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STABILITY OF MULTI-DELAY SYSTEMS OF DIFFERENTIAL EQUATIONS: CHARACTERISTIC EQUATIONS AND APPLICATIONS

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Abstract: One approach to stability analysis depends on identifying the signs of the real parts of roots of a characteristic function. Where the characteristic function for an ordinary differential equation typically is a polynomial with real coefficients, the characteristic function for a delay differential equation normally includes exponential terms that involve the delay quantities. These functions, therefore, are called "exponential polynomials" [3], or "transcendental characteristic functions" [5], [14]. Stability analysis of delay models has, for a main body of work, been on a case by case basis, using a technique for treating exponential polynomials that goes back to [8]. What is lacking, however, are results that would both (a) apply to the general multi-delay case; and (b) be useful for the clinical scientist. Toward the possibility of a practical general theory, we use a linear algebraic framework. In this context, the traditional technique

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[8] is related to a factorization of the general delay equation that is point-wise linear and offers insight into the general structure of the zero set. Necessary and sufficient conditions for roots are obtained that allow for a unified approach to multi-delay equations. Certain classical formulas for the one-delay equation are extended to the multi-delay equation. Results are illustrated with examples and applications from the literature. A general result on Hopf bifurcation in the multi-delay system is given. The paper concludes with an indication of further related lines of enquiry that emerge from the context.

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1. Introduction

One approach to stability analysis depends on identifying the signs of the real parts of roots of a characteristic function. The characteristic function for an ordinary differential equation typically is a polynomial with real coefficients. In a similar way, the characteristic function for a delay differential equation that depends on a finite number of delay terms of the form $t - \tilde{\tau}$ normally is of the form $\Delta_\tau(\lambda) = P(\lambda) + Q_1(\lambda)e^{-\lambda\tau_1} + \ldots + Q_k(\lambda)e^{-\lambda\tau_k}$, where the $Q_i(\lambda)$ are polynomials in λ with real coefficients. These characteristic functions are called "exponential polynomials" [3], or "transcendental characteristic functions" [5], [14]. Stability analysis of delay models and their transcendental characteristic functions has, for a main body of work, been on a case by case basis, using a technique for treating exponential polynomials that goes back to Hale [8]. The technique has enjoyed widespread application to modeling processes that involve delay quantities (see [11], [5], [14] for bibliographies) and recently has been useful in the analysis of delay models for HIV infection of CD4+ - T Cells [5], [6], [12], bacteriophage infection [4], and numerous other processes in biology and engineering.

As it stands, however, Hale's technique mainly is useful when there is only one delay (see, however, [14], [2], [7] for examples of how the technique can sometimes be adapted to special cases of two-delay and three-delay models; and [12] for applications and preliminary results that help lead toward the general case treated in the present paper). In view of the increasing use of delay models in biology and engineering, what would be helpful, therefore, would be results for the general multi-delay equation, in terms useful to the laboratory scientist.

A transcendental characteristic function is analytic. It is not, however, a polynomial in λ. It follows that, for stability analysis, the well known Routh-Hurwitz criteria for determining the signs of the real parts of roots do not apply. This situation is discussed in [2, p. 33], where it is pointed out that, while there are certain general tests for analyzing the roots of a transcendental equation ([15], for example), applying those tests can be non-trivial. Indeed, the classically formulated approach is based on Cauchy's Theorem and the resulting index theorem. One would therefore look to a suitable contour integral. In general, however, in order to evaluate the contour integral for a particular characteristic function, one would require detailed information on the values of the function along the contour – information which, frequently, is as unknown as information on root locations.

In [10], results on stability analysis of multi-delay equations are given in terms of special norms of the Jacobian matrices of the linearized equation; as well as certain contour integrals. The practical difficulty of appealing to contour integrals is as just mentioned; and evaluation of the matrix norms also requires more data on the polynomials and matrix coefficients than is easily available from the characteristic function. In fact, in many cases, being able to calculate these norms would require more data than merely having the roots of the characteristic equations.

A main purpose of the present paper, therefore, is to make some progress toward a general and practical theory of multi-delay equations. Results are given directly in terms of the coefficients of the transcendental characteristic function, and so are intended to be useful in a laboratory situation. We avoid the impasse involved when invoking the classical index theory, by exploiting algebraic structures specific to the framework of exponential polynomials. In particular, we advert to and generalize a point-wise matrix factorization of the transcendental characteristic equation that is implicit in Hale's technique. Among other things, consideration of domains of definition provides useful necessary and sufficient conditions for roots in terms of that factorization.

The paper outline is as follows: In Section 2, we restrict to equations that have one real delay. In Section 3, we generalize the results of Section 2 to the case of k real delays $\tau = (\tau_1, \ldots, \tau_k)$. Recall that in the one-delay transcendental characteristic function $\Delta(\lambda) = P(\lambda) + Q(\lambda)e^{-\lambda\tau}$, for purely imaginary roots there is the well-known necessary condition $|P|^2 - |Q|^2 = 0$. We obtain a natural generalization of this to the k -delay equation that pertains to arbitrary roots (imaginary or otherwise). For the purposes of illustration, throughout the paper we apply our results to certain classical results; to known contemporary results; and to certain introduced equations. In Section 4, we obtain a general theorem on Hopf bifurcation in the k-delay equation. In Section 5, we conclude by indicating further possible lines of enquiry that follow from the present context.

2. Characteristic Equations with One Delay

Taking real and complex parts, we find that $\lambda = \rho + i\omega$ is a root of the transcendental characteristic equation $\Delta(\lambda) = P(\lambda) + e^{-\lambda\tau}Q(\lambda) = 0$ if and only if

$$\begin{bmatrix} \operatorname{Re} P \\ \operatorname{Im} P \end{bmatrix} = e^{-\lambda\tau} \begin{bmatrix} -\operatorname{Re} Q & -\operatorname{Im} Q \\ -\operatorname{Im} Q & \operatorname{Re} Q \end{bmatrix} \begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix} = M \begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix}. \tag{2.1}$$

Note that the operator M has the special form

$$M = \begin{bmatrix} E & F \\ F & -E \end{bmatrix}, \tag{2.2}$$

where in the present case, $E = -e^{-\rho\tau} \operatorname{Re} Q$ and $F = -e^{-\rho\tau} \operatorname{Im} Q$.

Using polar decomposition, we obtain

$$M = (T^*T)^{1/2}\, V = \begin{pmatrix} \sqrt{E^2+F^2} & 0 \\ 0 & -\sqrt{E^2+F^2} \end{pmatrix} V, \tag{2.3}$$

where V is a (real) orthogonal matrix.

Suppose that $\det M \neq 0$, that is,

$$\det M = -\left[E^2 + F^2\right] = -e^{-2\rho\tau}\left[(\operatorname{Re} Q)^2 + (\operatorname{Im} Q)^2\right] \neq 0. \tag{2.4}$$

Note that if $\det M = 0$, then $M = 0$. That is, at the particular (ρ, ω) where the determinant is evaluated, the equation reduces to where there is no delay component. In the case of two or more delay terms, however, the rank of the matrix M need not be full, and it is this degree of freedom that plays into the structure of the solution set. This will be elaborated on in Section 3 below.

For the moment, then, we continue with the hypothesis that $\det M \neq 0$, and we formally solve for the right-hand-side of equation (2.1):

$$\begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix} = M^{-1} \begin{bmatrix} \operatorname{Re} P \\ \operatorname{Im} P \end{bmatrix} = \left(\frac{-1}{E^2+F^2}\right) \begin{bmatrix} -E & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \operatorname{Re} P \\ \operatorname{Im} P \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{2.5}$$

Using a suitable branch of one of the inverse trigonometric functions – $\arccos(\theta)$, $\arcsin(\theta)$, or $\arctan(\theta)$ – an implicit relation $W(\rho, \omega, \tau) = 0$ can be obtained, correlating the variables ρ, ω and τ. If, for example, $X \neq 0$ and $\omega \neq 0$, then

$$\tau = \frac{1}{\omega} \arctan\left(\frac{Y}{X}\right). \tag{2.6}$$

Since in general X and Y can depend on τ, the relation (2.6) is not necessarily a function.

