THE SECOND REGULARIZED TRACE OF A SECOND ORDER DIFFERENTIAL OPERATOR WITH UNBOUNDED OPERATOR COEFFICIENT

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Abstract: In this work, a formula for the second regularized trace of second order differential operator with unbounded operator coefficient is found.

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1. Introduction

Let \( H \) be a infinite dimensional separable Hilbert space. We denote the inner product in \( H \) by \((.,.)\) and the norm in \( H \) by \( \| \|. \). Let \( H_1 = L^2(H; \[0, \pi\]) \) denote the set of all functions \( f \) from \([0, \pi]\) into \( H \) which are strongly measurable and satisfy the condition \( \int_0^\pi \|f(x)\|^2 \, dx < \infty \). If the inner product of arbitrary two elements \( f \) and \( g \) of the space \( H_1 \) is defined as

\[
(f, g)_{H_1} = \int_0^\pi (f(x), g(x)) \, dx ,
\]

then \( H_1 \) becomes a infinite dimensional separable Hilbert space [13]. The norm in the space \( H_1 \) is denoted by \( \| \|. \) \( _1 \). \( \sigma_\infty(H) \) denotes the set of all compact
operators from $H$ into $H$. If $A \in \sigma_\infty(H)$, then $A^*A$ is a nonnegative self-adjoint operator and $(A^*A)^{1/2} \in \sigma_\infty(H)$ [7]. Let the non-zero eigenvalues of the operator $A^*A$ be $s_1 \geq s_2 \geq \ldots \geq s_k$ ($0 \leq k \leq \infty$). Here, each eigenvalue is repeated according to its own multiplicity number. Since $(A^*A)^{1/2}$ is nonnegative, $s_1, s_2, \ldots, s_k$ are positive numbers. The numbers $s_1, s_2, \ldots, s_k$ are called $s$-numbers of the operator $A$. If $k < \infty$, then $s_j = 0$; $j = k+1, k+2, \ldots$ will be accepted. $s$-numbers of the operator $A$ is also denoted by $s_j(A)$ $(j = 1, 2, \ldots)$. Here, $s_1(A) = \|A\|$. If $A$ is a normal operator, then $s_j(A) = |\lambda_j(A)|$ $(j = 1, 2, \ldots, k)$ [7]. Here, $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots \geq |\lambda_k(A)|$ are the non-zero eigenvalues of the operator $A$. $\sigma_p$ or $\sigma_p(H)$ is the set of all operators $A \in \sigma_\infty(H)$ the $s$-numbers of which satisfy the condition $\sum_{j=1}^\infty s_j^p(A) < \infty$ $(p \geq 1)$. The set $\sigma_p$ ($p \geq 1$) is a separable Banach space [7] with respect to the function $\|A\|_{\sigma_p(H)} = \left[\sum_{j=1}^\infty s_j^p(A)\right]^{1/p}$ ($A \in \sigma_p(H)$).

$\sigma_1(H)$ is the set of all the operators $A \in \sigma_\infty(H)$ the $s$-numbers of which satisfy the condition $\sum_{j=1}^\infty s_j(A) < \infty$. An operator is called a kernel operator if it belongs to $\sigma_1(H)$. If the operator $A \in \sigma_p(H)$ and $T \in B(H)$ then $AT, TA \in \sigma_p(H)$ and

$$\|AT\|_{\sigma_p(H)} \leq \|T\|\|A\|_{\sigma_p(H)}, \quad \|TA\|_{\sigma_p(H)} \leq \|T\|\|A\|_{\sigma_p(H)}.$$ 

If $A$ is a kernel operator and $\{e_j\}_1^\infty \subset H$ is any orthonormal basis, the series $\sum_{j=1}^\infty (Ae_j, e_j)$ is convergent and the sum of the series $\sum_{j=1}^\infty (Ae_j, e_j)$ does not depend on the choice of the basis $\{e_j\}_1^\infty$. The sum of the series $\sum_{j=1}^\infty (Ae_j, e_j)$ is said to be matrix trace and is denoted by $\text{tr} A$. It is known that (see [7]):

$$\text{tr} A = \sum_{j=1}^{\nu(A)} \lambda_j(A). \quad (1.1)$$

Here, each eigenvalue is added according to its own algebraic multiplicity number. $\nu(A)$ denotes the sum of algebraic multiplicity of non-zero eigenvalues of the operator $A$.

