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ON Ω-FUZZY IDEALS IN Ω-SEMIRINGS/HEMIRINGS

Young Bae Jun[1 §], Kyung Ho Kim[2]

[1]Department of Mathematics
College of Education
Gyeongsang National University
Chinju, 660-701, KOREA
e-mail: ybjun@gsnu.ac.kr

[2]Department of Mathematics
Chungju National University
Chungju, 380-702, KOREA
e-mail: ghkim@gukwon.chungju.ac.kr

Abstract: The Ω-fuzzy setting of an Ω-left k-ideal (resp. Ω-left h-ideal) in an Ω-semiring (resp. Ω-hemiring) is constructed, and basic properties are investigated. Using a collection of Ω-left k-ideals (resp. Ω-left h-ideals) of an Ω-semiring (resp. Ω-hemiring) S, Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S are established. The notion of a finite valued Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) is introduced, and its characterization is given. Fuzzy relations on an Ω-semiring (resp. Ω-hemiring) S are discussed.

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§Correspondence author

1. Introduction

Semirings play an important role in studying matrices and determinants. Many aspects of the theory of matrices and determinants over semirings have been studied by Beasley and Pullman [3, 4], Ghosh [9], and others. Although ideals in semirings are useful for many purposes, they do not in general coincide with the usual ring ideals if S is a ring and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. Henriksen [11] defined a more restricted class of ideals in semirings, which is called *k-ideals*, with the property that if the semiring S is a ring then a complex in S is a k-ideal if and only if it is a ring ideal. A still more restricted class of ideals in hemirings has been given by Iizuka [12]. However, a definition of ideal in any additively commutative semiring S can be given which coincides with Iizuka's definition provided S is a hemiring, and it is called *h-ideal*. Latorre [21] investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of familiar ring theorems. The concept of a fuzzy set, introduced in Zadeh [22], was applied to generalize some of the basic concepts of algebra. Several authors have discussed a fuzzy theory in semirings (see [1, 2, 6, 7, 8, 10, 13, 14, 15, 17, 19, 20]). In this paper, we consider the Ω-fuzzy setting of k-ideals (resp. h-ideals) in an Ω-semiring (resp. Ω-hemiring). Using a collection of left h-ideals of an Ω-semiring (resp. Ω-hemiring) S, we establish fuzzy left h-ideals of S. We give a characterization of a finite valued fuzzy left h-ideal in an Ω-semiring (resp. Ω-hemiring) S, and show that if an Ω-semiring (resp. Ω-hemiring) S is Ω-left k-Noetherian (resp. Ω-left h-Noetherian), then every Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S is finite valued. We prove that if μ and ν are Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of an Ω-semiring (resp. Ω-hemiring) S, then $\mu \times \nu$ is an Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of $S \times S$. Conversely, we show that if $\mu \times \nu$ is an Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of $S \times S$, then either μ or ν is an Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S. We prove that a fuzzy set ν in an Ω-semiring (resp. Ω-hemiring) S is an Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S if and only if the strongest fuzzy relation μ_ν on S is an Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of $S \times S$.

2. Preliminaries

A *semiring* S is a system consisting of a nonempty set S together with two binary operations on S called *addition* and *multiplication* (denoted in the usual manner) such that:

— S together with addition is a semigroup,

— S together with multiplication is a semigroup, and

— $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for all $a, b, c \in S$.

A semiring S is said to be *additively commutative* if $a + b = b + a$ for all $a, b \in S$. A zero element of a semiring S is an element 0 such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. By a *hemiring* we mean an additively commutative semiring with zero.

A subset A of a semiring S is called a *left ideal* of S if A is closed under addition and $SA \subseteq A$. A left ideal A of S is called a *left k-ideal* of S if $y, z \in A$, $x \in S$, and $x + y = z$ implies $x \in A$. Right (k-) ideals are defined similarly. A fuzzy set μ in a semiring S is called a *fuzzy left ideal* of S if it satisfies:

— $\mu(x + y) \geq \min\{\mu(x),\ \mu(y)\}$, $\forall x, y \in S$,

— $\mu(xy) \geq \mu(y)$, $\forall x, y \in S$.

Note that if μ is a fuzzy left ideal of a hemiring S, then $\mu(0) \geq \mu(x)$ for all $x \in S$.

A fuzzy left ideal μ of a semiring S is called a *fuzzy left k-ideal* of S (see [10]) if for any $x, y, z \in S$, $x + y = z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.

Fuzzy right (k-)ideals are defined similarly.

Lemma 2.1. (see [13, 15]) *A fuzzy set μ in a semiring S is a fuzzy left (k-)ideal of S if and only if the level subset $U(\mu; t)$, $t \in [0, 1]$, of μ is a left (k-)ideal of S whenever it is nonempty.*

3. Ω-Fuzzy Ideals in Ω-Semirings/Hemirings

In what follows let Ω denotes a nonempty set.

