A CLASS OF I-TOPOLOGICAL SPACES INDUCED FROM METRIC SPACES USING FUZZY POINTS

Jung-Chan Chang1§, Hsinjung Chen2, Wei-Cheng Lian3

1Department of Applied Mathematics
I-Shou University
Ta-Hsu, Kaohsiung, 84008, TAIWAN, R.O.C.
e-mail: jcchang@isu.edu.tw

2,3Department of Information Management
National Kaohsiung Marine University
Hai Chuan Road 142, Kaohsiung, 811, TAIWAN, R.O.C.
2 e-mail: htchen@mail.nkmu.edu.tw
3 e-mail: wclian@mail.nkmu.edu.tw

Abstract: In this article we discuss the corresponding relation between I-topology and crisp topology. We also establish some characteristic of the I-open and I-closed sets. The relations between continuous mappings and I-continuous mappings are also discussed.

AMS Subject Classification: 54A05, 54A40
Key Words: I-topology, I-continuous function, I-open mapping theorem

1. Introduction

The fundamental concepts of a fuzzy set, introduced by Zadeh in 1965 [15], provides a natural foundation for treating mathematically the fuzzy phenomena which exist pervasively in our real world and for building new branches of fuzzy mathematics. In the area of I-topology (here I denotes the interval [0, 1] in R), much research has been carried Chang [2] since 1968, which is called fuzzy topology due to Chang [2].

Earlier work on I-topology was concentrated on generalizing topology to set-
theoretic behavior almost identical to that of crisp sets. Therefore the concept of \( I \)-topology was collection of \( I \)-topology results valid in a fuzzy setting within this span of time, or has been able to not solve within the framework of general topology. For the detail, see Chang [3], [4], [11], [12], etc.

According to Chang’s definition, the \( I \)-topology is a direct generalization of the fuzzification crisp topology. But it is difficult to discuss the topological properties of the space. In order to avoid these difficulties, we derive the \( I \)-topology as the product topology by means of fuzzy points, it depends on the topological properties of the underlying interval \( I \). We construct a class of \( I \)-topology which is based on corresponding relation between \( I \)-topology and crisp topology (for the \( I \)-topology and fuzzy points, we refer the reader to the excellent book of Höhle and Rodabaugh(Eds) [9]). Using the crisp topology as a stepping-stone, it is more convenient to discuss the topological properties of this \( I \)-topology.

In Section 2, we construct the \( I \)-topology using fuzzy points, and establish some characteristics of \( I \)-open and \( I \)-closed sets. In Section 3, we define the class of \( I \)-continuous functions and discuss the relation of continuous functions and \( I \)-continuous functions.

### 2. The Construction of \( I \)-Topology

Suppose that \((X, d_X)\) is a metric space. A fuzzy set is a function from \( X \) into \( I \). The class of all fuzzy sets is denoted by \( I^X \). Next, we give the definition of fuzzy points which play an important role in our paper.

**Definition 1.** We say that \( x^*_\alpha, 0 < \alpha \leq 1, \) is a fuzzy point of \( X \) if

\[
x^*_\alpha(y) = \begin{cases} 
\alpha, & y = x, \\
0, & y \neq x,
\end{cases}
\]

for each \( x \in X \). We use \( X^* \) to denote the set of all fuzzy points.

It is obvious that \( X^* \) is a subset of \( I^X \).

Let us define \( B_{\epsilon, \delta}(x^*_\alpha) = \{y^*_\beta | d_X(x, y) < \epsilon, |\beta - \alpha| < \delta\} \), where \( x^*_\alpha \in X^* \) and \( \epsilon, \delta > 0 \). We have the following crisp topological property.

**Theorem 2.** Let \( \Gamma = \{B_{\epsilon, \delta}(x^*_\alpha) | x^*_\alpha \in X^* \text{and} \epsilon, \delta > 0\} \). Then \( \Gamma \) forms a crisp topology base on \( X^* \).