Let us consider the differential expression in the space $H_1 = L_2(H; [0, \pi])$,

$$l_0(y) = -y''(x) + Ay(x).$$
Here, an operator $A$ from $D(A) \subset H$ into $H$ satisfies the conditions

$$A = A^* \geq I, \quad A^{-1} \in \sigma_{\infty}(H).$$

Let $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq \ldots$ be the eigenvalues of the operator $A$ and $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be the orthonormal eigenvectors corresponding to these eigenvalues. Here, each eigenvalue is repeated according to its own multiplicity number.

Moreover, $D_0$ denotes the set of the functions $y(x) \in H_1$ satisfying the conditions:

1) $y(x)$ has continuous derivative of the second order with respect to the norm in the space $H$ in the interval $[0, \pi]$.
2) $Ay(x)$ continuous with respect to the norm in the space $H$.
3) $y'(0) = y'(\pi) = 0$.

Here, $\overline{D_0} = H_1$ and the operator $L_0' y = l_0(y)$ from $D_0$ into $H_1$ is symmetric. The eigenvalues of $L_0'$ are $k^2 + \gamma_j$ ($k = 0, 1; \ldots; j = 1, 2, \ldots$) and the orthonormal eigenvectors corresponding to these eigenvalues are $M_k \cos kx.\varphi_j(k = 0, 1; \ldots; j = 1, 2, \ldots)$. Here,

$$M_k = \begin{cases} 
\frac{1}{\sqrt{\pi}}, & \text{if } k = 0, \\
\frac{\sqrt{2}}{\sqrt{\pi}}, & \text{if } k = 1, 2, \ldots .
\end{cases}$$

As seen, the orthonormal eigenvectors system of the symmetric operator $L_0'$ is an orthonormal basis in the space $H_1$. The operator $L_0 = \overline{L_0'} : D(L_0) \to H_1$ is self-adjoint.

Let $Q(x)$ be an operator function satisfying the following conditions:

1) $Q(x)$ has weak derivative of the fourth order and $Q^{(2k-1)}(0) = Q^{(2k-1)}(\pi) = 0$, $k = 1, 2$.
2) For every $x \in [0, \pi]$, $Q^{(i)}(x)$ ($i = 0, 1, 2, 3, 4$) are self-adjoint operators from $H$ into $H$.
3) For every $x \in [0, \pi]$, $AQ(x), AQ''(x), Q^{IV}(x) \in \sigma_1(H)$ and the functions $\|AQ(x)\|_{\sigma_1(H)}, \|AQ''(x)\|_{\sigma_1(H)}, \|Q^{IV}(x)\|_{\sigma_1(H)}$ are bounded and measurable in the interval $[0, \pi]$.
4) For every $f \in H$, $\int_0^\pi (Q(x)f, f) dx = 0$.

Let us consider the self-adjoint operator $L = L_0 + Q$ from $D(L) = D(L_0)$ into $H_1$. The operators $L_0$ and $L$ have purely-discrete spectrum. Let the eigenvalues of the operators $L_0$ and $L$ be $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots$ and $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$.

$$\|AQ(x)\|_{\sigma_1(H)}, \|AQ''(x)\|_{\sigma_1(H)}, \|Q^{IV}(x)\|_{\sigma_1(H)}$$
... \leq \lambda_n \leq ... respectively. Let \( R_0^\lambda = (L_0 - \lambda I)^{-1} \), \( R_\lambda = (L - \lambda I)^{-1} \) be the resolvents of the operators \( L_0 \) and \( L \) respectively.