Definition 3.1. By an *Ω-semiring* (resp. *Ω-hemiring*) we mean an algebraic system consisting of a semiring (resp. hemiring) S, a set Ω and a function defined in the product set $\Omega \times S$ and having values in S such that, if αx denotes the element in S determined by the element x of S and the element α of Ω, then $\alpha(x + y) = \alpha x + \alpha y$ and $\alpha(xy) = (\alpha x)(\alpha y)$ holds for any $x, y \in S$ and $\alpha \in \Omega$.

Definition 3.2. (see [21]) A *left h-ideal* of a hemiring S is defined to be a left ideal A of S such that

$$(\forall x, z \in S)(\forall a, b \in A)(x + a + z = b + z \Rightarrow x \in A).$$

Right h-ideals are defined similarly. Note that every left (resp. right) h-ideal is a left (resp. right) k-ideal, but the converse may not be true (see [21]).

Definition 3.3. Let S be an Ω-semiring (resp. Ω-hemiring). By an *Ω-left (k-) ideal* (resp. *Ω-left h-ideal*) we mean a left (k-) ideal (resp. left h-ideal) A of S that satisfies $\alpha x \in A$ for all $\alpha \in \Omega$ and $x \in A$.

An Ω-right (k-) ideal (resp. Ω-right h-ideal) is defined similarly.

Definition 3.4. (see [16]) A *fuzzy left h-ideal* of a hemiring S is defined to be a fuzzy left ideal μ of S such that

$$(\forall a, b, x, z \in S)(x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}).$$

Fuzzy right h-ideals are defined similarly.

Definition 3.5. Let S be an Ω-semiring (resp. Ω-hemiring). By an *Ω-fuzzy left (k-) ideal* (resp. *Ω-fuzzy left h-ideal*) of S we mean a fuzzy left (k-) ideal (resp. fuzzy left h-ideal) μ of S that satisfies the inequality $\mu(\alpha x) \geq \mu(x)$ for all $\alpha \in \Omega$ and $x \in S$.

An Ω-fuzzy right (k-) ideal (resp. Ω-fuzzy right h-ideal) is defined similarly.

Example 3.6. Let S be an Ω-semiring (resp. Ω-hemiring) and let A be a nonempty subset of S. Then A is an Ω-left k-ideal (resp. Ω-left h-ideal) of S if and only if the characteristic function χ_A of A is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S.

The following theorem is a generalization of Example 3.6.

Theorem 3.7. *Let A be a nonempty subset of an Ω-semiring* (resp. *Ω-hemiring*) *S. Let μ be a fuzzy set in S defined by*

$$\mu(x) := \begin{cases} s & \text{if } x \in A, \\ t & \text{otherwise,} \end{cases}$$

where $s > t$ in $[0, 1]$. Then μ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S if and only if A is an Ω-left k-ideal (resp. Ω-left h-ideal) of S.

Proof. Assume that μ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S. Then A is a left k-ideal (resp. left h-ideal) of S (see [16, Theorem 3.3]).

Let $\alpha \in \Omega$ and $x \in A$. Then $\mu(\alpha x) \geq \mu(x) = s$, and thus $\mu(\alpha x) = s$. This means that $\alpha x \in A$. Therefore A is an Ω-left k-ideal (resp. Ω-left h-ideal) of S. Conversely suppose that A is an Ω-left k-ideal (resp. Ω-left h-ideal) of S. Then μ is a fuzzy left k-ideal (resp. fuzzy left h-ideal) of S (see [16, Theorem 3.3]). Let $x \in S$ and $\alpha \in \Omega$. If $x \in A$, then $\alpha x \in A$ and so $\mu(\alpha x) = s = \mu(x)$. If $x \in S \setminus A$, then $\mu(x) = t \leq \mu(\alpha x)$. Hence μ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S. □

Theorem 3.8. *Let μ be an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of an Ω-semiring (resp. Ω-hemiring) S. Given $\alpha \in \Omega$, the fuzzy set $\bar{\mu}$ in S defined by $\bar{\mu}(x) = \mu(\alpha x)$ for all $x \in S$ is a fuzzy left k-ideal (resp. fuzzy left h-ideal) of S.*

Proof. Let $x, y \in S$ and $\alpha \in \Omega$. Then

$$\begin{aligned}\bar{\mu}(x+y) &= \mu(\alpha(x+y)) = \mu(\alpha x + \alpha y)\\ &\geq \min\{\mu(\alpha x), \mu(\alpha y)\} = \min\{\bar{\mu}(x), \bar{\mu}(y)\},\end{aligned}$$

$$\bar{\mu}(xy) = \mu(\alpha(xy)) = \mu((\alpha x)(\alpha y)) \geq \mu(\alpha y) = \bar{\mu}(y).$$

Let $x, y, z \in S$ be such that $x + y = z$. Then $\alpha x + \alpha y = \alpha(x+y) = \alpha z$, and so

$$\bar{\mu}(x) = \mu(\alpha x) \geq \min\{\mu(\alpha y), \mu(\alpha z)\} = \min\{\bar{\mu}(y), \bar{\mu}(z)\}.$$