Note that with the identification $X = \cos(\omega\tau)$, $Y = \sin(\omega\tau)$, this formal solution given by (2.5) is not always defined. For, when the matrix M is invertible, the solution $[X, Y]^T$ to equation (2.5) is unique; and so if $X^2 + Y^2 \neq 1$, identifying the coordinates via the cosine and sine functions is not tenable. On the other hand, if both $X^2 + Y^2 = 1$ and the argument for the pair $[X, Y]^T$ is equivalent to $\omega\tau$ (mod2π), then $X = \cos(\omega\tau)$, $Y = \sin(\omega\tau)$ provides a solution.

For the general problem of obtaining the argument determined by a given vector, it is necessary to select branches of the multi-valued argument function. For the present situation, however, it is necessary only to determine whether or not two vectors are parallel. In this special case, therefore, we may appeal to the dot product to obtain explicit criteria for the coordinates.

These observations lead to the following proposition.

Proposition 2.1. *Consider the system of equations*

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \operatorname{Re} P \\ \operatorname{Im} P \end{bmatrix} = e^{-\rho\tau} \begin{bmatrix} -\operatorname{Re} Q & -\operatorname{Im} Q \\ -\operatorname{Im} Q & \operatorname{Re} Q \end{bmatrix} \begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix}. \tag{2.7}$$

Let M be defined by (2.2) and suppose that $\det M \neq 0$. Then, there is a solution to equation (2.7) if and only if there exists a triple (ρ, ω, τ) such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = M^{-1} \begin{bmatrix} U \\ V \end{bmatrix} \tag{2.8}$$

is a unit vector; and

$$\begin{bmatrix} X \\ Y \end{bmatrix} \cdot \begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix} = \left\| \begin{bmatrix} X \\ Y \end{bmatrix} \right\| = 1 > 0. \tag{2.9}$$

Proof. It only remains to comment on the second condition. But this is equivalent to the cosine of the angle between the two vectors being unity. In other words, the second condition is that the argument for $[X, Y]^T$ be equivalent to $\omega\tau$ mod 2π. ∎

Remark 2.2. Since M is angle preserving, $M^{-1}\begin{bmatrix} U \\ V \end{bmatrix}$ is a unit vector if and only if

$$\left\| \begin{bmatrix} U \\ V \end{bmatrix} \right\| = \|M\| = \sqrt{E^2 + F^2} = e^{-\rho\tau}\sqrt{(\operatorname{Re} Q)^2 + (\operatorname{Im} Q)^2}.$$

Recall that $[U, V]^T = [\operatorname{Re} P, \operatorname{Im} P]^T$.

So, more concisely, we obtain $|P| = e^{-\rho\tau}|Q|$, which for the non-trivial characteristic equation is equivalent to $F_\tau(\rho, \omega) = |P|^2 - |e^{-\rho\tau}Q|^2 = 0$. For the special case $\rho = 0$, this gives the well known necessary condition for purely imaginary roots, namely, $F(\omega) = |P|^2 - |Q|^2 = 0$ (see, e.g. [11], Theorem 4.1, p. 83).

Corollary 2.3. *Suppose that $[X, Y]^T$ is a unit vector that solves (2.7). If $\omega \neq 0, X \neq 0$ and the ratio $\frac{Y}{X}$ is independent of τ, then a family of critical delays is completely determined by (2.6).*

Proof. Let $\tau = \tau_c$ be the smallest positive solution for (2.6): That is, τ_c is the smallest positive number with $\omega_c\tau$ satisfying $\tan \omega_c\tau = \frac{Y}{X}$. The family of solutions is obtained from $\tau = \tau_c + \frac{k\pi}{\omega}$, where k is any non-negative integer. □

Example 2.4. (see [3], Lemma 3.2, p. 64) This classical result states that all roots of the equation

$$a\lambda + b + ce^{-\lambda\tau} = 0 \tag{2.10}$$

lie to the left of some vertical line in the complex plane.

We use our approach to re-derive this result.

Proof. If $c = 0$, then the characteristic equation is trivial with a unique solution. Suppose therefore that $c \neq 0$. In this case, equation (2.1) is

$$\begin{bmatrix} a\rho + b \\ a\omega \end{bmatrix} = \begin{bmatrix} -ce^{-\rho\tau} & 0 \\ 0 & ce^{-\rho\tau} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix}. \tag{2.11}$$

Solving for X, Y, using $X^2 + Y^2 = 1$ and completing the square in ρ, it follows that

$$\left(\rho + \frac{1}{a}\right)^2 + \omega^2 = \frac{c^2}{a^2e^{2\rho\tau}} + \frac{1 - b^2}{a^2}. \tag{2.12}$$

Isolating ω^2, we get

$$\omega^2 = \frac{c^2}{a^2e^{2\rho\tau}} + \frac{1 - b^2}{a^2} - \left(\rho + \frac{1}{a}\right)^2. \tag{2.13}$$

For large $\rho > 0$ the right hand side becomes negative. Hence, for ρ sufficiently large there can be no solution, as claimed.

More can be said. In order for a solution to exist, the right-hand-side of equation (2.12) must be non-negative. If $1 - b^2 \geq 0$, this is true for all ρ and τ. If $1 - b^2 < 0$, however, an additional constraint is required, for then it is necessary also that $\rho \leq \frac{1}{2\tau} \ln \left(\frac{c^2}{b^2 - 1} \right)$. In this case the quadratic equation gives two candidates for ω. We have not yet invoked the conditions imposed on the angles. Depending on the values of the various constants involved, for each ρ in the domain of (2.13), there will be two solutions, one solution, or no solution to equation to (2.10).

Note also that for any candidate solution $[X, Y]^T$ with $X \neq 0$, equation (2.11) shows that $\frac{Y}{X}$ is independent of the delay term τ. It follows from Corollary 2.3 that when $\omega \neq 0$, possible critical delays are obtained from a multi-valued function in ω. □

Example 2.5. In this example we recall another classical result (Hayes equation [9]) that appears as Theorem 13.8 of [3]. In [3], the theorem reads as follows: *All the roots of* $pe^{\lambda} + q - \lambda e^{\lambda} = 0$, *where* p *and* q *are real, have negative real parts if and only if:*

(a) $p < 1$, *and*

(b) $p < -q < \sqrt{\omega_1^2 + p^2}$,

where ω_1 *is a root of* $\omega = p \tan \omega$, *with* $0 < \omega < \pi$. *If* $p = 0$, *we take* $\omega_1 = \frac{\pi}{2}$.

We give an analysis of the Hayes equation based on the approach of the present paper.

By taking real and imaginary parts, the matrix formulation of the equation is:

$$\begin{bmatrix} q \\ 0 \end{bmatrix} = \begin{bmatrix} e^{\rho}(-p + \rho) & -e^{\rho}\omega \\ e^{\rho}\omega & e^{\rho}(-p + \rho) \end{bmatrix} \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}. \tag{2.14}$$

This becomes

$$\begin{gathered} \begin{bmatrix} q \\ 0 \end{bmatrix} = \begin{bmatrix} E & F \\ -F & E \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \quad X^2 + Y^2 = 1, \\ \operatorname{Arg} \begin{bmatrix} X \\ Y \end{bmatrix} \equiv \omega \bmod 2\pi, \end{gathered} \tag{2.15}$$

where $E = e^{\rho}(-p + \rho)$, $F = -e^{\rho}\omega$.

Using the inverse matrix to solve for $[X, Y]^T$, we obtain

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{E^2 + F^2} \begin{bmatrix} qE \\ qF \end{bmatrix}. \tag{2.16}$$

Imposing the norm criterion on $[X, Y]^T$, we obtain

$$q^2 = E^2 + F^2 . \tag{2.17}$$

We therefore get the following result: A complex number $\lambda = \rho + i\omega$ is a root of the Hayes equation if and only if

(a1) $e^{-2\rho}q^2 = (-p+\rho)^2 + \omega^2$, and

(b1) $\begin{bmatrix} e^{\rho}q\,(-p+\rho) \\ -e^{\rho}\omega \end{bmatrix} \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} = 1.$

As a corollary, we obtain: *The Hayes equation has a purely imaginary root* ($\rho = 0$) *if and only if:*

(a2) $q^2 = p^2 + \omega^2$, *and*

(b2) $\begin{bmatrix} -qp \\ -q\omega \end{bmatrix} \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} = 1.$

Note that for this situation there are two candidates for ω determined by the quadratic equation $\omega^2 = q^2 - p^2$. It follows that there are no purely imaginary roots if and only if $q^2 < p^2$ or the dot product criterion fails.