**Proof.** Let \( z^*_\gamma \in B_{\epsilon_1, \delta_1}(x^*_\alpha) \cap B_{\epsilon_2, \delta_2}(y^*_\beta) \). We have \( d_X(z, x) < \epsilon_1, d_X(z, y) < \epsilon_2, |\gamma - \alpha| < \delta_1 \) and \( |\gamma - \beta| < \delta_2 \). Let \( \epsilon_3 = \min(\epsilon_1 - d_X(z, x), \epsilon_2 - d_X(z, y)) \)
and let $\delta_3 = \min(\delta_1 - |\gamma - \alpha|, \delta_2 - |\gamma - \beta|)$. We shall show that $B_{\epsilon_3,\delta_3}(z^\ast) \subset B_{\epsilon_1,\delta_1}(x^\ast) \cap B_{\epsilon_2,\delta_2}(y^\ast)$. Then $\Gamma$ forms a crisp topology base. Suppose that $t^\ast_i \in B_{\epsilon_3,\delta_3}(z^\ast)$, then $d_X(t, z) < \epsilon_3$ and $|\gamma - \lambda| < \delta_3$. It follows that $d_X(t, x) \leq d_X(t, z) + d_X(z, x) \leq \epsilon_3 + d_X(z, x) \leq \epsilon_1 - d_X(z, x) + d_X(z, x) = \epsilon_1$. Similarly, we can show that $d_X(t, y) < \epsilon_2$, $|\lambda - \alpha| < \delta_1$ and $|\beta - \lambda| < \delta_2$. It follows that $\Gamma$ forms a crisp topology base.

**Notations.** We use $T_X^\ast$ to denote the crisp topology generated by $\Gamma$ on $X^*$ and $(X^*, T_X^\ast)$ denotes the crisp topological space $X^*$ with crisp topology $T_X^\ast$.

Now we recall the definition of $I$-topology on $X$.

**Definition 3.** A family $T_X \subseteq I^X$ is called an $I$-topology on $X$ if $T_X$ satisfies the following conditions:

1. $0_x, 1_x \in T_X$, where $0_x : x \mapsto 0$, $\forall x \in X$ and $1_x : x \mapsto 1$, $\forall x \in X$,
2. $A \cap B \in T_X$, whenever $A, B \in T_X$,
3. $\bigvee \{A_i\} \in T_X$, whenever each $A_i \in T_X$, $i \in J$ for any index family $J$.

**Definition 4.** Let $T_X \subseteq I^X$ be an $I$-topology on $X$. We call every $A \in T_X$ an $I$-open set. The complement $A' \in I^X$ of $A$ defined by $A'(x) := 1 - A(x)$ is called an $I$-closed set.

The following theorem is the main result of this section. It establishes a connection of $I$-topology on $X$ and the crisp topological space $(X^*, T_X^\ast)$.

**Theorem 5.** Suppose that $A \in I^X$ and $E(A)$ denotes the set $E(A) := \{x^\ast_\alpha | x^\ast_\alpha \in X^*, \alpha < A(x)\}$. Then the collection $T_X := \{A | A \in I^X, E(A) \text{ is open in } (X^*, T_X^\ast)\}$ forms an $I$-topology on $X$.

**Proof.** The fuzzy set $0_x$ and $1_x \in T_X$ since $E(0_x) = \emptyset$ and $E(1_x) = X^* \setminus \{x^\ast_1\}$ are open in $(X^*, T_X^\ast)$. Next, suppose that $A$ and $B$ belong to $T_X$, then $E(A)$ and $E(B)$ are open in $(X^*, T_X^\ast)$. Moreover,

$$E(A \cap B) = \{x^\ast_\alpha | \alpha < (A \cap B)(x), x \in X\}$$

$$= \{x^\ast_\alpha | \alpha < A(x) \text{ and } \alpha < B(x), x \in X\}$$

$$= \{x^\ast_\alpha | \alpha < A(x), x \in X\} \cap \{x^\ast_\alpha | \alpha < B(x), x \in X\} = E(A) \cap E(B)$$

is open in $(X^*, T_X^\ast)$.