Hence $\bar{\mu}$ is a fuzzy left k-ideal of S. Now let $a, b, x, z \in S$ be such that $x+a+z = b+z$. Then $\alpha x + \alpha a + \alpha z = \alpha b + \alpha z$, and thus

$$\bar{\mu}(x) = \mu(\alpha x) \geq \min\{\mu(\alpha a), \mu(\alpha b)\} = \min\{\bar{\mu}(a), \bar{\mu}(b)\}.$$

Therefore $\bar{\mu}$ is a fuzzy left h-ideal of S. □

Lemma 3.9. (see [16]) *A fuzzy set μ in a hemiring S is a fuzzy left h-ideal of S if and only if the level subset $U(\mu; t)$, $t \in [0,1]$, of μ is a left h-ideal of S whenever it is nonempty.*

Theorem 3.10. *Let S be an Ω-semiring (resp. Ω-hemiring). A fuzzy set μ in S is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S if and only if the level subset $U(\mu; t)$, $t \in [0,1]$, of μ is an Ω-left k-ideal (resp. Ω-left h-ideal) of S whenever it is nonempty.*

Proof. Let μ be an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S. Then the level subset $U(\mu; t)$, $t \in [0,1]$, of μ is a left k-ideal (resp. left h-ideal) of S whenever it is nonempty by Lemma 2.1 and Lemma 3.9. Let $x \in U(\mu; t)$

and $\alpha \in \Omega$. Then $\mu(\alpha x) \geq \mu(x) \geq t$, and so $\alpha x \in U(\mu; t)$. Therefore $U(\mu; t)$ is an Ω-left k-ideal (resp. Ω-left h-ideal) of S. Conversely let $t \in [0, 1]$ be such that $U(\mu; t)$ is nonempty and is an Ω-left k-ideal (resp. Ω-left h-ideal) of S. Then μ is a fuzzy left k-ideal (resp. fuzzy left h-ideal) of S (see Lemma 2.1 and Lemma 3.9). We now claim that $\mu(\alpha x) \geq \mu(x)$ for all $\alpha \in \Omega$ and $x \in S$. If not, then there exist $\beta \in \Omega$ and $y \in S$ such that $\mu(\beta y) < s < \mu(y)$, where $s := \frac{1}{2}(\mu(\beta y) + \mu(y))$. It follows that $y \in U(\mu; s)$, but $\beta y \notin U(\mu; s)$. This is a contradiction. Hence $\mu(\alpha x) \geq \mu(x)$ for all $\alpha \in \Omega$ and $x \in S$. This completes the proof. □

It is clear that every Ω-fuzzy left h-ideal is an Ω-fuzzy left k-ideal of an Ω-hemiring S. Let A be an Ω-left k-ideal of an Ω-hemiring S which is not an Ω-left h-ideal of S. Let μ be the fuzzy set given in Theorem 3.7. Then

$$U(\mu; r) := \begin{cases} \emptyset & \text{if } s < r, \\ A & \text{if } t < r \leq s, \\ S & \text{if } r \leq t, \end{cases}$$

and so μ is an Ω-fuzzy left k-ideal of S which is not an Ω-fuzzy left h-ideal of S.

Theorem 3.11. *Let μ and ν be fuzzy sets in an Ω-semiring (resp. Ω-hemiring) S. If they are Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S, then so is $\mu \cap \nu$, where $\mu \cap \nu$ is defined by*

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}, \ x \in S.$$

Proof. If μ and ν are Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S, then $\mu \cap \nu$ is a fuzzy left k-ideal (resp. fuzzy left h-ideal) of S (see [16, Proposition 3.6]). For any $x \in S$ and $\alpha \in \Omega$, we have

$$(\mu \cap \nu)(\alpha x) = \min\{\mu(\alpha x), \nu(\alpha x)\} \geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x).$$

Hence $\mu \cap \nu$ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S. □

We note that the intersection of all Ω-left k-ideals (resp. Ω-left h-ideals) of a semiring (resp. hemiring) S is also a Ω-left k-ideal (resp. Ω-left h-ideal) of S. Let Λ be a totally ordered set and let $\{A_t \mid t \in \Lambda\}$ be a collection of Ω-left k-ideals (resp. Ω-left h-ideals) of S such that for all $t, s \in \Lambda$, $t < s$ if and only if $A_s \subset A_t$. Then $\bigcup_{t \in \Lambda} A_t$ is a Ω-left k-ideal (resp. Ω-left h-ideal) of S. For any subset A of S, denote by $\langle A \rangle_k$ (resp. $\langle A \rangle_h$) the intersection of all Ω-left k-ideals (resp. Ω-left h-ideals) containing A. It is obvious that $\langle A \rangle_k$ (resp. $\langle A \rangle_h$) is the smallest Ω-left k-ideal (resp. Ω-left h-ideal) of S containing A. We call it the *Ω-left k-ideal* (resp. *Ω-left h-ideal*) *generated* by A.