Regarding the issue of roots with negative real parts, we may appeal to equation (a1) above. For fixed ω, the left quantity is exponential in ρ, while the right quantity is quadratic in ρ. If, at $\rho = 0$, the exponential intercept falls below the quadratic intercept, then, modulo the dot product criterion, there will be a root candidate with $\rho < 0$.

Further analysis is possible, but our present purpose is only to illustrate the method.

Example 2.6. (see [13]) Investigating the possibility of purely imaginary roots $\lambda = i\omega$, the real and imaginary parts of the characteristic equation (29) in [13] are

$$\begin{aligned} -A\omega^2 - \delta c\omega\sin(\omega\tau) + \delta c\rho - \delta c(\rho - \psi')\cos(\omega\tau) = 0\,, \\ -\omega^3 + B\omega - bc\omega\cos(\omega\tau) + \delta c(\rho - \psi')\sin(\omega\tau) = 0 \end{aligned} \tag{2.18}$$

(in [13], a complex root is written $\lambda = \mu + iv$; and ρ is a constant in their equation. In order to connect notation with the present paper, we have used ω in place of ν for the imaginary part of the possible root. We have, however, followed their use of ρ as a constant of the equation).

As in (2.1), we express this as a matrix system to obtain

$$\begin{bmatrix} -A\omega^2 + \delta c\rho \\ -\omega^3 + B\omega \end{bmatrix} = \begin{bmatrix} -\delta c(\rho - \psi') & -\delta c\omega \\ -\delta c\omega & \delta c(\rho - \psi') \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \tag{2.19}$$

with

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix} . \tag{2.20}$$

Using the notation of (2.2),

$$E = -\delta c(\rho - \psi') \,, \; F = -\delta c\omega, \; U = -A\omega^2 + \delta c\rho, \; V = -\omega^3 + B\omega \,,$$

and

$$\det M(\omega) = -(E^2 + F^2) = -(\delta c)^2 \left[(\rho - \psi')^2 + \omega^2\right] \neq 0 \,.$$

We assume that the product $\delta c\,(\rho - \psi') \neq 0$, so that the matrix M is non-singular for all ω (from (2.3), if M is singular then, because of its special structure, it must be the zero matrix).

Suppose that for the unique solution $[X(\omega), Y(\omega)]^T$ to equation (2.19) we have $X \neq 0$. Then, since the coefficients of the equation do not depend on the delay τ, we may use (2.6). In other words, we obtain τ as a multi-function of ω.

Recall also that this is conditional, subject to the constraints that $[X, Y]^T$ be a unit vector and that its argument be equivalent to $\omega\tau$ mod 2π (see (2.9)).

Altogether, this gives four equations for two unknowns (ω, τ). One may then investigate that (when the constants satisfy the Routh-Hurwitz criteria as given in [13]) this particular system is over-determined with no solution. See also Remark 3.2 below.

The following Proposition 2.7 and its Corollary 2.8 constitute a refinement of Theorem 1.1 from [11], p. 64. The roots of (2.21) (below) depend not only on the relative magnitude but also on the relative sign of the leading coefficients a, b. When suitably indexed, we also obtain stability parameters (see Remark 2.9). The methods used are traditional, and do not depend on Proposition 2.1 as such. The result, however, does bear on the general topic of this section of the paper, namely, zeros of single-delay characteristic functions. In addition, while the result is special, the basis of the proof helps reveal the need for new methods. That is, the traditional approach of factoring and then using Rouché's Theorem is not adequate for a sum of numerous delay terms as in the general multi-delay equation. The last results of this section therefore also implicitly point to the need for new methods, and it is to that topic that we direct to Section 3.

Proposition 2.7. *Consider a characteristic equation of the form*

$$a + be^{-\lambda\tau} = 0 \,, \tag{2.21}$$

where a and b are real constants with $ab \neq 0$, and $\tau > 0$.

(i) If $ab > 0$, then the roots $\lambda = \rho + i\omega$ are given by

$$\rho = \frac{1}{\tau} \ln \left| \frac{b}{a} \right| , \quad \omega = \frac{1}{\tau} (2k+1)\, \pi \, . \tag{2.22}$$

(ii) If $ab < 0$, then the roots are given by

$$\rho = \frac{1}{\tau} \ln \left| \frac{b}{a} \right| , \quad \omega = \frac{1}{\tau} (2k\pi) \, . \tag{2.23}$$

Proof. Dividing (3.7) by b, we obtain

$$1 + \frac{b}{a} e^{-\rho\tau} \left[\cos(\omega\tau) - i \sin(\omega\tau) \right] = 0 \, .$$

For the imaginary part to be zero we obtain $\omega\tau = (2k+1)\,\pi$ or $\omega\tau = 2k\pi$.

(i) Suppose that $ab > 0$. If $\omega\tau = (2k+1)\,\pi$, then $\cos(\omega\tau) = -1$. In that case ρ is given by (2.22). If $\omega\tau = 2k\pi$, then $\cos(\omega\tau) = 1$ and there is no solution.

(ii) The argument is similar for $ab < 0$. □

Corollary 2.8. *The ratio $\frac{b}{a} \neq 0$ provides a stability parameter. That is:*

(i) If $\left|\frac{b}{a}\right| < 1$, then all roots of (2.21) have negative real part, for all τ;

(ii) If $\left|\frac{b}{a}\right| > 1$, then all roots of (2.21) have strictly positive real part, for all τ;

(iii) If $\left|\frac{b}{a}\right| = 1$, the all roots of (2.21) are purely imaginary, for all τ.

Proof. Substitute these cases into equations (2.22) and (2.23), and the result follows. □

Remark 2.9. Note that τ can be taken as a parameterization of equations. Suppose that σ is some additional parameter from some topological space. If the real valued functions sign $\left(\frac{b_{(\sigma,\tau)}}{a_{(\sigma,\tau)}}\right)$, sign $\left(1 - \left|\frac{b_{(\sigma,\tau)}}{a_{(\sigma,\tau)}}\right|\right)$ are continuous, then, for non-zero roots, we obtain continuous set maps in the complex plane that for all (σ, τ) have real parts strictly positive; or strictly negative. In applications, this typically is a local property in the parameters (σ, τ).

Much as in [11] (Theorem 1.1, p. 64), Proposition 2.7 may be extended to a result for characteristic equations of the form $\Delta(\lambda) = P(\lambda) + Q(\lambda)e^{-\lambda\tau}$, where the polynomials $P(\lambda), Q(\lambda)$ are of arbitrary, but equal, degree.

Corollary 2.10. *Consider the characteristic equation*

$$\Delta(\lambda) = P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \, ,$$

where, as in Section 1, $P(\lambda), Q(\lambda)$ *are polynomials in* λ *with real coefficients. Suppose that* $\deg P(\lambda) = \deg Q(\lambda) = n \geq 0$*, and that the leading coefficients of* P, Q *are* a_n, b_n *respectively.*

(i) If $\left|\frac{b_n}{a_n}\right| < 1$*, then there are infinitely many roots of arbitrarily large norm, asymptotic (in both the positive and negative imaginary directions) to a vertical line with negative real part;*

(ii) If $\left|\frac{b_n}{a_n}\right| > 1$*, then there are infinitely many roots of arbitrarily large norm, asymptotic (in both the positive and negative imaginary directions) to a vertical line with positive real part; and*

(iii) If $\left|\frac{b_n}{a_n}\right| = 1$*, then there are infinitely many roots of arbitrarily large norm, asymptotic (in both the positive and negative imaginary directions) to the imaginary axis.*

Proof. As in Theorem 1.1 of [11], the proof is based on Rouché's Theorem. A brief description is as follows:

First, for convenience, we recall a version of Rouché's Theorem: *Suppose that* $f(\lambda)$ *and* $g(\lambda)$ *are analytic on a region* Ω *and satisfy* $|f(\lambda) - g(\lambda)| < |f(\lambda)|$ *on a circle* Γ*. Then* $f(\lambda)$ *and* $g(\lambda)$ *have the same number of zeros enclosed by* Γ (in the classical texts, Rouché's Theorem often is given as a direct consequence of the Argument Principle/Index Theorem, see e.g. [1], Section 5.2).