Finally, we claim that $E(\bigvee_{i \in J} A_i) = \bigcup_{i \in J} E(A_i)$ for each $A_i \in T_X, i \in J$ and any index set $J$. If it is true, then $E(\bigvee_{i \in J} A_i)$ will be open in $(X^*, T_X^\ast)$ since $\bigcup_{i \in J} E(A_i)$ is open in $(X^*, T_X^\ast)$. It follows that $T_X$ forms an $I$-topology on $X$.  

Let $x^*_\alpha \in E(\bigvee_{i \in J} A_i)$. Then $\alpha < \bigvee_{i \in J} (A_i)(x)$. It follows that $\alpha < (A_i)(x)$ for some $i \in J$ by the definition of union of fuzzy sets. Thus $x^*_\alpha \in E(A_i)$ for some $i \in J$. Then $x^*_\alpha \in \bigcup_{i \in J} E(A_i)$. Hence, $E(\bigvee_{i \in J} A_i) \subseteq \bigcup_{i \in J} E(A_i)$. Conversely, let $x^*_\alpha \in \bigcup_{i \in J} E(A_i)$. Then $x^*_\alpha \in E(A_i)$ and $\alpha < (A_i)(x)$ for some $i \in J$. Thus, $\alpha < (\bigvee_{i \in J} (A_i))(x)$, i.e. $x^*_\alpha \in E(\bigvee_{i \in J} A_i)$ and $\bigcup_{i \in J} E(A_i) \subseteq E(\bigvee_{i \in J} A_i)$. Therefore, we complete the proof. 

**Remark 6.** Theorem 5 does not hold for general lattice $L$. Since $E(\bigvee_{i \in J} A_i) = \bigcup_{i \in J} E(A_i)$ may not be true.

**Notation.** We use $(I^X, T_X)$ to denote the $I$-topological space $X$ with the $I$-topology $T_X$ in Theorem 5.

In the Theorem 5, we introduce a new technique which can replace the $I$-open set by the open set of crisp topology. By similar tools we also can establish the relation between $I$-closed sets by means of the closed sets in $(X^*, T_X^*)$.

**Definition 7.** Let $A \in I^X$. Define $C(A) = \{x^*_\alpha | x^*_\alpha \in X^*, \alpha \leq A(x)\}$.

**Remark 8.** (1) Define $\bigvee \emptyset = 0$. Then $A(x) = \bigvee \{\alpha | x^*_\alpha \in C(A)\} = \bigvee \{\alpha | x^*_\alpha \in E(A)\}$ for fixed $x \in X$ and each $A \in I^X$.

(2) $E(A) = E(B) \iff C(A) = C(B) \iff A = B$ for $A, B \in I^X$.

(3) It is obvious that $(x_n)^*_\alpha \rightarrow x^*_\alpha$ in $(X^*, T_X^*)$ iff $x_n \rightarrow x$ in $(X, d)$ and $\alpha_n \rightarrow \alpha$ in $(I, | \cdot |)$ for each sequence $\{(x_n)^*_\alpha\} \subseteq X^*$ (Here $| \cdot |$ denotes the absolute value function).

**Theorem 9.** Let $A \in I^X$. $A$ is $I$-closed in $(I^X, T_X)$ iff $C(A)$ is closed in $(X^*, T_X^*)$.

**Proof.** "\Rightarrow" Suppose that $A \in I^X$ is an $I$-closed set and $\{(x_n)^*_\alpha\}$ is a convergent sequence in $C(A)$ such that $\{(x_n)^*_\alpha\}$ converges to $x^*_\alpha$ in the crisp topology $(X^*, T_X^*)$. We shall show that $x^*_\alpha \in C(A)$, and it will imply $C(A)$ is closed in $(X^*, T_X^*)$. Suppose not, $x^*_\alpha \notin C(A)$, then it will imply $\alpha > A(x)$. So, $1-\alpha < A'(x)$ and $x_{1-\alpha} \in E(A')$. By the definition of $A$ and Theorem 5, $E(A')$ is open. Hence, there exists an open ball $B_{\epsilon, \delta}(x_{1-\alpha})$ such that $B_{\epsilon, \delta}(x_{1-\alpha}) \subset E(A')$. Since $\{(x_n)^*_\alpha\}$ converges to $x^*_\alpha$, we see that $\{(x_n)^*_\alpha\}$ converges to $x^*_\alpha$. Hence, there exists a nature number $N$ such that $(x_n)^*_1_{-\alpha} \in B_{\epsilon, \delta}(x_{1-\alpha}) \subset E(A')$ for each $n > N$. That is, $1-\alpha_n < A'(x_n)$ for $n > N$. Hence, $\alpha_n > A(x_n)$, i.e., $(x_n)^*_\alpha \notin C(A)$ for $n > N$, which contradicts that $\{(x_n)^*_\alpha\}$ is a convergent
sequence in \( C(A) \). Therefore \( x^*_\alpha \in C(A) \). We derive that \( C(A) \) is closed in \( (X^*, T^*_X) \).