If \( \gamma_j \sim a_j^\alpha \) as \( j \to \infty \) that is \( \lim_{j \to \infty} \frac{\gamma_j}{a_j^\alpha} = 1 \) then as \( n \to \infty \) (see [11])

\[
\lambda_n, \mu_n \sim dn^{\frac{2\alpha}{2+\alpha}}.
\]

(1.2)

Here \( d > 0 \) is constant. By using this relation, it is seen that the sequence \( \{\mu_n\}_{n=1}^\infty \) has a subsequence \( \{\mu_n\}_{m=1}^\infty \) such that

\[
\mu_k - \mu_{n_m} \geq d\left(k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}}\right) \quad (k = n_m, n_m + 1, ...).
\]

In [3], the formula in the form

\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4} [\text{tr } Q(0) + \text{tr } Q(\pi)]
\]

is found for the regularized trace of the operator \( L \). In this work, we obtain a formula in the following form:

\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^{p} (-1)^{j-1} \text{Res } \lambda^j \text{tr } [\lambda^j(QR_\lambda^0)])
\]

\[
= \frac{1}{2} [\text{tr } AQ(0) + \text{tr } AQ(\pi)] - \frac{1}{8} [\text{tr } Q''(0) + \text{tr } Q''(\pi)]. \quad (1.3)
\]

Here \( \alpha > 2 \) is a constant and \( p = \left\lfloor \frac{5\alpha + 6}{\alpha - 2} \right\rfloor + 1 \). The limit

\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^{p} (-1)^{j-1} \text{Res } \lambda^j \text{tr } [\lambda^j(QR_\lambda^0)])
\]

is called the second regularized trace of operator \( L \).

The regularized trace formulas for scalar differential operators are studied in [8], [10], [12] and in many other works. The list of the works on the subjects is given in [9] and [14], but a small number of these works are on the regularized trace of differential operators with operator coefficient. In [6], the regularized trace of Sturm-Liouville operator with bounded operator coefficient is calculated. In [1], a formula for the regularized trace of the difference of two Sturm-Liouville operators which is given in half-axis with the bounded operator coefficient is found. In [15], a formula for the regularized trace of the Sturm-Liouville operator under Dirichlet boundary conditions with unbounded
operator coefficient, is found. In [5], the regularized trace of a singular differential operator of second order with bounded operator coefficient is investigated. In [4] and [2], the formulas for the regularized traces of differential operators with bounded operator coefficient are found.

2. Some Relations about the Eigenvalues

If \( \alpha > 2 \) and \( \lambda \neq \lambda_k, \mu_k \) \((k = 1, 2, \ldots)\), then by (1.2) the serieses 
\[
\sum_{k=1}^{\infty} \frac{1}{|\mu_k - \lambda|} \text{ and } \sum_{k=1}^{\infty} \frac{1}{|\lambda_k - \lambda|}
\]
are convergent. Therefore, \( R_0^\lambda \) and \( R_\lambda \) are kernel operators. In this case we obtain
\[
\text{tr} (R_\lambda - R_0^\lambda) = \text{tr} R_\lambda - \text{tr} R_0^\lambda = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right). \tag{2.1}
\]

If the equality (2.1) is multiplied with \( \frac{\lambda^2}{2\pi i} \) and integrated on the circle \(|\lambda| = b_m = 2^{-1}(\mu_{n_m} + \mu_{n_m+1})\) then the following is obtained:
\[
\frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} (R_\lambda - R_0^\lambda) \, d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=b_m} \sum_{k=1}^{\infty} \left( \frac{\lambda^2}{\lambda_k - \lambda} \right) d\lambda - \frac{1}{2\pi i} \int_{|\lambda|=b_m} \sum_{k=1}^{\infty} \left( \frac{\lambda^2}{\mu_k - \lambda} \right) d\lambda.
\]

It is easily seen that for the large values of \( m \)
\[
\{\lambda_k, \mu_k\}_{1}^{n_m} \subset B(0, b_m) = \{\lambda : |\lambda| < b_m\}, \quad \lambda_k, \mu_k \notin B[0, b_m] = \{\lambda : |\lambda| \leq b_m\} \quad (k \geq n_m + 1).
\]

Therefore by (2.1) we have
\[
\sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} (R_\lambda - R_0^\lambda) d\lambda. \tag{2.2}
\]