Now, the function $\Delta(\lambda)$ can be written as

$$\lambda^n \left(a_n + b_n e^{-\lambda\tau}\right) + R(\lambda, \tau),$$

where

$$R(\lambda, \tau) = A(\lambda) + B(\lambda)e^{-\lambda\tau}$$

and $A(\lambda), B(\lambda)$ are polynomials in λ of degree at most $n-1$. Therefore $\Delta(\lambda) = 0$ if and only if $g(\lambda, \tau) = a + be^{-\lambda\tau} + \frac{1}{\lambda^n}R(\lambda, \tau) = 0$. Evidently, for λ of large modulus, $\frac{1}{\lambda^n}R(\lambda, \tau)$ converges uniformly to zero, and so $f(\lambda) = a_n + b_n e^{-\lambda\tau}$ and $g(\lambda) = a_n + b_n e^{-\lambda\tau} + \frac{1}{\lambda^n}R(\lambda, \tau)$ are mutually asymptotic. It is, therefore, to this pair of functions that Rouché's Theorem will be applied.

A first step is to enclose each of the zeros λ_k of $f(\lambda) = a_n + b_n e^{-\lambda\tau}$. Use Proposition 2.7 to obtain radius $r > 0$ sufficiently small so that $f(\lambda) \neq 0$ on the circle $\Gamma_k(r)$ of radius r centered at the isolated λ_k. Let $\delta > 0$ be the maximum of $f(\lambda)$ restricted to $\Gamma_k(r)$. On this circle, the difference $|f(\lambda) - g(\lambda)| = \left|\frac{1}{\lambda^n}R\right|$, which, as mentioned above, converges uniformly to zero for λ of large modulus (note that the fact that this convergence is uniform is sufficient for present purposes, but not necessary). Since the norms of the zeros λ_k increase without bound in $|k|$, it follows that for sufficiently large $|k|$, the functions $f(\lambda)$ and $g(\lambda)$ satisfy the hypotheses of Rouché's Theorem. The result follows. □

3. Characteristic Equations with k Delays

Proposition 2.1 can be extended to the case of multi-delay equations in a natural way. For simplicity of notation, we restrict to the case of two delays. The argument for k delays is similar.

For two delays τ_1 and τ_2, the (transcendental) characteristic equation is of the form

$$\Delta(\lambda) = P_\tau(\lambda) + Q_1(\lambda)e^{-\lambda\tau_1} + Q_2(\lambda)e^{-\lambda\tau_2} = 0\,, \tag{3.1}$$

where $\tau = (\tau_1, \tau_2)$.

Taking real and imaginary parts we obtain the underlying linear equation of the form

$$\begin{bmatrix} \operatorname{Re} P \\ \operatorname{Im} P \end{bmatrix} = \begin{bmatrix} E_1 & F_1 & E_2 & F_2 \\ F_1 & -E_1 & F_2 & -E_2 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix} = M \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{bmatrix}. \tag{3.2}$$

As in Section 2, the matrix coefficient functions E_1, E_2, F_1, F_2 are functions of (ρ, ω, τ) determined by

$$E_i = -e^{-\lambda\tau_i} \operatorname{Re} Q_i, \quad F_i = -e^{-\lambda\tau_i} \operatorname{Im} Q_i.$$

If $\operatorname{rank}(M) = 1$, then in order for a solution to exist, it is necessary that $\begin{bmatrix} U \\ V \end{bmatrix}$ be in the real linear span of the column space of M. So, for each column $\begin{bmatrix} R \\ S \end{bmatrix}$ of M, the inner-product with the column vector $\begin{bmatrix} U \\ V \end{bmatrix}$ of the left-hand-side must satisfy

$$\begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} = \pm \left\| \begin{bmatrix} U \\ V \end{bmatrix} \right\| \left\| \begin{bmatrix} R \\ S \end{bmatrix} \right\| \tag{3.3}$$

(since the rank of $M(\omega)$ is one, it is necessary and sufficient that this hold for any one of the non-zero column vectors).

If $\operatorname{rank}(M) = 2$, then for every $\begin{bmatrix} U \\ V \end{bmatrix}$, there necessarily exist solutions $\begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ y_2 \end{bmatrix}$ to the linear equation (3.2).

Suppose then that solutions to (3.2) exist. Using Gaussian elimination, these solutions can be given explicitly by equations of the form

$$a^i X_1 + b^i Y_1 + c^i X_2 + d^i Y_2 + e^i = 0, \quad 1 \le i \le 2\,, \tag{3.4}$$

where the coefficients $a^i, \ldots, e^i$ are elementary combinations of E, F, G, H from equation (3.2).

In order to obtain solutions of the form $[\cos(\omega\tau_1), \sin(\omega\tau_1), \cos(\omega\tau_2)\,,\ \sin(\omega\tau_2)]^T$, it is necessary also that for each solution $[X_1, Y_1, X_2, Y_2]$ to (3.4) we have: (i) $X_i^2 + Y_i^2 = 1$, for $i = 1, 2$; and (ii) that the argument for each $[X_i, Y_i]^T$ be equivalent to $\omega\tau_i \mathrm{mod} 2\pi$ for each i.

For (i), the solution set (3.4) is projected to the $[X_i, Y_i]$ coordinate plane for each $i = 1, 2$. The distance from the i-th origin to this projected set must then be less than or equal to one (since the solution set and its projections are known from (3.4), this distance may be calculated using classical formulas from coordinate geometry).

For (ii), (modulo periodicity) possible combinations of τ_1 and τ_2 must satisfy $\mathrm{Arg}[X_i, Y_i]^T \equiv \omega\tau_i$, $i = 1, 2$. This is obtained if and only if $[X_i, Y_i]^T\, [\cos(\omega\tau_i), \sin(\omega\tau_i)] = 1$, for $i = 1, 2$.

Note that, as in the case for one delay τ (Corollary 2.3), if $X_1 \neq 0, X_2 \neq 0$, and the ratios $\frac{Y_i}{X_i}$, for $i = 1, 2$ are independent of (τ_1, τ_2) (for example, if the coefficients of the characteristic equation polynomials do not depend explicitly on delay factors), then for each i, critical delays τ_i may be obtained as explicit multi-functions of $\omega \neq 0$.

These results may be stated now for the general case.

Theorem 3.1. *Suppose that a transcendental characteristic equation involves k delays $(\tau_1, \tau_2, \ldots, \tau_k)$ and is of the form*

$$\Delta(\lambda) = P(\lambda) + Q_1(\lambda)e^{-\lambda\tau_1} + Q_2(\lambda)e^{-\lambda\tau_2} + \cdots + Q_k(\lambda)e^{-\lambda\tau_k} = 0\,. \tag{3.5}$$

As in equation (3.2) for the case of two delays, consider the underlying linear system obtained from the real and imaginary parts of the transcendental characteristic equation $\Delta(\lambda) = 0$:

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} E_1 & F_1 & \cdots & E_k & F_k \\ F_1 & -E_1 & \cdots & F_k & -E_k \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ \vdots \\ X_k \\ Y_k \end{bmatrix} = M \begin{bmatrix} X_1 \\ Y_1 \\ \vdots \\ X_k \\ Y_k \end{bmatrix}\,. \tag{3.6}$$

Here $\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} \operatorname{Re} P \\ \operatorname{Im} P \end{bmatrix}$ and $\begin{bmatrix} E_i \\ F_i \end{bmatrix} = \begin{bmatrix} -e^{-\rho\tau} \operatorname{Re} Q_i \\ -e^{-\rho\tau} \operatorname{Im} Q_i \end{bmatrix}$ for $i = 1, 2, \dots, k$.