"\( \Leftarrow \)" Suppose that \( C(A) \) is closed. If \( A' \) is not an \( I \)-open set, then \( E(A') \) will be not open in \( (X^*, T^*_X) \). There exist \( x^*_\alpha \in E(A') \) and \( N \in \mathcal{N} \) such that \( B_{\frac{1}{N}}(x^*_\alpha) \) is not contained in \( E(A') \) for each \( n > N \). Choose \( (x_n)^*_\alpha \in B_{\frac{1}{n}}(x^*_\alpha) \) but \( (x_n)^*_\alpha \notin E(A') \). Then it is clear that \( \{(x_n)^*_\alpha\} \) converges to \( x^*_\alpha \). Thus \( \{(x_n)^*_{1-\alpha}\} \) converges to \( x^*_{1-\alpha} \) in \( (X^*, T^*_X) \). Since \( (x_n)^*_\alpha \notin E(A') \) for \( n > N \), it follows that \( 1 - \alpha_n < A(x_n) \). So, \( (x_n)^*_{1-\alpha_n} \in E(A) \) for each \( n \in \mathcal{N} \). Since \( C(A) \) is closed and \( \{(x_n)^*_{1-\alpha}\} \) converges to \( x^*_{1-\alpha} \), we have that \( x^*_{1-\alpha} \in C(A) \). It implies that \( 1 - \alpha \leq A(x) \), i.e. \( \alpha \geq A'(x) \). Thus \( x^*_\alpha \notin E(A') \). It contradicts to \( x^*_\alpha \in E(A') \). So, \( E(A') \) is open. \( \Box \)

Finally, it follows from Theorem 5 and Theorem 9 that

**Theorem 10.** Let \( (X^*, T^*_X) \) be the crisp topology generated by the base \( \Gamma \) and let \( T_X \) be the \( I \)-topology generated by \( (X^*, T^*_X) \). Then the followings are equivalent:

(a) \( A \) is \( I \)-open in \( (I^X, T_X) \).
(b) \( E(A) \) is open in \( (X^*, T^*_X) \).
(c) \( A' \) is \( I \)-closed in \( (I^X, T_X) \).
(d) \( C(A') \) is closed in \( (X^*, T^*_X) \).

In the following theorem, we shall show that an \( I \)-open set \( A \) in \( (I^X, T_X) \) can be viewed as a crisp open set in \( X \times (0, 1] \) by using \( E(A) \).

**Theorem 11.** Let \( A \) be a fuzzy set of \( X \). Then \( S = \{(x, \alpha)| x^*_\alpha \in E(A)\} \) is open in the crisp topological space \( X \times (0, 1] \) with the product topology iff \( A \) is an \( I \)-open set in \( (I^X, T_X) \).

**Proof.** By the definition of \( (I^X, T_X) \) and crisp product topology, we see that

\[
S \text{ is open in } X \times (0, 1] \\
\iff \text{ for each } (x, \alpha) \in S \text{ there exist } \delta, \epsilon > 0 \text{ s.t. } \\
\{(y, \beta)| d_X(x, y) < \epsilon \text{ and } |\alpha - \beta| < \delta\} \subset S \\
\iff \text{ there exist } \delta, \epsilon > 0 \text{ s.t. } B_{\epsilon, \delta}(x^*_\alpha) \subset E(A) \\
\iff A \text{ is an } I \text{-open set in } (I^X, T_X). \quad \Box
\]

Finally, we give some examples.