This is a well-known formula for the resolvents of the operators \( L_0 \) and \( L \):
\[
R_\lambda = R_0^\lambda - R_\lambda Q R_0^\lambda \quad (\lambda \in \rho(L) \cap \rho(L_0)).
\]
By using this formula we obtain

\[ R_\lambda - R_\lambda^0 = \sum_{j=1}^{p} (-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{p+1} R_\lambda (QR_\lambda^0)^{p+1}, \]

for every \( p \) positive integer. By (2.2) and the last equality we have

\[ \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \left[ \sum_{j=1}^{p} (-1)^{j+1} R_\lambda^0 (QR_\lambda^0)^j + (-1)^p R_\lambda (QR_\lambda^0)^{p+1} \right] d\lambda, \]

or

\[ \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \sum_{j=1}^{p} D_{mj} + D_m^{(p)}. \quad (2.3) \]

Here,

\[ D_{mj} = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \left[ R_\lambda^0 (QR_\lambda^0)^{j} \right] d\lambda \quad (j = 1, 2, \ldots), \quad (2.4) \]

\[ D_m^{(p)} = \frac{(-1)^p}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \left[ R_\lambda (QR_\lambda^0)^{p+1} \right] d\lambda. \]

**Theorem 2.1.** If \( r_j \sim a j^\alpha (a > 0, \alpha > 2) \) as \( j \to \infty \) then

\[ D_{mj} = \frac{(-1)^j}{\pi i j} \int_{|\lambda|=b_m} \lambda \text{tr} \left[ (QR_\lambda^0)^j \right] d\lambda. \]

**Proof.** It can be shown that the operator function \( (QR_\lambda^0)^j \) is analytic with respect to the norm in the space \( \sigma_1(H_1) \) in the region \( \rho(L_0) \) and

\[ \text{tr} \left\{ [(QR_\lambda^0)^j]' \right\} = j \cdot \text{tr}[ (QR_\lambda^0)' (QR_\lambda^0)^{j-1}], \quad (QR_\lambda^0)' = Q(R_\lambda^0)^2. \]

Therefore, we have

\[ \text{tr} \left\{ [(QR_\lambda^0)^j]' \right\} = j \cdot \text{tr}[ (QR_\lambda^0)^{j-1} Q(R_\lambda^0)^2] = j \cdot \text{tr}[ (QR_\lambda^0)^j R_\lambda^0] = j \cdot \text{tr}[ R_\lambda^0 (QR_\lambda^0)^j]. \]
From (2.4) and the last equality we obtain

\[ D_{mj} = \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_m} \lambda^2 \text{tr} \{ [(QR^0_\lambda)^j]' \} d\lambda. \]

This formula can be written in the following form also:

\[ D_{mj} = \frac{(-1)^j}{\pi ij} \int_{|\lambda|=b_m} \lambda \text{tr} (QR^0_\lambda)^j d\lambda + \frac{(-1)^{j+1}}{2\pi ij} \int_{|\lambda|=b_m} \text{tr} [\lambda^2 (QR^0_\lambda)^j]' d\lambda. \quad (2.5) \]

It is easy to see that

\[ \text{tr} \{ [\lambda^2 (QR^0_\lambda)^j]' \} = \{ \text{tr} [\lambda^2 (QR^0_\lambda)^j] \}' \]

and

\[ \int_{|\lambda|=b_m} \{ [\lambda^2 (QR^0_\lambda)^j]' \} d\lambda = \int_{|\lambda|=b_m} \{ [\lambda^2 (QR^0_\lambda)^j] \}' d\lambda. \]

We write the right hand side integral of this equality in the following way:

\[ \int_{|\lambda|=b_m} \{ [\lambda^2 (QR^0_\lambda)^j]' \} d\lambda = \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR^0_\lambda)^j] \}' d\lambda = \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR^0_\lambda)^j] \}' d\lambda. \quad (2.6) \]

