.073	0.073	0.071	0.065	0.06
95	0.052	0.069	0.075	0.069	0.068	0.065	0.061	0.053	0.051	0.049
96	0.048	0.065	0.072	0.074	0.078	0.077	0.076	0.076	0.073	0.074
97	0.051	0.073	0.081	0.086	0.093	0.099	0.102	0.097	0.092	0.089
98	0.053	0.071	0.075	0.077	0.072	0.073	0.073	0.072	0.076	0.075
99	0.049	0.069	0.071	0.072	0.074	0.07	0.071	0.076	0.076	0.077

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $m1_1 :=\text{mean}\left[\left(B1^{\langle 1\rangle}\right)^2\right]$.

$m1^T =$

	4	5	6	7	8	9	10
0	$5.162\,10^{-3}$	$5.124\,10^{-3}$	$4.959\,10^{-3}$	$4.771\,10^{-3}$	$4.523\,10^{-3}$	$4.281\,10^{-3}$	$4.068\,10^{-3}$

Its graph:



Finally get strength of SR: $SH := \max(m1)$, $SH = 5.162 \times 10^{-3}$.
For the case of $A = 0.4$: $A := A$, $A := 0.4$.
Call program and define approximate solution as matrix $B2$: $B2 := Bp(a, b, T, n, m, A)$.

$B2 =$

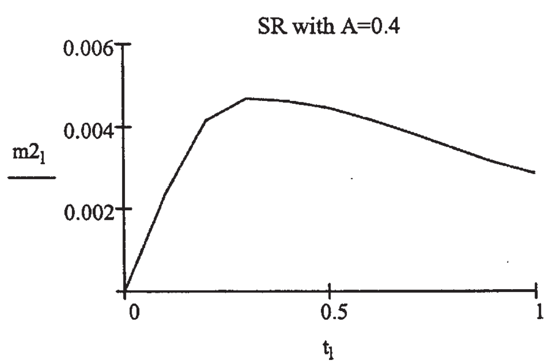
	1	2	3	4	5	6	7	8	9	10
88	0.049	0.063	0.072	0.075	0.076	0.073	0.073	0.07	0.068	0.068
89	0.051	0.069	0.069	0.068	0.064	0.059	0.051	0.044	0.041	0.041
90	0.049	0.06	0.06	0.059	0.057	0.061	0.062	0.062	0.059	0.056
91	0.05	0.066	0.074	0.068	0.065	0.06	0.055	0.054	0.049	0.041
92	0.046	0.063	0.069	0.07	0.071	0.071	0.07	0.067	0.067	0.064
93	0.049	0.06	0.065	0.067	0.07	0.067	0.065	0.059	0.053	0.052
94	0.049	0.056	0.057	0.054	0.051	0.053	0.05	0.04	0.043	0.038
95	0.051	0.068	0.071	0.07	0.07	0.075	0.071	0.068	0.062	0.062
96	0.045	0.057	0.06	0.056	0.05	0.037	0.035	0.03	0.023	0.023
97	0.05	0.066	0.067	0.062	0.059	0.054	0.047	0.045	0.045	0.049
98	0.051	0.068	0.074	0.07	0.065	0.056	0.053	0.048	0.044	0.035
99	0.051	0.064	0.07	0.066	0.065	0.058	0.053	0.046	0.036	0.033

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $m2_1 :=$mean$\left[\left(B2^{\langle 1\rangle}\right)^2\right]$.

$m2^T =$

	4	5	6	7	8	9
0	$4.469\,10^{-3}$	$4.184\,10^{-3}$	$3.858\,10^{-3}$	$3.509\,10^{-3}$	$3.155\,10^{-3}$	$2.878\,10^{-3}$

Its graph:



Finally get strength of SR: $SH := \max(m2)$, $SH = 4.704 \times 10^{-3}$.

For the case of $A = 0.5$: $A := A$, $A := 0.5$.

Call program and define approximate solution as matrix $B3$: $B3 := Bp(a, b, T, n, m, A)$.