**Example 1.** Let \( X = \mathcal{R} \) with the metric \( d(x, y) = |x - y| \). Let \( A \) be the fuzzy set defined by \( A(x) = \frac{1}{2} \) for each \( x \in \mathcal{R} \). We can find \( S = \{(x, \alpha)| x^*_\alpha \in E(A)\} = \mathcal{R} \times (0, \frac{1}{2}) \). By Theorem 11, \( A \) is an \( I \)-open set in \( (I^X, T_X) \).
Example 2. Let $X = \mathbb{R}$ with the metric $d_X(x, y) = |x - y|$. Let $A$ be the fuzzy set defined by

$$A(x) = \begin{cases} 
\frac{1}{2}, & x \leq 0, \\
1, & x > 0.
\end{cases}$$

We can find that $S = \{(x, \alpha)|x^*_α \in E(A)\} = (-\infty, 0] \times (0, \frac{1}{2}) \cup (0, \infty) \times (0, 1]$ is an open set in $\mathbb{R} \times (0, 1]$. By Theorem 11, $A$ is an $I$-open set in $(I^X, T_X)$.

Define

$$B(x) = \begin{cases} 
\frac{1}{2}, & x \neq 0, \\
1, & x = 0.
\end{cases}$$

We can find that $S = \{(x, \alpha)|x^*_α \in E(B)\} = (-\infty, 0) \times (0, \frac{1}{2}) \cup (0, \infty) \times (0, \frac{1}{2}) \cup \{0\} \times (0, 1]$ is not an open set in $\mathbb{R} \times (0, 1]$. By Theorem 11, $B$ is not an $I$-open set in $(I^X, T_X)$.

3. $I$-Continuous Functions

In this section, all topologies are the same as those in Section 2. Suppose that $(X, d_X)$ and $(Y, d_Y)$ are two metric spaces. If $f$ is a function from $X$ into $Y$, then we can define the function $f^- : I^X \to I^Y$ by

$$f^- (A)(y) = \bigvee_{f(x) = y} A(x),$$

for $A \in I^X$ by Zadeh extension. In particular, we use $(f^-)^*$ to denote the restriction of $f^-$ on $X^*$, i.e. $(f^-)^* : X^* \to Y^*$ is defined by $(f^-)^*(x^*_α) = y^*_α$ if $f(x) = y$.

We want to show that some crisp topological properties of functions will be preserved under Zadeh extension, for example, continuity and open mapping.

**Theorem 12.** If $f : (X, d_X) \to (Y, d_Y)$ is a continuous function, then $(f^-)^*$ is a continuous function from $(X^*, T^*_X)$ into $(Y^*, T^*_Y)$.

**Proof.** Let $\{(x_n)_{α_n}\}$ be a sequence which converges to $x^*_α$ in $(X^*, T^*_X)$. So, $x_n \to x$ and $α_n \to α$. By the continuity of $f$, we see that $f(x_n) \to f(x)$. Hence, $(f^-)^*(x_n)_{α_n} \to (f^-)^*(x^*_α)$ and it follows that $(f^-)^*$ is a continuous function from $(X^*, T^*_X)$ into $(Y^*, T^*_Y)$.

Now, we recall the definition of the $I$-continuous functions.
Definition 13. Let \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \) be a function. \( f \) is called \( I \)-continuous if \( f^{-1}(B) \in \mathcal{T}_X \) for each \( B \in \mathcal{T}_Y \), where \( f^{-1}(B) \in I^X \) is the fuzzy set in \( X \) defined by \( f^{-1}(B)(x) = B(f(x)) \) for each \( x \in X \).

We also define \((f^{-1})^*(y^*_\alpha)\) to be the crisp set \( \{x^*_\alpha | f(x) = y\} \). Moreover, we define
\[
(f^{-1})^*(E(A)) = \{(f^{-1})^*(x^*_\alpha)|x^*_\alpha \in E(A)\}
\]
for each \( A \in I^X \) and
\[
(f^{-1})^*(E(B)) = \{x^*_\alpha|(f^{-1})^*(x^*_\alpha) \in E(B)\}
\]
for each \( B \in I^Y \).