Let T be the generic unit circle $X^2 + Y^2 = 1$ in two real dimensions. Then there exists solutions $(\omega, \tau_1, \tau_2, \dots, \tau_k)$ of equation (3.6) of the form $[X_i, Y_i]^T = [\cos(\omega\tau_i), \sin(\omega\tau_i)]^T$, $i = 1, 2, \dots, k$ if and only if:

(a) The affine subspace of $\mathbb{R}^{2k}$ at $(\omega, \tau_1, \tau_2, \dots, \tau_k)$ determined by equation (3.6) has non-empty intersection with the k-fold Cartesian product $\prod_1^k T$; and

(b) There are solutions $\begin{bmatrix} X_i \\ Y_i \end{bmatrix}$ obtained from (a) that satisfy $\begin{bmatrix} X_i \\ Y_i \end{bmatrix} \begin{bmatrix} \cos(\omega\tau_i) \\ \sin(\omega\tau_i) \end{bmatrix} = 1$, for all $i = 1, 2, \dots, k$. Equivalently, there are solutions from (a) that satisfy the congruence system $\theta_i = \operatorname{Arg}[X_i, Y_i]^T \equiv \omega\tau_i \bmod 2\pi$, $i = 1, 2, \dots, k$.

Remark 3.2. (i) The full system of equations in Theorem 3.1 consists of $2k + 2$ equations for $k + 1$ quantities $(\omega, \tau_1, \tau_2, \dots, \tau_k)$: Two equations from equation (3.6); k equations from (a); and k equations from (b). So, in general, we can expect there to be situations where the solution set is over-determined and empty.

(ii) It is sometimes useful to write the congruence system explicitly:

$$\begin{aligned} &\omega\tau_1 = \theta_1 + 2m_1\pi\,, \\ &\omega\tau_2 = \theta_2 + 2m_2\pi\,, \\ &\vdots \\ &\omega\tau_k + \theta_k + 2m_k\pi\,, \end{aligned} \tag{3.7}$$

for some integers $m_1, m_2, \dots, m_k$.

Note also that $\frac{(k-1)k}{2}$ derived equations may be obtained as follows: Consider a pair $i \neq j$. The i-th equation gives $\omega = \frac{\theta_i + 2m_i\pi}{\tau_i}$. Substituting this into the j-th equation, we obtain

$$(\tau_j\theta_i - \tau_i\theta_j) = (m_j - m_i)\,2\pi\,. \tag{3.8}$$

Example 3.3. Suppose that $k = 2$, and that the second delay $\tau_2 = l\tau_1$ is some integer multiple of τ_1. If $l\theta_1 - \theta_2$ is not equivalent to $0 \bmod 2\pi$, then there is no solution. For, otherwise, after dividing out the common factor τ_1, (3.8) gives $[\theta_1 l - \theta_2] = -[lm_1 - m_2]\,2\pi$. Since l is an integer, this would be a contradiction.

Remark 3.4. For the case of one delay, it was pointed out in Remark 2.2 that if there is a solution to the characteristic equation, then $M^{-1}\,[X,Y]^T$ must be a unit vector. Theorem 3.1 may be used to formulate the analogous result for k-delays. Indeed, for $[\mathbf{X},\mathbf{Y}]^T = [X_1, Y_1 \ldots X_k, Y_k]^T$, define

$$\|[\mathbf{X},\mathbf{Y}]\|_{k-\sup} = \sup_{1\le i\le k} \left\|[X_i, Y_i]^T\right\| = \sup_{1\le i\le k} \sqrt{X_i^2 + Y_i^2}.$$

If there is a solution to $\Delta(\lambda) = 0$, then the set $M^{-1}\,[U,V]^T$ has non-empty intersection with the boundary of the $\|{*}\|_{k-\sup}$ unit ball of $\overset{k}{\underset{i=1}{\oplus}}\left(\mathrm{R}^2\right)$.

Theorem 3.1 also can be used to obtain a generalization of the one-delay formula $|P|^2 = |e^{-\rho\tau}Q|^2$ (see Remark 2.2).

Proposition 3.5. *Let*

$$\begin{array}{l}\mathbf{P} = [\,\mathrm{Re}\,P, \mathrm{Im}\,P]^T\,, \quad \Lambda_i = [X_i, Y_i]^T\,, \quad \mathbf{G_i} = [E_i, F_i]^T\,, \quad i = 1,2,\ldots,k\,,\\ \mathbf{E} = [E_1, E_2, \ldots, E_k]\,, \quad \mathbf{F} = [F_1, F_2, \ldots, F_k]\,,\\ \mathbf{X} = [X_1, X_2, \ldots, X_k]\,, \quad \mathbf{Y} = [Y_1, Y_2, \ldots, Y_k]\,.\end{array} \tag{3.9}$$

Furthermore, let

$$\begin{array}{l}P^2 = \mathbf{P}\cdot\mathbf{P}, E^2 = \mathbf{E}\cdot\mathbf{E}, F^2 = \mathbf{F}\cdot\mathbf{F}, G_i^2 = \mathbf{G}_i\cdot\mathbf{G}_i\,,\\ G^2 = E^2 + F^2 = \sum\limits_{i=1}^{k} G_i^2\,,\\ \text{and}\ \ \|[\mathbf{E},\mathbf{F}]\| = \|[\mathbf{F},-\mathbf{E}]\| = \xi\,,\end{array} \tag{3.10}$$

where we use the canonical l^2-norm on R^{2k} to obtain ξ.

If there exist vectors $\Lambda_i = (\cos(\omega\tau_i), \sin(\omega\tau_i))$, $i = 1,2,\ldots,k$ that yield a solution to (3.6), then

$$\mathrm{Re}\,P = k\xi\cos(\theta_+)\,, \qquad \mathrm{Im}\,Q = k\xi\cos(\theta_-)\,, \tag{3.11}$$

where $\cos(\theta_+)$ and $\cos(\theta_-)$ are the cosines for the angles in R^{2k} between the pairs $[\mathbf{E},\mathbf{F}]$ and $[\mathbf{X},\mathbf{Y}]$; and $[\mathbf{F},-\mathbf{E}]$ and $[\mathbf{X},\mathbf{Y}]$ respectively.

Furthermore,

$$P^2 = G^2 + 2\Bigg\{\sum_{i<j} (\mathbf{G}_i\cdot\mathbf{G}_j)\,(\Lambda_i\cdot\Lambda_j) + \sum_{i<j} \det\begin{bmatrix}\mathbf{G}_i & \mathbf{G}_j\end{bmatrix} \det\begin{bmatrix}\Lambda_i & \Lambda_j\end{bmatrix}\Bigg\}. \tag{3.12}$$

Proof. First, rewrite (3.6) in block matrix form as

$$\mathbf{P} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{F} & -\mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{F} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix} \\ \begin{bmatrix} \mathbf{F} & -\mathbf{E} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix} \end{bmatrix}.$$

The first formulas follow from the dot product in R^{2k} and the fact that $\left\| \begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix} \right\|^2 = \sum\limits_{i=1}^{k} \|\Lambda_i\|^2 = \sum\limits_{i=1}^{k} 1 = k$.