$B3 =$

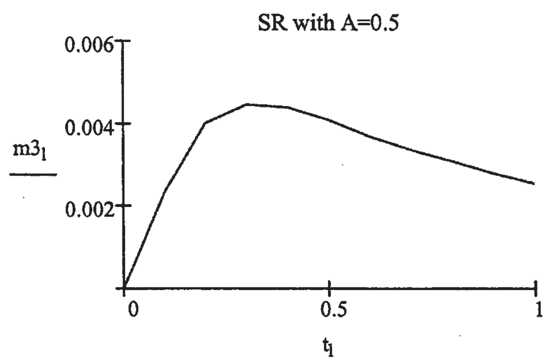
	1	2	3	4	5	6	7	8	9	10
88	0.05	0.064	0.064	0.066	0.065	0.059	0.055	0.058	0.058	0.055
89	0.049	0.061	0.06	0.063	0.062	0.057	0.051	0.048	0.045	0.041
90	0.047	0.062	0.069	0.064	0.059	0.052	0.048	0.04	0.038	0.039
91	0.05	0.061	0.06	0.054	0.049	0.043	0.038	0.036	0.031	0.034
92	0.05	0.066	0.068	0.069	0.072	0.072	0.068	0.062	0.056	0.057
93	0.048	0.058	0.056	0.054	0.051	0.05	0.052	0.047	0.044	0.041
94	0.048	0.056	0.053	0.045	0.039	0.031	0.023	0.016	0.016	0.015
95	0.05	0.066	0.066	0.067	0.064	0.062	0.054	0.05	0.048	0.045
96	0.051	0.064	0.068	0.069	0.068	0.065	0.063	0.06	0.058	0.056
97	0.051	0.067	0.069	0.069	0.071	0.069	0.064	0.062	0.063	0.058
98	0.048	0.063	0.063	0.061	0.059	0.054	0.054	0.049	0.05	0.053
99	0.046	0.064	0.069	0.071	0.072	0.066	0.055	0.045	0.041	0.039

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $m3_1 :=$mean$\left[\left(B3^{\langle 1\rangle}\right)^2\right]$.

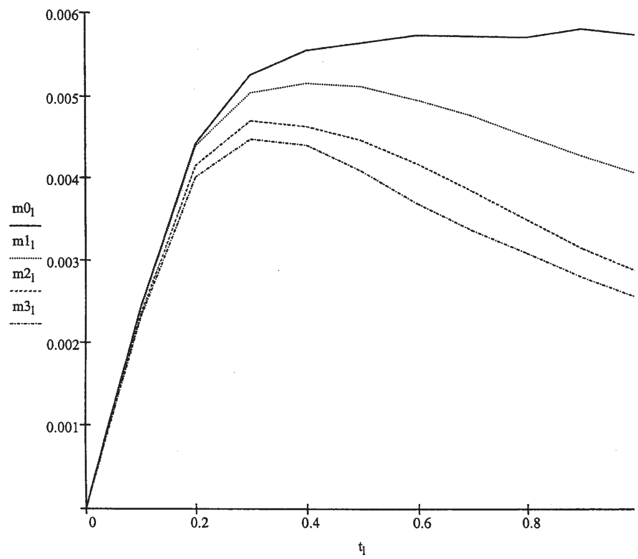
$m3^T =$

	4	5	6	7	8	9
0	$4.094\,10^{-3}$	$3.702\,10^{-3}$	$3.380\,10^{-3}$	$3.103\,10^{-3}$	$2.807\,10^{-3}$	$2.559\,10^{-3}$

Its graph:



Finally get strength of SR: $SH := \max(m3)$, $SH = 4.481 \times 10^{-3}$.
Comparison of SR in different cases of A's.



Remarks. (1) From above numerical and visual analysis we can see that: under the above selection of parameters of the system SR is suppressed by external feedback of binary pulses. If A is bigger the suppressed is bigger. Of course, this is just one example; if choose different parameters the results will be different.

(2) This approach provides us with a basis to do mathematics experiment. Since we can select different parameters of the system and get different results in short time we can test selections of parameters of the system easily and make decision by test. This is a kind of use of technology for mathematics experiment.

4. Controlled Noisy Bistable Oscillator with Feedback of Restoring Force

In this section, we shall assume the external feedback is restoring force: $F_c(x) = -gx$. Thus, the equation of oscillator becomes:

$$\frac{d^2}{dt^2}x(t) + 10\frac{d}{dt}x(t) = x^3 - x + 0.1N(t) + 0.04\sin(\omega t) - gx\,.$$

Let $\frac{d}{dt}x(t) = y$, then $\frac{d}{dt}y = x^3 - x - 10y - gx + 0.04\sin(\omega t) + 0.1N(t)$, where $k(t,x,y) = x^3 - x - 10y - gx + 0.04\sin(\omega t)$ and $s = 0.1$.