Theorem 3.12. *Let S be an Ω-semiring (resp. Ω-hemiring) and let $\{A_t \mid t \in \Lambda \subset [0,1]\}$ be a collection of Ω-left k-ideals (resp. Ω-left h-ideals) of S such that:*

(i) $S = \bigcup_{t\in\Lambda} A_t$,

(ii) $t < s$ if and only if $A_s \subset A_t$ for all $t, s \in \Lambda$.

Define a fuzzy set μ in S by $\mu(x) := \sup\{t \in \Lambda \mid x \in A_t\}$ for all $x \in S$. Then μ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S.

Proof. For any $s \in [0,1]$, we consider the following two cases:

$$s = \sup\{t \in \Lambda \mid t < s\},\ \ s \neq \sup\{t \in \Lambda \mid t < s\}$$

For the first case, we know that $x \in U(\mu; s)$ if and only if $x \in A_t$ for all $t < s$ if and only if $x \in \bigcap_{t<s} A_t$. Hence $U(\mu; s) = \bigcap_{t<s} A_t$, which is an Ω-left k-ideal (resp. Ω-left h-ideal) of S. The second case implies that there exists $\varepsilon > 0$ such that $(s-\varepsilon, s) \cap \Lambda = \emptyset$. We claim that $U(\mu; s) = \bigcup_{t\geq s} A_t$. If $x \in \bigcup_{t\geq s} A_t$, then $x \in A_t$ for some $t \geq s$. It follows that $\mu(x) \geq t \geq s$. Hence $x \in U(\mu; s)$, showing that $\bigcup_{t\geq s} A_t \subset U(\mu; s)$. Conversely, if $x \notin \bigcup_{t\geq s} A_t$, then $x \notin A_t$ for all $t \geq s$, which implies that $x \notin A_t$ for all $t > s - \varepsilon$. This shows that if $x \in A_t$, then $t \leq s - \varepsilon$. Thus $\mu(x) \leq s - \varepsilon$, and so $x \notin U(\mu; s)$. Therefore $U(\mu; s) \subset \bigcup_{t\geq s} A_t$. Hence μ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S by Theorem 3.10. □

Using the similar method to the proof of Theorems 3.8, 3.9, and 3.10 in [16], we have the following theorems respectively.

Theorem 3.13. *Let S be an Ω-semiring (resp. Ω-hemiring) and let μ be a fuzzy set in S. Then a fuzzy set μ^* in S defined by*

$$\mu^*(x) = \sup\{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle_k\},\ \forall x \in S$$

$$(\text{resp. } \mu^*(x) = \sup\{t \in [0,1] \mid x \in \langle U(\mu; t)\rangle_h\},\ \forall x \in S)$$

is the least Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S that contains μ.

Theorem 3.14. *Let S be an Ω-semiring (resp. Ω-hemiring) and let $\{A_n \mid n \in \mathbb{N}\}$ be a family of left k-ideals (resp. left h-ideals) of S which is nested, that is, $S = A_1 \supset A_2 \supset A_3 \supset \cdots$. Let μ be a fuzzy set in S defined by*

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{for } x \in A_n \setminus A_{n+1},\quad n = 1, 2, 3, \cdots, \\ 1 & \text{for } x \in \bigcap\limits_{n=1}^{\infty} A_n \end{cases}$$

for all $x \in S$. Then μ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S.

Theorem 3.15. *Let S be an Ω-semiring (resp. Ω-hemiring) and let μ be an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S with $\mathrm{Im}(\mu) = \{t_i \mid i \in \Lambda\}$ and let $\mathbb{U} := \{U(\mu; t) \mid t \in \mathrm{Im}(\mu)\}$. Then:*

(i) S is the set-theoretic union of all $U(\mu; t) \in \mathbb{U}$.

(ii) The members of $\mathbb{U}$ form a chain.

(iii) $\mathbb{U}$ contains all level Ω-left k-ideals (resp. Ω-left h-ideals) of μ if and only if μ attains its infimum on all Ω-left k-ideals (resp. Ω-left h-ideals) of S.

Let S be an Ω-semiring (resp. Ω-hemiring) and let $\mu : S \to [0,1]$ be a fuzzy set. The smallest Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) containing μ is called the *Ω-fuzzy left k-ideal* (resp. *Ω-fuzzy left h-ideal*) *generated by* μ, and μ is said to be *n-valued* if $\mu(S)$ is a finite set of n elements. When no specific n is intended, we call μ a *finite valued* fuzzy set.