For the second formula, note that the block matrix equation just given implies that

$$\begin{aligned} (\operatorname{Re} P)^2 &= (\mathbf{E}\cdot\mathbf{X})^2 + 2\,(\mathbf{E}\cdot\mathbf{X})\,(\mathbf{F}\cdot\mathbf{Y}) + (\mathbf{F}\cdot\mathbf{Y})^2 \\ (\operatorname{Im} P)^2 &= (\mathbf{F}\cdot\mathbf{X}) - 2\,(\mathbf{F}\cdot\mathbf{X})\,(\mathbf{E}\cdot\mathbf{Y}) + (\mathbf{E}\cdot\mathbf{Y})^2 \end{aligned}$$

Now observe the following: For each $i = 1, 2, \ldots, k$: (a) $X_i^2 + Y_i^2 = 1$ implies that $E_i^2 X_i^2 + E_i^2 Y_i^2 = E_i^2$; and (b) The terms $E_i F_i X_i Y_i$ appear in both $(\operatorname{Re} P)^2$ and $(\operatorname{Im} P)^2$, but with opposite signs. Adding the equations therefore gives

$$\begin{aligned} P^2 = {} & \left(E^2 + F^2\right) \\ & + 2\left\{ \sum_{i<j} E_i E_j\,(X_i X_j + Y_i Y_j) + \sum_{i<j} F_i F_j\,(Y_i Y_j + X_i X_j) \right\} \\ & \quad + 2\left\{ \sum_{i<j} E_i F_j\,(X_i Y_j - Y_i X_j) + \sum_{i<j} F_i E_j\,(Y_i X_j - X_i Y_j) \right\}. \end{aligned}$$

Collecting terms and using the symbols, $G^2, \mathbf{G_i}$ and Λ_j, the result follows.

Note. The special case for a single delay is obtained by setting $k = 1$ and using the symbol Q in place of G.

Corollary 3.6. *Suppose that $k = 2$ and that $F_i = 0, E_i \neq 0$ for $i = 1, 2$. If there exists a solution of (3.6) the form $\Lambda_i = (\cos(\omega\tau_i), \sin(\omega\tau_j))$, $i = 1, 2$, then*

$$|P^2 - E^2| \leq 2|E_1 E_2|\,. \tag{3.13}$$

Proof. In this case, $\mathbf{G_i} = [E_i, 0]^T$ and (3.12) becomes

$$\begin{aligned} P^2 = {} & E^2 + 2\,(\mathbf{G}_1\cdot\mathbf{G}_2)\,(\Lambda_1\cdot\Lambda_2) \\ & + 2\det\begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix} \det\begin{bmatrix} \Lambda_1 & \Lambda_2 \end{bmatrix}. \end{aligned} \tag{3.14}$$

Let θ_{12} be a representative angle between Λ_1 and Λ_2. Then, using $\det\left[\begin{array}{cc}\mathbf{G}_1 & \mathbf{G}_2\end{array}\right] = 0$ and $\|\Lambda_i\| = 1$, we obtain

$$P^2 - E^2 = 2E_1E_2\cos(\theta_{12})\,. \tag{3.15}$$

Taking the absolute value, the result follows. □

Remark 3.7. For $k = 2$, Corollary 3.6 can be used in applications toward solving the converse problem. Let R_θ be the rotation operator acting on two dimensions. Every solution of (3.14) that consists of unit vectors Λ_1, Λ_2 is obtained from the pair $\left\{R_\theta[1,0]^T, R_\theta[\cos(\theta_{12}),\ \sin(\theta_{12})]^T\right\}$. Finally, as in Theorem 3.1, impose the further criteria that $\mathrm{Arg}\Lambda_i \equiv \omega\tau_i$, $i = 1, 2$.

Example 3.8. (Distinct Delays $\tau_1 \neq \tau_2$) *Suppose that a transcendental equation is of the form*

$$\Delta(\lambda) = a_0 + a_1e^{-\lambda\tau_1} + a_2e^{-\lambda\tau_2} = 0\,. \tag{3.16}$$

If there is a purely imaginary root $\lambda = i\omega$*, then*

$$\left|a_1^2 + a_2^2 - a_0^2\right| \leq |2a_1a_2|\ . \tag{3.17}$$

Proof. For this characteristic equation, $E_i = -a_ie^{-\rho\tau_i}$, $F_i = 0$ for $i = 1, 2$ and $\mathbf{P} = [a_0, 0]^T$. Substitute into (2.8) and set $\rho = 0$. □

Example 3.9. The characteristic equation in [14] is

$$\lambda + a + be^{-\lambda\tau_1} + be^{-\lambda\tau_2} = 0\,, \tag{3.18}$$

where the coefficients on the delay terms are equal to some non-zero real constant b. In [14], the investigation focuses on the possible existence of purely imaginary roots.

For $\lambda = \rho + i\omega$, the real and imaginary parts of (3.18) give the system

$$-be^{-\rho\tau_1}X_1 - be^{-\rho\tau_2}X_2 = (\rho + a)\ , \quad be^{-\rho\tau_1}Y_1 + be^{-\rho\tau_2}Y_2 = \omega\,. \tag{3.19}$$

In this case, $E_i = -be^{-\rho\tau_i}$, $i = 1, 2$ and $\mathbf{P} = [(a+\rho)\,, \omega]^T$. Evaluating (3.13), and setting$\rho = 0$, we obtain

$$\left|2b^2 - \left(a^2 + \omega^2\right)\right| \leq \left|2b^2\right|\ .$$

This can be re-expressed as

$$0 \leq \omega^2 \leq 4b^2 - a^2\,. \tag{3.20}$$

Note. In particular, if $4b^2 - a^2 < 0$, then there is no purely imaginary root, for all delays τ_1, τ_2.

Example 3.10. In [7], the system of delay equations involves three delay terms, $\tau_1 = \tau$, $\tau_2 = \xi$ and $\tau_3 = \tau + \xi = \tau_1 + \tau_2$. The transcendental characteristic equation is

$$\Delta(\lambda) = \left[\lambda^2 + a\lambda + b\right] + [c\lambda + p]\, e^{-\tau_1\lambda} + [d\lambda + q]\, e^{-\tau_2\lambda} + re^{-\tau_3\lambda}\,, \tag{3.21}$$

where a, b, c, p, q, r are real constants.

When there are three delay terms, (2.7) yields

$$\left|P^2 - G^2\right| \le 2\Big\{ |\mathbf{G}_1\mathbf{G}_2| + |\mathbf{G}_1\mathbf{G}_3| + |\mathbf{G}_2\mathbf{G}_3| + \left|\det\left[\begin{array}{cc} \mathbf{G}_1 & \mathbf{G}_2 \end{array}\right]\right| + \left|\det\left[\begin{array}{cc} \mathbf{G}_1 & \mathbf{G}_3 \end{array}\right]\right| + \left|\det\left[\begin{array}{cc} \mathbf{G}_2 & \mathbf{G}_3 \end{array}\right]\right| \Big\}. \tag{3.22}$$

Consequently, if there is a purely imaginary root $\lambda = i\omega$ of the characteristic equation in [7], then

$$\left(-\omega^2 + b\right)^2 + (a\omega)^2 \le 2\left\{\left|pq + cd\omega^2\right| + |pr| + |qr| + |pd - cq| + |r\omega\,(c + d)|\right\}. \tag{3.23}$$

4. Hopf Bifurcation

Much as for parameterization by a single real variable, in the multi-delay system we may enquire into the stability profile when there is a purely imaginary root $\lambda_\tau = i\omega_\tau$, $\tau = (\tau_1, \ldots, \tau_k)$. It turns out that when $k \ge 2$, for all non-singular cases there is a Hopf bifurcation (see Theorem 4.6 below). To see this requires adapting the Hopf Bifurcation Theorem to the k-fold parameterization given by $\tau = (\tau_1, \ldots, \tau_k)$. Key quantities to be determined are $S = \text{sign}\left[\text{Re}\, \frac{\partial\lambda}{\partial\tau_i}\right]$, $i = 1, ..., k$. As is well known, these may be obtained through implicit differentiation of the defining characteristic equation (3.5).

For a first illustration, we look to Example 3.8 (above), where $k = 2$ and the coefficients of the characteristic equation (3.16) are arbitrary but constant. Theorem 4.4 (below) regards this situation, and in preparation for that we start with various calculations.