Thus, its approximate solution produced by approximate scheme of order 3 in (1.3):

$$x_{i+1} = x_i + y_i h + \frac{0.2}{3} h\Delta w_i + \frac{1}{2} k_i h^2 - \frac{4}{15} h^2 \Delta w_i \\ + \frac{1}{6}\left[0.04\omega \sin(\omega t_i) + y_i\left[3(x_i)^2 - 1 - g\right] - 10k_i\right] h^3 ,$$

$$y_{i+1} = y_i + 0.1\Delta w_i + k_i h \\ + \frac{1}{2}\left[0.04\omega \sin(\omega t_i) + y_i\left[3(x_i)^2 - 1 - g\right] - 10k_i\ \right] h^2 \\ + \left(\frac{0.016}{15}\omega\cos(\omega t_i) + \frac{8}{3}\right) h^2 \Delta w_i + \frac{1}{6}\Bigg[\left[3(x_i)^2 - 1 - g\right] k_i \\ + 0.04\left[3(x_i)^2 - 1 - g\right]\omega\cos(\omega t_i) - 10y_i\left[3(x_i)^2 - 1 - g\right] \\ + k_i\left[3(x_i)^2 - 1 - g\right]^2 - 0.04\omega^2 \sin(\omega t_i) + 6x_i(y_i)^2\Bigg] .$$

In expression above, $\{x_n\}$ is our desired solution which is the approximate stochastic process over time interval $[0, T]$.

Now, we use a PC program to perform the approximate scheme to get its approximate solution over interval $[0, T]$ with n steps and step size h. Also we assume the initial conditions: $x_0 = a = 0$ and $y_0 = b = 1$.

We proceed to our investigation with the cases: $g = 0, 0.25, 0.5, 0.75$. Their solutions are expressed by matrices $D0$, $D1$, $D2$, $D3$ with size 100×11 which are produced by programs $Rs(a, b, T, m, n, g)$:

```
Rs(a,b,T,n,m,g) := | x_0 ← a
                   | y_0 ← b
                   | h ← T/n
                   | t_0 ← 0
                   | ω ← 10·π·2^(−9)
                   | for i ∈ 0..n − 1
                   |   | t_(i+1) ← t_i + h
                   |   | x_(i+1) ← x_i + y_i·h + (0.2/3)·h·rnorm(m,0,1)·√h − (4/15)·h^2·√h·rnorm(m,0,1) ...
                   |   |      + (1/2)·[(x_i)^3 − x_i − 10·y_i − g·x_i + 0.04·sin(ω·t_i)]·h^2 ...
                   |   |      + (1/6)·[ 0.04·ω·cos(ω·t_i) + y_i·[3·(x_i)^2 − 1 − g] ...
                   |   |               + (−10)·[(x_i)^3 − x_i − 10·y_i − g·x_i + 0.04·sin(ω·t_i)] ]·h^3
                   |   | y_(i+1) ← y_i + rnorm(m,0,1)·√h / 10 + [(x_i)^3 − x_i − 10·y_i − g·x_i + 0.04·sin(ω·t_i)]·h .
                   |   |      + (1/2)·[ 0.04·ω·cos(ω·t_i) + y_i·[3·(x_i)^2 − 1 − g] ...
                   |   |               + (−10)·[(x_i)^3 − x_i − 10·y_i − g·x_i + 0.04·sin(ω·t_i)] ]·h^2 ...
                   |   |      + ((0.016/15)·ω·cos(ω·t_i) + 8/3)·h^2·√h·rnorm(m,0,1) ...
                   |   |      + (1/6)·[ [3·(x_i)^2 − 1 − g]·[ [(x_i)^3 − x_i − 20·y_i − g·x_i ...
                   |   |                                       + 0.04·sin(ω·t_i)] ...
                   |   |                                     + 0.04·ω·cos(ω·t_i) ] ...
                   |   |               + y_i·[3·(x_i)^2 − 1 − g]^2 − 0.04·ω^2·sin(ω·t_i) + 6·x_i·(y_i)^2 ]·h^3
                   | x ← augment(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_10)
                   | x
```

Define parameters: $n := 10$, $m := 100$, $T := 1$, $h := \frac{T}{n}$.
Define initial condition: $j := 1..m-1$, $a_{j,0} := 0$, $b_{j,0} := 0$.
For the case of $g = 0$ (without external feedback): $g := g$, $g := 0$.
Call program and define approximate solution as matrix $D0$: $D0 := Rs(a, b, T, n, m, g)$.