Theorem 3.16. *Let S be an Ω-semiring* (resp. *Ω-hemiring*). *An Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) ν of S is finite valued if and only if it is generated by a finite valued fuzzy set μ in S.*

Proof. If $\nu : S \to [0,1]$ is a finite valued Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S, then one may choose $\mu = \nu$. Conversely, assume that $\mu : S \to [0,1]$ is an n-valued fuzzy set with n distinct values $t_1, t_2, \cdots, t_n$, where $t_1 > t_2 > \cdots > t_n$. Let G^i be the inverse image of t_i under μ, i.e., $G^i = \mu^{-1}(t_i)$. Obviously, $\cup_{i=1}^{j} G^i \subset \cup_{i=1}^{r} G^i$ when $j < r$. Denote by A^j the Ω-left k-ideal (resp. Ω-left h-ideal) of S generated by the set $\cup_{i=1}^{j} G^i$. Then we have the following chain of Ω-left k-ideals (resp. Ω-left h-ideals):

$$A^1 \subseteq A^2 \subseteq \cdots \subseteq A^n = S.$$

Define a fuzzy set $\nu : S \to [0,1]$ by

$$\nu(x) := \begin{cases} t_1 & \text{if } x \in A^1, \\ t_j & \text{if } x \in A^j \setminus A^{j-1};\ j = 2, 3, \cdots, n. \end{cases}$$

We claim that ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S generated by μ. Let $x, y \in S$ and let i and j be the smallest integers such that $x \in A^i$ and $y \in A^j$. We may assume that $i > j$ without loss of generality. Then $x + y \in A^i$ and $yx \in A^j$, and so $\nu(yx) \geq t_j = \nu(y)$ and

$$\nu(x + y) \geq t_i = \min\{t_i, t_j\} = \min\{\nu(x), \nu(y)\}.$$

Hence ν is a fuzzy left ideal of S. Let $x, y, z \in S$ be such that $x + y = z$. If $y \in A^i$ and $z \in A^j$ for some $i < j$, then $y, z \in A^j$ and thus $x \in A^j$ since A^j is a left k-ideal of S. Hence $\nu(x) \geq t_j = \min\{\nu(y), \nu(z)\}$. If $y, z \in A^j \setminus A^{j-1}$ for $j = 2, 3, \cdots, n$, then $x \in A^j$ and so $\nu(x) \geq t_j = \min\{\nu(y), \nu(z)\}$. Thus ν is a fuzzy left k-ideal of S. Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. If $a \in A^i$ and $b \in A^j$ for some $i < j$, then $a, b \in A^j$ and so $x \in A^j$ as A^j is a left h-ideal of S. Thus $\nu(x) \geq t_j = \min\{\nu(a), \nu(b)\}$. If $a, b \in A^j \setminus A^{j-1}$ for $j = 2, 3, \cdots, n$, then $x \in A^j$. Hence $\nu(x) \geq t_j = \min\{\nu(a), \nu(b)\}$. Therefore ν is a fuzzy left h-ideal of S. Let $\alpha \in \Omega$ and $x \in S$. If $x \in A^1$, then $\alpha x \in A^1$ and so $\nu(\alpha x) = \nu(x)$. If $x \in A^j \setminus A^{j-1}$, then $\alpha x \in A^j$ and $\nu(\alpha x) \geq t_j = \nu(x)$. Consequently ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S. If $x \in S$ and $\mu(x) = t_j$, then $x \in G^j$ and so $x \in A^j$. But we get $\nu(x) \geq t_j = \mu(x)$. Consequently, $\mu \subseteq \nu$. Let γ be any Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S containing μ. Then $\cup_{i=1}^{j} G^i = U(\mu; t_j) \subseteq U(\gamma; t_j)$, and thus $A^j \subseteq U(\gamma; t_j)$. Hence $\nu \subseteq \gamma$ and ν is generated by μ. Note that $|\mathrm{Im}(\mu)| = n = |\mathrm{Im}(\nu)|$, completing the proof. □

An Ω-semiring (resp. Ω-hemiring) S is said to be *Ω-left k-Noetherian* (resp. *Ω-left h-Noetherian*) if it satisfies the ascending chain condition on Ω-left k-ideals (resp. Ω-left h-ideals) of S.

Theorem 3.17. *If An Ω-semiring (resp. Ω-hemiring) S is Ω-left k-Noetherian* (resp. *Ω-left h-Noetherian*), *then every Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S is finite valued.*

Proof. Let $\mu : S \to [0, 1]$ be an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S which is not finite valued. Then there exists an infinite sequence of distinct numbers $\mu(0) = t_1 > t_2 > \cdots > t_n > \cdots$ where $t_i = \mu(x_i)$ for some $x_i \in S$. This sequence induces an infinite sequence of distinct Ω-left k-ideals (resp. Ω-left h-ideals) of S:

$$U(\mu; t_1) \subset U(\mu; t_2) \subset \cdots \subset U(\mu; t_n) \subset \cdots .$$

This is a contradiction. □

Combining Theorem 3.16 and Theorem 3.17, we have the following corollary.