**Lemma 14.** Suppose that \( f \) is a function from \( X \) into \( Y \). Then \( f^{-} \) has the following properties:

(a) \( E(f^{-1}(A)) = (f^{-})^*(E(A)) \) for each \( A \in I^X \).

(b) \( E(f^{-1}(B)) = (f^{-})^*(E(B)) \) for each \( B \in I^Y \).

**Proof.** (a) Suppose that \( y^*_\alpha \in (f^{-})^*(E(A)) \), then there exists an \( x^*_\alpha \in E(A) \) such that \((f^{-})^*(x^*_\alpha) = y\). Hence, \( f(x) = y \) and \( \alpha < A(x) \). It implies \( \alpha < \bigvee A(x) \). Therefore, \( \alpha < (f^{-}(A))(y) \). Thus \( y^*_\alpha \in E(f^{-1}(A)) \) and
\[
E(f^{-1}(A)) \supset (f^{-})^*(E(A)).
\]
Conversely, suppose that \( y^*_\alpha \in E(f^{-1}(A)) \), we have \( \alpha < (f^{-}(A))(y) \). So, \( \alpha < A(x) \) for some \( x \) with \( f(x) = y \). Therefore, there exists an \( x^*_\alpha \) such that \( x^*_\alpha \in E(A) \) and \( y^*_\alpha = (f^{-})^*(x^*_\alpha) \in (f^{-})^*(E(A)) \). We obtain that \( E(f^{-1}(A)) \subset (f^{-})^*(E(A)) \).

(b) Let \( B \in I^Y \). Then
\[
x^*_\alpha \in (f^{-})^*(E(B)) \iff (f^{-})^*(x^*_\alpha) \in E(B)
\]
\[
\iff (f(x))^*_\alpha \in E(B) \iff \alpha < B(f(x))
\]
\[
\iff \alpha < (f^{-}(B))(x) \iff x^*_\alpha \in E(f^{-}(B)).
\]
This shows that \( E(f^{-1}(B)) = (f^{-})^*(E(B)) \) for each \( B \in I^Y \).

Now, we discuss the main results of this section.

**Theorem 15.** If \( f : (X, d_X) \to (Y, d_Y) \) is a continuous function, then \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \) is an \( I \)-continuous function.

**Proof.** Suppose that \( B \in \mathcal{T}_Y \), then \( E(B) \in \mathcal{T}_Y^* \). We see that \((f^{-})^*(E(B)) \in \mathcal{T}_X^* \) by Theorem 12. Moreover, by Lemma 14(b), we see that \( E(f^{-1}(B)) = (f^{-})^*(E(B)) \in \mathcal{T}_X^* \) and it shows that \( f \) is an \( I \)-continuous function.
We say that \( f : (X, d_X) \to (Y, d_Y) \) is an open mapping if \( f(V) \) is open in \( (Y, d_Y) \) whenever \( V \) is open in \( (X, d_X) \). Similarly, we say that \( f \) is an \( I \)-open mapping if \( f^{-1}(A) \) is an \( I \)-open set whenever \( A \) is an \( I \)-open set. In the rest of this section, we will show that \( f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \) is an \( I \)-open mapping whenever \( f \) is an open mapping from \( (X, d_X) \) to \( (Y, d_Y) \).

**Lemma 16.** Suppose that \( f : (X, d_X) \to (Y, d_Y) \) is an open mapping, then \( (f^-)^* : (X^*, \mathcal{T}_X^*) \to (Y^*, \mathcal{T}_Y^*) \) is an open mapping.