$D0 =$

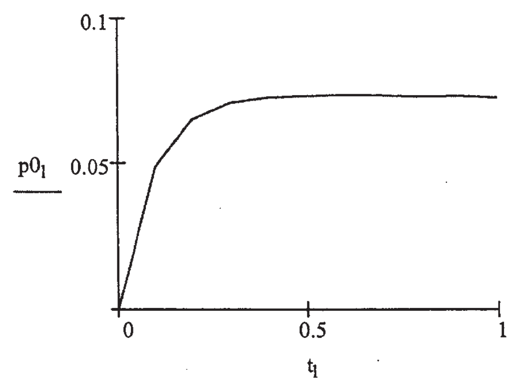
	1	2	3	4	5	6	7	8	9	10
88	0.042	0.063	0.065	0.074	0.076	0.083	0.083	0.092	0.093	0.094
89	0.052	0.069	0.073	0.073	0.072	0.07	0.073	0.072	0.075	0.074
90	0.051	0.068	0.076	0.078	0.08	0.082	0.082	0.082	0.083	0.082
91	0.051	0.066	0.071	0.074	0.078	0.077	0.081	0.085	0.094	0.096
92	0.049	0.069	0.078	0.075	0.078	0.081	0.083	0.082	0.084	0.085
93	0.048	0.064	0.077	0.087	0.088	0.088	0.087	0.086	0.082	0.082
94	0.047	0.064	0.065	0.071	0.072	0.07	0.067	0.068	0.068	0.067
95	0.05	0.067	0.075	0.078	0.079	0.084	0.092	0.09	0.087	0.085
96	0.05	0.064	0.068	0.066	0.062	0.065	0.068	0.069	0.08	0.087
97	0.052	0.071	0.083	0.086	0.092	0.091	0.091	0.091	0.09	0.088
98	0.047	0.063	0.071	0.071	0.067	0.07	0.069	0.07	0.074	0.07
99	0.052	0.069	0.072	0.069	0.063	0.057	0.054	0.047	0.048	0.044

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $p0_1$:=mean$\left(D0\right)^{\langle 1\rangle}\right)$.
To find mean square:

$p0^T =$

	1	2	3	4	5	6	7	8	9	10
0	0.049	0.066	0.071	0.073	0.074	0.074	0.074	0.073	0.074	0.073

Graph of mean square:



To find strength of SR: $SHR0 := \max(p0)$, $SHR0 = 0.074$.
For the case of $g = 0$ (without external feedback): $g := g$, $g := 0.25$.
Call program and define approximate solution as matrix $D1$: $D1 := Rs(a, b, T, n, m, g)$.

$D1 =$

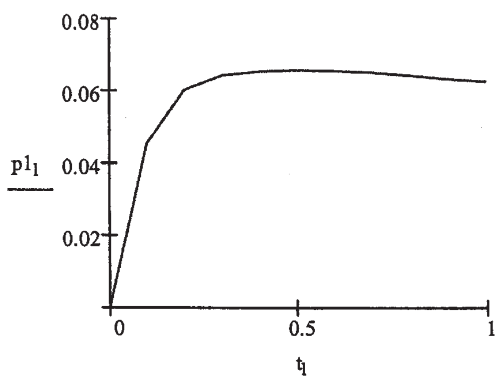
	1	2	3	4	5	6	7	8	9	10
88	0.048	0.063	0.067	0.072	0.071	0.068	0.07	0.066	0.066	0.067
89	0.043	0.061	0.062	0.059	0.06	0.064	0.066	0.066	0.074	0.08
90	0.046	0.062	0.068	0.073	0.08	0.085	0.083	0.082	0.088	0.088
91	0.046	0.066	0.073	0.074	0.078	0.086	0.086	0.087	0.092	0.091
92	0.046	0.063	0.068	0.068	0.065	0.063	0.066	0.069	0.071	0.07
93	0.049	0.067	0.073	0.07	0.07	0.059	0.058	0.058	0.058	0.06
94	0.047	0.063	0.069	0.076	0.077	0.07	0.07	0.07	0.065	0.061
95	0.049	0.063	0.069	0.063	0.064	0.061	0.058	0.052	0.051	0.05
96	0.045	0.06	0.066	0.069	0.073	0.074	0.073	0.075	0.073	0.075
97	0.048	0.068	0.075	0.081	0.088	0.095	0.1	0.095	0.092	0.09
98	0.049	0.066	0.069	0.072	0.068	0.07	0.07	0.07	0.076	0.076
99	0.046	0.063	0.066	0.067	0.069	0.066	0.068	0.074	0.076	0.078