Corollary 3.18. *If An Ω-semiring (resp. Ω-hemiring) S is Ω-left k-Noetherian (resp. Ω-left h-Noetherian), then every Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S is generated by a finite fuzzy set in S.*

4. Cartesian Product of Ω-Fuzzy Left *h*-Ideals

Let S and T be Ω-semirings (resp. Ω-hemirings). We define two operations: addition and multiplication in $S \times T$ as follows:

$$(\forall x_1, y_1 \in S)\,(\forall x_2, y_2 \in T)\,((x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)),$$

$$(\forall x_1, y_1 \in S)\,(\forall x_2, y_2 \in T)\,((x_1, x_2) \cdot (y_1, y_2) = (x_1 y_1, x_2 y_2)).$$

It is easy to verify that $S \times T$ is a semiring (resp hemiring). Moreover, if we define $\alpha(x, y) = (\alpha x, \alpha y)$ for all $\alpha \in \Omega$ and $(x, y) \in S \times T$, then $S \times T$ is an Ω-semiring (resp. Ω-hemiring).

Definition 4.1. [5] A *fuzzy relation* on any set S is a fuzzy set

$$\mu : S \times S \to [0, 1].$$

Definition 4.2. (see [5]) If μ is a fuzzy relation on a set S and ν is a fuzzy set in S, then μ is a *fuzzy relation* on ν if

$$\mu(x, y) \leq \min\{\nu(x), \nu(y)\}, \quad \forall x, y \in S.$$

Definition 4.3. (see [5]) Let μ and ν be fuzzy sets in a set S. The *Cartesian product* of μ and ν is defined by

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}, \quad \forall x, y \in S.$$

Lemma 4.4. (see [5]) *Let μ and ν be fuzzy sets in a set S. Then*

(i) $\mu \times \nu$ is a fuzzy relation on S,

(ii) $U(\mu \times \nu; t) = U(\mu; t) \times U(\nu; t)$ for all $t \in [0, 1]$.

Definition 4.5. (see [5]) If ν is a fuzzy set in a set S, the *strongest fuzzy relation* on S that is a fuzzy relation on ν is μ_ν, given by

$$\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\}, \quad \forall x, y \in S.$$

Lemma 4.6. (see [5]) *For a given fuzzy set ν in a set S, let μ_ν be the strongest fuzzy relation on S. Then for $t \in [0, 1]$, we have that $U(\mu_\nu; t) = U(\nu; t) \times U(\nu; t)$.*

Proposition 4.7. *For a given fuzzy set ν in an Ω-semiring (resp. Ω-hemiring) S, let μ_ν be the strongest fuzzy relation on S. If μ_ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S \times S$, then $\nu(x) \leq \nu(0)$ for all $x \in S$.*

Proof. See [16], Proposition 4.7. □

Proposition 4.8. *Let S be an Ω-semiring (resp. Ω-hemiring). If ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S, then the level Ω-left k-ideals (resp. Ω-left h-ideals) of μ_ν are given by $U(\mu_\nu;t) = U(\nu;t) \times U(\nu;t)$ for all $t \in [0,1]$.*

Proof. Straightforward. □

Theorem 4.9. *Let S be an Ω-semiring (resp. Ω-hemiring) and let μ and ν be Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S. Then $\mu \times \nu$ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S \times S$.*

Proof. Let $(x_1,x_2),(y_1,y_2) \in S \times S$. Then

$$\begin{aligned}
&(\mu\times\nu)((x_1,x_2)+(y_1,y_2)) = (\mu\times\nu)(x_1+y_1, x_2+y_2)\\
&\quad = \min\{\mu(x_1+y_1),\ \nu(x_2+y_2)\}\\
&\quad \geq \min\{\min\{\mu(x_1),\mu(y_1)\},\ \min\{\nu(x_2),\nu(y_2)\}\}\\
&\quad = \min\{\min\{\mu(x_1),\nu(x_2)\},\ \min\{\mu(y_1),\nu(y_2)\}\}\\
&\quad = \min\{(\mu\times\nu)(x_1,x_2),\ (\mu\times\nu)(y_1,y_2)\},
\end{aligned}$$

and

$$\begin{aligned}
&(\mu\times\nu)((x_1,x_2)(y_1,y_2)) = (\mu\times\nu)(x_1y_1, x_2y_2)\\
&\qquad = \min\{\mu(x_1y_1),\ \nu(x_2y_2)\} \geq \min\{\mu(y_1),\nu(y_2)\} = (\mu\times\nu)(y_1,y_2).
\end{aligned}$$

Hence $\mu\times\nu$ is a fuzzy left ideal of $S\times S$. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in S\times S$ be such that $(x_1,x_2)+(y_1,y_2) = (z_1,z_2)$. Then $x_1+y_1 = z_1$ and $x_2+y_2 = z_2$, which imply that

$$\begin{aligned}
&(\mu\times\nu)(x_1,x_2) = \min\{\mu(x_1),\nu(x_2)\}\\
&\quad \geq \min\{\min\{\mu(y_1),\mu(z_1)\},\min\{\nu(y_2),\nu(z_2)\}\}\\
&\quad = \min\{\min\{\mu(y_1),\nu(y_2)\},\min\{\mu(z_1),\nu(z_2)\}\}\\
&\quad = \min\{(\mu\times\nu)(y_1,y_2),(\mu\times\nu)(z_1,z_2)\}.
\end{aligned}$$

