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A CONSTRUCTION OF LIMITS AND COLIMITS OF TOPOLOGICAL STRUCTURES

Martin Dowd
1613 Wintergreen Pl.
Costa Mesa, CA 92626, USA
e-mail: MartDowd@aol.com

Abstract: A construction of limits and colimits in the category of topological structures over a first order language is given. The construction readily specializes to the full subcategory of models of an equational theory.

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1. Introduction

Let L be a first order language, which for simplicity will be assumed not to have relation symbols. Let Struct_L be the category of structures in L. An object is a set S, with each n-ary function of L interpreted as a an n-ary function on S. A morphism is a homomorphism of structures (i.e., "preserves the functions").

Let TopS_L denote the category where:

1. S is equipped with a topology,

2. the functions of the structure are continuous (the structure and the topology are "compatible"), and

3. morphisms are continuous as well as structure preserving.

It is well-known that a diagram in Struct_L has a limit or colimit. We will show that this can be equipped with a topology, such that the resulting topological structure, together with the limit or colimit cone in Struct_L, comprise a limit or colimit in TopS_L.

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Let T be a set of (implicitly universally quantified) equations in L. Let Mdl_T denote the full subcategory of Struct_L consisting of the models of T. Let TopM_T denote the full subcategory of TopS_L. The construction of limits and colimits in Struct_L can be adapted to Mdl_T. We will show that in a similar way the construction of limits and colimits in TopS_L can be adapted to TopM_T.

In particular a simple description results of the amalgamated product in the category of topological groups, which is stated to be lacking in Khan and Morris [1].

2. Substructures

Lemma 1. *Given X in TopS_L and a substructure $W \subseteq X$, the subspace topology is compatible with the substructure.*

Proof. Suppose $V \subseteq W$ is open, and n is an n-ary function; let f_r denote the restriction to W^n. Then $w \in f_r^{-1}[V]$ iff $w \in W^n$ and $w \in f^{-1}[V]$. That is, $f_r^{-1}[V] = W^n \cap f^{-1}[V]$, so $f_r^{-1}[V]$ is open in W^n. □

3. Products

Lemma 2. *Given objects X_i in TopS_L, the product topology is compatible with the product structure. The product topology is that induced by the projections. The product structure equipped with the product topology is the product in TopS_L.*

Proof. Let U be the subbasic open set which is U_j in component j and X_i for $i \neq j$; suppose f is n-ary. Then $f(\langle x_{1i}\rangle, \ldots, \langle x_{ni}\rangle) \in U$ iff $f(x_{1j} \ldots, x_{nj}) \in U_j$. So $f^{-1}[U]$ is in fact the subbasic open set which is $f^{-1}[U_j]$ in component j and X_i^n for $i \neq j$, and in particular is open. The product topology is that induced by the projections since this is true of the topological spaces. Thus, given a cone from the X_i to any $Y \in \mathrm{TopS}_L$, the induced map from Y to $\times_i X_i$ (the product in Struct_L) is continuous. □

4. Limits

Theorem 3. *The limit of a diagram in TopS_L is the limit in Struct_L, equipped with the topology induced by the limit cone.*

Proof. The equalizer in Set is the equalizer in TopS_L. □

5. Quotients

Lemma 4. *Given X in TopS_L and a congruence relation $\equiv$ on X, the quotient topology is compatible with the quotient structure.*

Proof. For $x \in X$ let $\bar{x}$ denote the image under the canonical epimorphism, and extend the notation to subsets of X, and to the functions of the structure. Suppose $\bar{f}(\bar{x}_1, \ldots, \bar{x}_n) \in \bar{V}$ where $\bar{V}$ is open in $\bar{X}$. Then $f(x_1, \ldots, x_n) \in V$ (where $V = \cup\bar{V}$ is the saturated open set whose image is $\bar{V}$). Thus there are open sets $U_1, \ldots, U_n$ such that if $x_k \in U_k$ for $1 \le k \le n$ then $f(x_1, \ldots, x_n) \in V$. Since V is saturated and $\equiv$ is a congruence relation, U_k may be replaced by its saturation. Then if $\bar{x}_k \in \bar{U}_k$ for $1 \le k \le n$ then $\bar{f}(\bar{x}_1, \ldots, \bar{x}_n) \in \bar{V}$. □

6. Coproducts

So far everything has been determined by forgetful functors; this is no longer the case for the coproduct. The underlying structure is the coproduct structure, but it must be given a topology specific to the new category.