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $p1_1 :=\text{mean}\left(D1)^{\langle 1\rangle}\right)$. To find mean square:

$p1^T =$

	1	2	3	4	5	6	7	8	9	10
0	0	0.045	0.060	0.064	0.065	0.066	0.065	0.065	0.064	0.063

Graph of mean square:



To find strength of SR: $SHR1 := \max(p1)$, $SHR1 = 0.066$.

For the case of $g = 0$ (without external feedback): $g := g$, $g := 0.5$.

Call program and define approximate solution as matrix $D2$: $D2 := Rs(a, b, T, n, m, g)$.

$D2 =$

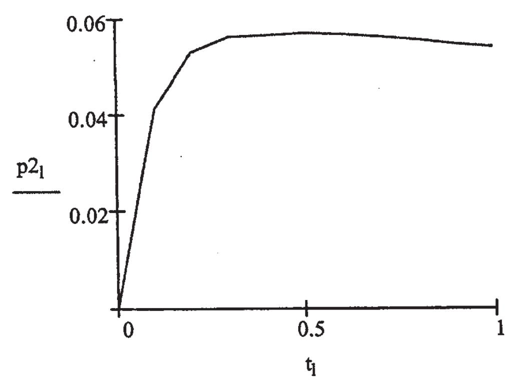
	1	2	3	4	5	6	7	8	9	10
88	0.043	0.052	0.061	0.065	0.068	0.067	0.069	0.069	0.069	0.072
89	0.044	0.058	0.058	0.058	0.056	0.053	0.047	0.042	0.043	0.044
90	0.042	0.049	0.049	0.049	0.049	0.055	0.058	0.06	0.06	0.06
91	0.043	0.056	0.063	0.058	0.057	0.054	0.051	0.052	0.05	0.045
92	0.04	0.053	0.058	0.06	0.063	0.065	0.066	0.066	0.068	0.068
93	0.042	0.049	0.054	0.057	0.061	0.061	0.062	0.057	0.054	0.056
94	0.042	0.045	0.046	0.044	0.043	0.047	0.047	0.039	0.045	0.041
95	0.044	0.058	0.06	0.06	0.062	0.069	0.068	0.066	0.063	0.065
96	0.038	0.046	0.049	0.046	0.042	0.032	0.032	0.029	0.025	0.027
97	0.043	0.056	0.056	0.053	0.051	0.048	0.044	0.044	0.046	0.052
98	0.044	0.057	0.063	0.061	0.056	0.05	0.05	0.047	0.045	0.039
99	0.044	0.054	0.059	0.057	0.057	0.052	0.049	0.044	0.037	0.037

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $p2_1 :=\text{mean}\big(D2)^{\langle 1\rangle}\big)$. To find mean square:

$p2^T =$

	1	2	3	4	5	6	7	8	9	10
0	0.041	0.053	0.056	0.057	0.057	0.057	0.056	0.056	0.055	0.054

Graph of mean square:



To find strength of SR: $SHR2 := \max(p2)$, $SHR1 = 0.057$.

For the case of $g = 0$ (without external feedback): $g := g$, $g := 0.75$.

Call program and define approximate solution as matrix $D3$: $D3 := Rs(a, b, T, n, m, g)$.