Thus $\mu\times\nu$ is a fuzzy left k-ideal of S. Now let (a_1,a_2), (b_1,b_2), (x_1,x_2), $(z_1,z_2) \in S\times S$ be such that

$$(x_1,x_2)+(a_1,a_2)+(z_1,z_2) = (b_1,b_2)+(z_1,z_2),$$

i.e., $(x_1+a_1+z_1, x_2+a_2+z_2) = (b_1+z_1, b_2+z_2)$. It follows that $x_1+a_1+z_1 = b_1+z_1$ and $x_2+a_2+z_2 = b_2+z_2$ so that

$$\begin{aligned}(\mu\times\nu)(x_1,x_2) &= \min\{\mu(x_1),\nu(x_2)\}\\ &\geq \min\{\min\{\mu(a_1),\mu(b_1)\},\ \min\{\nu(a_2),\nu(b_2)\}\}\\ &= \min\{\min\{\mu(a_1),\nu(a_2)\},\ \min\{\mu(b_1),\nu(b_2)\}\}\\ &= \min\{(\mu\times\nu)(a_1,a_2),\ (\mu\times\nu)(b_1,b_2)\}.\end{aligned}$$

Therefore $\mu\times\nu$ is a fuzzy left h-ideal of $S\times S$. Let $\alpha\in\Omega$ and $(x,y)\in S\times S$. Then $\alpha(x,y)=(\alpha x,\alpha y)$, and so

$$\begin{aligned}(\mu\times\nu)(\alpha(x,y)) &= (\mu\times\nu)(\alpha x,\alpha y) = \min\{\mu(\alpha x),\nu(\alpha y)\}\\ &\geq \min\{\mu(x),\nu(y)\} = (\mu\times\nu)(x,y).\end{aligned}$$

This completes the proof. □

As the converse of Theorem 4.9, we have a question as follows: If $\mu\times\nu$ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S\times S$, then are both μ and ν Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S? We know that if $\mu\times\nu$ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S\times S$, then μ and ν need not both be Ω-fuzzy left k-ideals (resp. Ω-fuzzy left h-ideals) of S. In fact, let S be an Ω-semiring (resp. Ω-hemiring) with $|S|\geq 2$ and let $s,t\in[0,1)$ be such that $s\leq t$. Define fuzzy sets μ and ν in S by $\mu(x)=s$ and

$$\nu(x)=\begin{cases} t & \text{if } x=0,\\ 1 & \text{otherwise,}\end{cases}$$

for all $x\in S$, respectively. Then $(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}=s$ for all $(x,y)\in S\times S$, that is, $\mu\times\nu$ is a constant function and so $\mu\times\nu$ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S\times S$. Now μ is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S, but ν is not an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S.

Theorem 4.10. *Let S be an Ω-semiring (resp. Ω-hemiring). Let ν be a fuzzy set in S and let μ_ν be the strongest fuzzy relation on S. Then ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S if and only if μ_ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S\times S$.*

Proof. Assume that ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S. Let (x_1, x_2), $(y_1, y_2) \in S \times S$. Then

$$\begin{aligned}\mu_\nu((x_1, x_2) + (y_1, y_2)) &= \mu_\nu(x_1 + y_1, x_2 + y_2)\\ &= \min\{\nu(x_1 + y_1),\ \nu(x_2 + y_2)\}\\ &\geq \min\{\min\{\nu(x_1), \nu(y_1)\},\ \min\{\nu(x_2), \nu(y_2)\}\}\\ &= \min\{\min\{\nu(x_1), \nu(x_2)\},\ \min\{\nu(y_1), \nu(y_2)\}\\ &= \min\{\mu_\nu(x_1, x_2),\ \mu_\nu(y_1, y_2)\}\end{aligned}$$

and

$$\begin{aligned}\mu_\nu((x_1, x_2)(y_1, y_2)) &= \mu_\nu(x_1y_1, x_2y_2) = \min\{\nu(x_1y_1),\ \nu(x_2y_2)\}\\ &\geq \min\{\nu(y_1), \nu(y_2)\} = \mu_\nu(y_1, y_2).\end{aligned}$$

Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in S \times S$ be such that $(x_1, x_2) + (y_1, y_2) = (z_1, z_2)$. Then $x_1 + y_1 = z_1$ and $x_2 + y_2 = z_2$, which imply that

$$\begin{aligned}\mu_\nu(x_1, x_2) &= \min\{\nu(x_1), \nu(x_2)\}\\ &\geq \min\{\min\{\nu(y_1), \nu(z_1)\}, \min\{\nu(y_2), \nu(z_2)\}\}\\ &= \min\{\min\{\nu(y_1), \nu(y_2)\}, \min\{\nu(z_1), \nu(z_2)\}\}\\ &= \min\{\mu_\nu(y_1, y_2), \mu_\nu(z_1, z_2)\}.\end{aligned}$$