Let t denote a term over L. We suppose that the distinct variables from left to right are $x_1, \ldots, x_r$. Let H denote the set of closed terms $\hat{t}$ of the form $t(c_1, \ldots c_r)$, where $c_l \in X_{i_l}$. This is a structure in an obvious and well-known manner. We define the topology T_H on H as that whose subbasic open sets are the sets $U_{t,U_1,\ldots U_r}$, where U_l is an open set of X_{i_l}. The closed terms in this open set are those, where $c_l \in U_l$. From hereon H is supposed to be equipped with this topology.

Lemma 5. *If f is an n-ary function of L then f is a continuous function on H.*

Proof. A closed term $f(\hat{t}_1, \ldots, \hat{t}_n)$ is in a subbasic open set iff the term t has the closed term as an instance, and the open sets contain the constants. An (subbasic) open set for $\hat{t}_k$ is obtained by taking t_k as the term, and the open sets for its constants as for t. □

Lemma 6. *The map $X_i \mapsto H$ mapping c to itself is continuous.*

Proof. The inverse image of $U_{c,W}$ is W.

Let $\equiv$ be the equivalence relation on H, where two closed terms are equal iff all their constants are from the same structure and they are equal in the structure. This is easily seen to be a congruence relation in the structure H. It is not difficult to show that $H/\equiv$ is the coproduct in Struct_L (see Dowd [1] for example). □

Theorem 7. *$H/\equiv$, equipped with the quotient topology, is the coproduct in TopS_L.*

Proof. Given $K \in \mathrm{TopS}_L$ and TopS_L morphisms $\nu_i : X_i \mapsto K$, there is a unique Struct_L morphism $\phi : H \mapsto K$ such that $\phi(c) = \nu_i(c)$ for each $c \in X_i$. Suppose $V \subseteq K$ is open and $\phi(\hat{t}) \in V$. Write $\hat{t}$ as $t(c_1, \ldots, c_r)$ where $c_l \in X_{i_l}$. Let t_K be the function on K interpreting t. Since t_K is continuous there are open sets $U_l \subseteq X_{i_l}$ such that for $\langle c'_1, \ldots, c'_r \rangle \in U_1 \times \cdots \times U_r$, $t_K(\nu_{i_1}(c'_1), \ldots, \nu_{i_r}(c'_r)) \in V$. Thus, $\phi[U_{t,U_1,\ldots U_r}] \subseteq V$, and ϕ has been shown to be continuous. If $d = f(c_1, \ldots, c_n)$ in X_i then $\phi(d) = \phi(f(c_1, \ldots, c_n))$, so ϕ respects $\equiv$. Thus, ϕ factors through the canonical epimorphism in Top, and the theorem follows. □

7. Colimits

Theorem 8. *The colimit of a diagram in* TopS_L *is obtained from the coproduct and the coequalizer in the usual way.*

Proof. Immediate. □

8. Equational Theories

Theorem 3 holds in TopM_T because substructures and products of models are models. Theorem 7 must be modified, in a standard manner, namely, the relation $\equiv$ is modified to include the equivalences arising from T.

References

[1] M. Dowd, Higher type categories, *Mathematical Logic Quarterly*, **39** (1993), 251-254.

[2] M.S. Khan, Sidney A. Morris, Free Products of Topological Groups with Central Amalgamation, I, *Trans. Amer. Math. Soc.*, **273** (1982), 405-416.