**Proof.** Let \( U \) be an open set in \((X^*, \mathcal{T}_X^*)\) and let \( y_\alpha^* \in (f^-)^*(U) \). There exists an \( x_\alpha^* \in U \) such that \((f^-)^*(x_\alpha^*) = y_\alpha^*\). Moreover, let \( B_{\epsilon, \delta}(x_\alpha^*) \) be a open ball of \( x_\alpha^* \) such that \( B_{\epsilon, \delta}(x_\alpha^*) \subseteq U \) and let \( V = \{ b \in X \mid \text{there exists } \gamma \text{ such that } b_\gamma^* \in B_{\epsilon, \delta}(x_\alpha^*) \} \). Then \( V \) is an open set in \((X, d_X)\) by the definition of \( B_{\epsilon, \delta}(x_\alpha^*) \). Since \( f \) is an open mapping, we have that \( f(V) \) is an open set in \( Y \). So, we can obtain that the set \( S = \{ b_\gamma^* ; b \in f(V), \ |\gamma - \alpha| < \delta \} \) is open in \((Y^*, \mathcal{T}_Y^*)\). We claim that \( S \subseteq (f^-)^*(U) \). If it is true, then \( S \) is an open neighborhood of \( y_\alpha^* \), this shows that \((f^-)^*(U) \) is open. And then we can conclude that \((f^-)^* \) is an open mapping.

Let \( b_\gamma^* \in S \). Then there exists \( a \in V \) such that \( f(a) = b \) and \(|\gamma - \alpha| < \delta\). So, \( a_\gamma \in B_{\epsilon, \delta}(x_\alpha^*) \). It follows that

\[
b_\gamma^* = (f(a))_\gamma^* = (f^-)^*(a_\gamma^*) \in (f^-)^*(B_{\epsilon, \delta}(x_\alpha^*)) \subseteq (f^-)^*(U).
\]

So, \( S \subseteq (f^-)^*(U) \).

**Theorem 17.** Suppose that \( f : (X, d_X) \to (Y, d_Y) \) is an open mapping, then \( f \) is also an \( I \)-open mapping from \((X, \mathcal{T}_X)\) to \((Y, \mathcal{T}_Y)\).

**Proof.** Let \( A \) be an \( I \)-open set in \((X, \mathcal{T}_X)\). Then \( E(A) \subseteq \mathcal{T}_X \). By Lemma 16 it follows that \((f^-)^*(E(A)) \subseteq \mathcal{T}_Y \). Moreover, by Lemma 14(a), we see that \( E(f^-)(A) \subseteq \mathcal{T}_Y \) and it follows that \( f^- \) is an \( I \)-open set in \((I^Y, \mathcal{T}_Y)\). So, \( f \) is an \( I \)-open mapping from \((X, \mathcal{T}_X)\) to \((Y, \mathcal{T}_Y)\).

**Corollary 18.** Suppose that \( X \) and \( Y \) are two Banach spaces over \( \mathcal{R} \) and \( f : X \to Y \) is a continuous linear surjection, then \( f \) is an \( I \)-open mapping.

**Proof.** It is obvious from the Open Mapping Theorem and Theorem 17.

**Definition 19.** Let \( A \) be a fuzzy set of a vector space over \( \mathcal{R} \) and let \( t \) be a scalar. Then:

(i) for \( t \neq 0 \), \( (tA)(x) = A(t^{-1}x) \) for all \( x \in X \),
(ii) for $t = 0$, 

$$(tA)(x) = \begin{cases} 
0, & \text{if } x \neq 0, \\
\bigvee_y A(y) & \text{if } x = 0. 
\end{cases}$$

**Corollary 20.** Suppose that $X$ is a Banach space over $\mathbb{R}$ and define the mapping $f : X \to X$ by $f(x) = tx$, where $t \in \mathbb{R}$ is a scalar, then $f$ is an $I$-open mapping.

**Proof.** $f$ is a linear surjection. It follows that $f$ is an $I$-open mapping by Corollary 18. $\square$

In the following corollary, we show that the translation is an $I$-open mapping.

**Corollary 21.** Suppose that $X$ is a Banach space over $\mathbb{R}$ and define the mapping $f_u : X \to X$ by $f_u(x) = u + x$, for any fixed $u \in X$. Then $f$ is an $I$-open mapping.

**Proof.** It is easy to see that $f_u(U) = u + U$ is an open set for each open set $U$. So, $f_u$ is an open mapping. Thus it follows that $f_u$ is an $I$-open mapping by Theorem 17 $\square$
References