To evaluate the partial derivatives $\frac{\partial\lambda}{\partial\tau_1}$ and $\frac{\partial\lambda}{\partial\tau_2}$, implicit differentiation yields

$$\begin{aligned}\frac{\partial\lambda}{\partial\tau_1} &= \frac{-\lambda a_1 e^{-\lambda\tau_1}}{a_1\tau_1 e^{-\lambda\tau_1} + a_2\tau_2 e^{-\lambda\tau_2}}\,,\\ \frac{\partial\lambda}{\partial\tau_2} &= \frac{-\lambda a_2 e^{-\lambda\tau_2}}{a_1\tau_1 e^{-\lambda\tau_1} + a_2\tau_2 e^{-\lambda\tau_2}}\,.\end{aligned} \tag{4.1}$$

At a purely imaginary root $\lambda = i\omega$, these equations become

$$\frac{\partial\lambda}{\partial\tau_1} = \frac{1}{\tau_1}\frac{-i\omega\left[\left(1+\frac{a_2\tau_2}{a_1\tau_1}\cos(\omega\Delta\tau)\right)+i\left(\frac{a_2\tau_2}{a_1\tau_1}\sin(\omega\Delta\tau)\right)\right]}{\left[\left(1+\frac{a_2\tau_2}{a_1\tau_1}\cos(\omega\Delta\tau)\right)^2+\left(\frac{a_2\tau_2}{a_1\tau_1}\sin(\omega\Delta\tau)\right)^2\right]}\,, \tag{4.2}$$

where $\Delta\tau = \tau_2 - \tau_1$, and

$$\frac{\partial\lambda}{\partial\tau_2} = \frac{1}{\tau_2}\frac{-i\omega\left[\left(1+\frac{a_1\tau_1}{a_2\tau_2}\cos(\omega\tilde{\Delta}\tau)\right)+i\left(\frac{a_1\tau_1}{a_2\tau_2}\sin(\omega\tilde{\Delta}\tau)\right)\right]}{\left[\left(1+\frac{a_1\tau_1}{a_2\tau_2}\cos(\omega\tilde{\Delta}\tau)\right)^2+\left(\frac{a_1\tau_1}{a_2\tau_2}\sin(\omega\tilde{\Delta}\tau)\right)^2\right]}\,, \tag{4.3}$$

where $\tilde{\Delta}\tau = \tau_1 - \tau_2$.

We now obtain

$$\begin{aligned}&\operatorname{sign}\operatorname{Re}\frac{\partial\lambda}{\partial\tau_1}\\ &\quad= \operatorname{sign}\left\{\frac{1}{\tau_1}\frac{\omega\frac{a_2\tau_2}{a_1\tau_1}\sin(\omega\Delta\tau)}{\left[\left(1+\frac{a_2\tau_2}{a_1\tau_1}\cos(\omega\Delta\tau)\right)^2+\left(\frac{a_2\tau_2}{a_1\tau_1}\sin(\omega\Delta\tau)\right)^2\right]}\right\}\,,\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}&\operatorname{sign}\operatorname{Re}\frac{\partial\lambda}{\partial\tau_2}\\ &\quad= \operatorname{sign}\left\{\frac{1}{\tau_2}\frac{\omega\frac{a_1\tau_1}{a_2\tau_2}\sin(\omega\tilde{\Delta}\tau)}{\left[\left(1+\frac{a_1\tau_1}{a_2\tau_2}\cos(\omega\tilde{\Delta}\tau)\right)^2+\left(\frac{a_1\tau_1}{a_2\tau_2}\sin(\omega\tilde{\Delta}\tau)\right)^2\right]}\right\}\,.\end{aligned} \tag{4.5}$$

Since the sine function is odd and the delays are strictly positive, we obtain

$$\operatorname{sign}\operatorname{Re}\frac{\partial\lambda}{\partial\tau_1} = \operatorname{sign}\left[\frac{a_2}{a_1}(\Delta\tau)\right], \tag{4.6}$$

and

$$\operatorname{sign}\operatorname{Re}\frac{\partial\lambda}{\partial\tau_2} = \operatorname{sign}\left[\frac{a_1}{a_2}\left(\tilde{\Delta}\tau\right)\right]. \tag{4.7}$$

Since $\Delta\tau = -\tilde{\Delta}\tau$, we have that

$$\operatorname{sign}\operatorname{Re}\frac{\partial\lambda}{\partial\tau_1} = -\operatorname{sign}\operatorname{Re}\frac{\partial\lambda}{\partial\tau_2}. \tag{4.8}$$

Next we consider directional derivatives in τ_1 and τ_2. Specifically, we seek the real part of the directional derivative $D_v\lambda$, in the (τ_1, τ_2)-parameter direction given by a unit vector $v = [\cos(\phi), \sin(\phi)]$.

Write $\tau = (\tau_1, \tau_2)$ and let $\lambda = \lambda(\tau(s)) = \rho(\tau(s)) + i\omega(\tau(s))$ be a smooth path of zeros, where the delays are parameterized by an interval of real numbers s. Evaluating the derivative with respect to s, we obtain

$$\begin{aligned}\frac{d\lambda}{ds} = \frac{d}{ds}\left[\rho(\tau(s)) + i\omega(\tau(s))\right] = \frac{d}{ds}\rho(\tau(s)) + \frac{d}{ds}i\omega(\tau(s))\\ = \nabla_\tau(\rho)\frac{d\tau}{ds} + i\nabla_\tau(\omega)\frac{d\tau}{ds}.\end{aligned}$$

Consequently,

$$\operatorname{Re}\frac{d\lambda}{ds} = \nabla_\tau(\rho)\frac{d\tau}{ds} = \nabla_\tau(\operatorname{Re}\lambda)\frac{d\tau}{ds} = \frac{d}{ds}\operatorname{Re}\lambda. \tag{4.9}$$

In other words, to determine the sign of the real part of the directional derivative, it is enough to calculate the sign of the directional derivative of the real part.

For Theorem 4.4, the following notation will be used: Let $\frac{a_2\tau_2}{a_1\tau_1} = \alpha = \tilde{\alpha}^{-1}$, $\theta = \omega\Delta\tau$ and $\tilde{\theta} = \omega\tilde{\Delta}\tau = -\theta$ (see (4.2) and (4.3)).

Theorem 4.4. *Let all terms be defined as above. Suppose that $\lambda = i\omega$ is an isolated root of the transcendental characteristic equation, with a conjugagate root $-i\omega$; that the remaining roots have strictly negative real parts; that $\omega \neq 0$, $\Delta\tau \neq 0$ and that $\sin(\omega\Delta\tau) \neq 0$. Suppose also that $\lambda(\tau(s))$ is a smooth path in the complex plane such that $\lambda(\tau(0)) = i\omega$. Write the unit tangent vector of $\tau(s)$ at $s = 0$ by $\mathbf{v} = (\cos(\phi), \sin(\phi))$. Then, for all ϕ that satisfy*

$$\tan(\phi) \neq \frac{\tau_2\alpha\left(1 + 2\tilde{\alpha}\cos(\tilde{\theta}) + \tilde{\alpha}^2\right)}{\tau_1\tilde{\alpha}\left(1 + 2\alpha\cos(\theta) + \alpha^2\right)}, \tag{4.10}$$

there is a Hopf bifurcation as $\lambda(\tau)$ passes through the purely imaginary root $\lambda = i\omega = i\omega_c$.

Note. Since $\tilde{\theta} = -\theta$, we have $\cos(\tilde{\theta}) = \cos(\theta)$.

Proof. The proof is a calculation, for it is enough to show that the real part of the directional derivative is not zero. From the remarks above,

$$\operatorname{Re} D_{\mathbf{v}}(\lambda) = \left(\operatorname{Re} \frac{\partial \lambda}{\partial \tau_1}, \operatorname{Re} \frac{\partial \lambda}{\partial \tau_2} \right) \cdot \mathbf{v}\,.$$

Substitute from (4.4) and (4.5) to obtain

$$\operatorname{Re} D_{\mathbf{v}}(\lambda) = \omega \Delta \tau \left[\frac{\alpha \cos(\varphi)}{\tau_1 \left(1 + 2\alpha \cos(\theta) + \alpha^2\right)} - \frac{\tilde{\alpha} \sin(\varphi)}{\tau_2 \left(1 + 2\tilde{\alpha} \cos(\tilde{\theta}) + \tilde{\alpha}^2\right)} \right]\,.$$

By hypothesis, $\omega \Delta \tau \neq 0$. Therefore, $\operatorname{Re} D_{\mathbf{v}}(\lambda) = 0$ if and only if $\tan(\varphi)$ is given by the right hand side of (4.10), and the result follows. ☐

Remark 4.5. It follows that an empirical system that is approximated by a model whose transcendental characteristic equation is of the form (3.16) will in most cases be unstable at any non-zero purely imaginary root satisfying the hypotheses of the theorem. For then there can be Hopf bifurcation in all delay-directions but possibly two. These two exceptional directions are mathematical but, other things being equal, rarely could be maintained in an empirical setting.