Now let (a_1, a_2), (b_1, b_2), (x_1, x_2), $(z_1, z_2) \in S \times S$ be such that

$$(x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2).$$

Then $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$. Thus

$$\begin{aligned}\mu_\nu(x_1, x_2) &= \min\{\nu(x_1), \nu(x_2)\}\\ &\geq \min\{\min\{\nu(a_1), \nu(b_1)\},\ \min\{\nu(a_2), \nu(b_2)\}\}\\ &= \min\{\min\{\nu(a_1), \nu(a_2)\},\ \min\{\nu(b_1), \nu(b_2)\}\}\\ &= \min\{\mu_\nu(a_1, a_2),\ \mu_\nu(b_1, b_2)\}.\end{aligned}$$

Let $\alpha \in \Omega$ and $(x, y) \in S \times S$. Then

$$\begin{aligned}\mu_\nu(\alpha(x, y)) &= \mu_\nu(\alpha x, \alpha y)\\ &= \min\{\nu(\alpha x), \nu(\alpha y)\} \geq \min\{\nu(x), \nu(y)\} = \mu_\nu(x, y).\end{aligned}$$

Hence μ_ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S \times S$. Conversely, suppose that μ_ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of $S \times S$. Let x_1, x_2, y_1, $y_2 \in S$. Then

$$\begin{aligned}\min\{\nu(x_1 + y_1),\ \nu(x_2 + y_2)\} &= \mu_\nu(x_1 + y_1, x_2 + y_2)\\ &\geq \min\{\mu_\nu(x_1, x_2),\ \mu_\nu(y_1, y_2)\}\\ &= \min\{\min\{\nu(x_1), \nu(x_2)\},\ \min\{\nu(y_1), \nu(y_2)\}\},\end{aligned}$$

which implies that

$$\nu(x_1+y_1) \geq \min\{\min\{\nu(x_1),\nu(x_2)\},\ \min\{\nu(y_1),\nu(y_2)\}\}.$$

In this inequality, we specialize the values of x_1, x_2, y_1, and y_2 as follows: $x_1 = x$, $x_2 = 0$, $y_1 = y$, and $y_2 = 0$. Then we have

$$\nu(x+y) \geq \min\{\min\{\nu(x),\nu(0)\},\ \min\{\nu(y),\nu(0)\}\} = \min\{\nu(x),\nu(y)\}$$

by using Proposition 4.7. Next, we have

$$\begin{aligned}\min\{\nu(x_1y_1),\ \nu(x_2y_2)\} &= \mu_\nu(x_1y_1, x_2y_2)\\ &= \mu_\nu((x_1,x_2)(y_1,y_2)) \geq \mu_\nu(y_1,y_2) = \min\{\nu(y_1),\nu(y_2)\},\end{aligned}$$

and so $\nu(x_1y_1) \geq \min\{\nu(y_1),\ \nu(y_2)\}$. Taking $x_1 = x$, $y_1 = y$, and $y_2 = 0$, and using Proposition 4.7, we get

$$\nu(xy) \geq \min\{\nu(y),\ \nu(0)\} = \nu(y).$$

Hence ν is a fuzzy left ideal of S. Let $x, y, z \in S$ be such that $x + y = z$. Then $(x, 0) + (y, 0) = (z, 0)$. Since μ_ν is a fuzzy left k-ideal of $S \times S$, we have

$$\begin{aligned}\nu(x) &= \min\{\nu(x),\nu(0)\} = \mu_\nu(x,0) \geq \min\{\mu_\nu(y,0),\mu_\nu(z,0)\}\\ &= \min\{\min\{\nu(y),\nu(0)\},\min\{\nu(z),\nu(0)\}\} = \min\{\nu(y),\nu(z)\}.\end{aligned}$$

Let $a, b, x, z \in S$ be such that $x+a+z = b+z$. Then $(x,0)+(a,0)+(z,0) = (b,0)+(z,0)$. Since μ_ν is a fuzzy left h-ideal of $S\times S$, it follows from Proposition 4.7 that

$$\begin{aligned}\nu(x) &= \min\{\nu(x),\ \nu(0)\} = \mu_\nu(x,0) \geq \min\{\mu_\nu(a,0),\ \mu_\nu(b,0)\}\\ &= \min\{\min\{\nu(a),\nu(0)\},\ \min\{\nu(b),\nu(0)\}\} = \min\{\nu(a),\nu(b)\}.\end{aligned}$$

Let $\alpha \in \Omega$ and $x, y \in S$. Then

$$\begin{aligned}\min\{\nu(\alpha x),\nu(\alpha y)\} &= \mu_\nu(\alpha x,\alpha y) = \mu_\nu(\alpha(x,y)) \geq \mu_\nu(x,y)\\ &= \min\{\nu(x),\nu(y)\},\end{aligned}$$

and so $\nu(\alpha x) \geq \nu(x)$ by taking $x = y$. Consequently, ν is an Ω-fuzzy left k-ideal (resp. Ω-fuzzy left h-ideal) of S. This completes the proof. □

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