$D3 =$

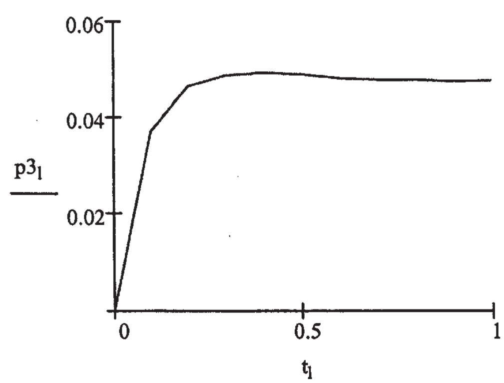
	1	2	3	4	5	6	7	8	9	10
88	0.039	0.048	0.048	0.05	0.051	0.048	0.048	0.054	0.056	0.056
89	0.039	0.045	0.043	0.048	0.049	0.047	0.043	0.044	0.043	0.043
90	0.037	0.046	0.053	0.048	0.046	0.042	0.04	0.036	0.037	0.041
91	0.04	0.045	0.043	0.039	0.036	0.033	0.031	0.032	0.03	0.036
92	0.039	0.05	0.051	0.054	0.059	0.061	0.06	0.057	0.054	0.058
93	0.037	0.042	0.039	0.039	0.038	0.04	0.045	0.043	0.042	0.043
94	0.037	0.04	0.037	0.03	0.026	0.021	0.016	0.013	0.015	0.017
95	0.039	0.05	0.049	0.051	0.051	0.051	0.046	0.046	0.046	0.046
96	0.04	0.048	0.051	0.053	0.055	0.054	0.055	0.056	0.056	0.057
97	0.04	0.051	0.052	0.054	0.058	0.059	0.057	0.057	0.061	0.059
98	0.038	0.047	0.046	0.046	0.046	0.044	0.047	0.045	0.049	0.054
99	0.036	0.048	0.052	0.056	0.059	0.055	0.047	0.041	0.039	0.04

Now, we can use it to find the strength of SR. First, find mean square of responce by routine of *Mathcad* as follows: $l := 0..n$, $t_1 := 1h$, $p3_1 := \text{mean}\big(D3)^{\langle 1 \rangle}\big)$.
To find mean square:

$p3^T =$

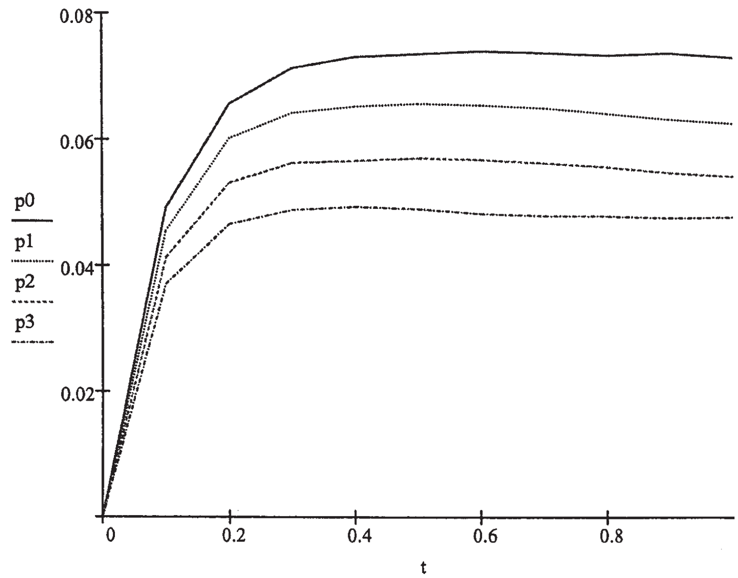
	1	2	3	4	5	6	7	8	9	10
0	0.037	0.046	0.049	0.049	0.049	0.048	0.048	0.048	0.048	0.048

Graph of mean square:



To find strength of SR: $SHR3 := \max(p3)$, $SHR1 = 0.049$.

Comparison of SR of 4 cases of g:



Remark. From above numerical and visual analysis we can see: under above selection of parameters of the system the external feedback suppreesses SR. Of course, this is one example, if select the different parameters of the system we may get different results.

5. Conclusion

(1) Adding of external feedback is the invasive control of SR system and it is different from the interna and external control. Beside the binary pulses and negative proportional there are other types of feedback, such as, binary hysteresis, binary windowed pulses, etc.

(2) SR can be controled by external feedback, either to suppress or enhance by appropriate selection of the parameters of the system and and external feedback.

(3) Using approximate schemes for stochastic oscillators is another approach to study SR of noninvasive control. This approach is easy to perform by computer technology and can be used as initial and supplementary study of other approaches.

(4) This approach is a way to do mathematics experiment. Since this approach is based on the computational and graphing ability of computer technology it can get results of massive selections of parameters of the system in

short time so that you can select desired one.

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