For a general result, consider a transcendental characteristic equation of the form $\Delta(\lambda) = P(\lambda) + Q_1(\lambda)e^{-\lambda\tau_1} + Q_2(\lambda)e^{-\lambda\tau_2} + \cdots + Q_k(\lambda)e^{-\lambda\tau_k} = 0$, where as above $\tau(s) = (\tau_1(s), \tau_1(s), \ldots \tau_k(s))$ is a path in the τ-parameter space.

Again, we obtain the gradient $\nabla_\tau(\lambda) = \left(\frac{\partial\lambda}{\partial\tau_1}, \frac{\partial\lambda}{\partial\tau_2}, \ldots, \frac{\partial\lambda}{\partial\tau_k}\right)$ by implicit differentiation. A straightforward calculation gives that for each $j = 1, \ldots, k$

$$\frac{\partial \lambda}{\partial \tau_j} = \frac{\lambda Q_j e^{-\lambda \tau_j}}{\left[\frac{dP}{d\lambda} + \sum_{m=1}^{k} \left(\frac{dQ_m}{d\lambda} - Q_m \tau_m \right) e^{-\lambda \tau_m} \right]}\,.$$

We are assuming that the analytic denominator is not zero. The zeros of the denominator are, of course, isolated, for otherwise the analytic function would be identically zero.

Now, from (4.9) we have the general result that

$$\operatorname{Re} \left(\frac{d}{ds} \lambda(\tau(s)) \right) = \operatorname{Re} \left(\nabla_\tau (\lambda) \cdot \frac{d\tau}{ds} \right) = \left(\operatorname{Re} \nabla_\tau \lambda \right) \cdot \frac{d\tau}{ds}\,.$$

Therefore, the real quantity Re $\left(\frac{d}{ds}\lambda(\tau(s))\right)$ is of the form $A_1\frac{d\tau_1}{ds}+A_2\frac{d\tau_2}{ds}+\cdots+A_k\frac{d\tau_k}{ds}$, where $A_j = A_j\left(\lambda(\tau)\right)$ are real functions for $j = 1, 2, \ldots, k$. Hence, Re $\left(\frac{d}{ds}\lambda(\tau(s))\right)=0$ if and only if $A_1\frac{d\tau_1}{ds}+A_2\frac{d\tau_2}{ds}+\cdots+A_k\frac{d\tau_k}{ds}=0$. This equation determines a subspace $\mathbf{B}_{\tau(0)}$ of the τ-parameter tangent space at $\tau(0)$. For $A_i = A_i\left(\tau(0)\right)$ not the zero-vector, the dimension of the subspace is $k-1$.

This leads to the following theorem.

Theorem 4.6. *Let all terms be defined as above and suppose that* $\operatorname{Re}\nabla_\tau\lambda = \nabla_\tau \operatorname{Re}\lambda \neq \mathbf{0}$ *at* $\tau(0)$. *Then for each path* $\tau(s)$ *whose tangent vector at* $s=0$ *is in the set theoretic complement of* $\mathbf{B}_{\tau(0)} = \{\mathbf{v}\,|\mathbf{v}\cdot \operatorname{Re}\left(\nabla_\tau\lambda\right)=0\}$, *there is a Hopf bifurcation as* $\lambda\left(\tau(s)\right)$ *passes through the purely imaginary point* $\lambda\left(\tau(0)\right)$.

5. Concluding Remarks

The transcendental characteristic $\Delta(\lambda)=0, \lambda=\rho+i\omega$ is equivalent to the point-wise linear factorization $\mathbf{P}=\mathbf{MZ}$,where

$$\mathbf{P}=\left[\operatorname{Re}P,\,\operatorname{Im}P\right]^T,\quad \mathbf{M}=\begin{bmatrix} E_1 & F_1 & \cdots\cdots & E_k & F_k \\ F_1 & -E_1 & & F_k & -E_k \end{bmatrix}$$

and

$$\mathbf{Z}=\begin{bmatrix} \cos(\omega\tau_1) & \sin(\omega\tau_1) & \cdots\cdots & \cos(\omega\tau_k) & \sin(\omega\tau_k) \end{bmatrix}^T.$$

Hence, a solution to the characteristic equation exists if and only if there exists (ρ,ω,τ) such that $\mathbf{Z}\left(\omega,\tau\right)$ is in the inverse image $\mathbf{M^{-1}}\left(\rho,\omega,\ \tau\right)\mathbf{P}\left(\rho,\omega,\tau\right)$. Equivalently, there is a solution if and only if $\mathbf{M}^{-1}\mathbf{P}\cap\prod_{i=1}^{k}T\neq\emptyset$ (T the generic unit circle in two variables), and at least one of the elements of this inverse image has the appropriate orientation with respect to the k-tuple $(\omega\tau_1,\ldots,\omega\tau_k)$.

As was illustrated in Section 3 and Section 4, adverting to this structure can reveal detailed features of the zero set of the characteristic equation; and can be used as the basis for a general approach that will, for instance, be useful in the analysis of multi-delay equations arising in the applied sciences.

We conclude the paper by indicating what we think would be worthwhile further related lines of enquiry that emerge naturally from the factorization.

1. Investigate the geometric structure (both commutative and non-commutative) of the field of linear maps associated with a transcendental characteristic equation.

2. Determine appropriate sub-classes of analytic functions that contain transcendental characteristic functions as key representatives.

3. Investigate possible generalizations of the k-circle map

$$[\cos(\omega\tau_1), \sin(\omega\tau_2), \dots, \cos(\omega\tau_k), \sin(\omega\tau_k)]^T,$$

with formulations that include algebraic structure. For example, suppose that Ψ is a set of functions $f(\lambda, \tau) : \mathbb{C} \to \mathbb{C}$, parameterized by the real k-parameter $\tau = (\tau_1, \dots, \tau_k), \tau_i \in \mathbb{R}$ for $i = 1, \dots, k$ (the set Ψ could be a singleton, or a set with algebraic structure). Investigate cases where there would be a canonically associated field of mutually orthogonal projections $P_i : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ ($\dim P_i = 2$) that would be linked to a factorization of the form $[\operatorname{Re} f, \operatorname{Im} f]^T = \mathbf{M}(\lambda, \tau)\mathbf{Z}(\omega, \tau)$. Identify circumstances where there would be a map $\mathbf{Z} : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^{2k}$ with the property that for each $i = 1, \dots, k$, we would have $\mathbf{Z}_i(\omega, \tau_i) = \mathbf{Z}(\omega, (0, \dots, 0, \tau_i, 0, \dots, 0))$ with the property that $\mathbf{P}_i\mathbf{Z}_i = \mathbf{Z}_i$.

4. Investigate transformations of functions (geometric and more general operators) that (a) leave the classes in 2. invariant; (b) leave the congruence relations (3.7) invariant; and (c) more generally, leave selected relations between the delay terms invariant (in [7], for example, $\tau_3 = \tau_1 + \tau_2$).

5. Investigate transformations that leave (or do not leave) the parameterized zero sets (spectra) of a characteristic function invariant.

6. Investigate the natural tensor and/or cross-product structures $\tau \times_\alpha X$ that pertain to multi-delay models.

7. In empirical processes, multi-delays can occur at different scales. For instance, in mathematical biology, a first delay in seconds or hours could refer to an intracellular process; while a second delay of months or even years could refer to a population density. So, investigate multi-delay models as a source of multi-scale models, as described in [16].

8. In view of Remark 4.5 and Theorem 4.6, investigate bifurcation (a) at purely imaginary roots that are critical points of the delay gradient $\nabla_\tau\lambda$; and (b) for parameterization paths whose tangent vector at the isolated purely imaginary root is tangent to the "boundary space" $\mathbf{B}_{\tau(0)}$.

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