Let \( \varepsilon_0 \) be a positive number such that \( b_m + \varepsilon_0 < \mu_{m+1} \). Considering the fact that the function \( \text{tr} [\lambda^2 (QR^0_\lambda)^j] \) is analytic in the simply connected regions

\[ G_1 = \{ \lambda : b_m - \varepsilon_0 < |\lambda| < b_m + \varepsilon_0, \text{Im} \lambda > -\varepsilon_0 \}, \]

\[ G_2 = \{ \lambda : b_m - \varepsilon_0 < |\lambda| < b_m + \varepsilon_0, \text{Im} \lambda < \varepsilon_0 \}, \]

and

\[ \{ \lambda : |\lambda| = b_m, \text{Im} \lambda \geq 0 \} \subset G_1, \quad \{ \lambda : |\lambda| = b_m, \text{Im} \lambda \leq 0 \} \subset G_2, \]

using Leibnitz formula, by (2.6), we get

\[ \int_{|\lambda|=b_m} \{ \text{tr} [\lambda^2 (QR^0_\lambda)^j] \}' d\lambda = \text{tr} [b_m^2 (QR_{-b_m}^0)^j] - \text{tr} [b_m^2 (QR_{b_m}^0)^j] \]
\[
+ \text{tr} \left[ b_m^2 (QR_{b_m}^0)^j \right] - \text{tr} \left[ b_m^2 (QR_{-b_m}^0)^j \right] = 0.
\]

From (2.5) and the last equality we have

\[
D_{mj} = \left( \frac{-1}{\pi i j} \right) \frac{1}{|\lambda| = b_m} \lambda \text{tr} \left[ (QR_\lambda)^j \right] d\lambda.
\]

The proof of the theorem is finished.

**Theorem 2.2.** If the operator function \( Q(x) \) satisfies the condition 1 and for every \( x \in [0, \pi] \), \( AQ''(x), Q''(x) \in \sigma_1(H) \) and the functions \( \| AQ''(x) \|_{\sigma_1(H)}, \| Q''(x) \|_{\sigma_1(H)} \) are bounded and measurable in the interval \([0, \pi]\) then

\[
\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |(k^2 + \gamma_j) \int_0^\pi \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx| < \infty.
\]

**Proof.** Let \( f_i(x) = (Q(x)\varphi_i, \varphi_i) \). Using the partial integration method and considering the condition \( f_i'''(0) = f_i''(\pi) = 0 \) we have

\[
\int_0^\pi f_i(x) \cos 2kx dx = \int_0^\pi f_i(x) \left( \frac{1}{2k} \sin 2kx \right)' dx = \frac{1}{2k} f_i(x) \sin 2kx |_0^\pi
\]

\[
- \frac{1}{2k} \int_0^\pi f_i(x) \sin 2kx dx = \frac{1}{2k} \int_0^\pi f_i(x) \left( \frac{1}{2k} \cos 2kx \right)' dx
\]

\[
= \frac{1}{4k^2} f_i'(x) \cos 2kx |_0^\pi - \frac{1}{4k^2} \int_0^\pi f_i''(x) \cos 2kx dx = -\frac{1}{4k^2} \int_0^\pi f_i''(x) \cos 2kx dx.
\]

By the last relation we obtain

\[
(k^2 + \gamma_i) \int_0^\pi f_i(x) \cos 2kx dx
\]

\[
= -\frac{1}{4} \int_0^\pi f_i''(x) \cos 2kx dx - \frac{\gamma_i}{4k^2} \int_0^\pi f_i''(x) \cos 2kx dx.
\]

Using the partial integration method again and considering the conditions \( f_i'''(0) = 0, f_i''(\pi) = 0 \)

\[
(k^2 + \gamma_i) \int_0^\pi f_i(x) \cos 2kx dx = -\frac{1}{4} \int_0^\pi f_i''(x) \cos 2kx dx
\]

\[
- \frac{\gamma_i}{4k^2} \int_0^\pi f_i''(x) \cos 2kx dx = -\frac{1}{4} \int_0^\pi f_i''(x) \sin 2kx dx - \frac{1}{2k} \int_0^\pi f_i''(x) \sin 2kx dx
\]

\[
- \frac{\gamma_i}{4k^2} \int_0^\pi f_i''(x) \cos 2kx dx = -\frac{1}{4} \left[ -\frac{1}{2k} \int_0^\pi f_i''(x) \sin 2kx dx \right]
\]
- \frac{\gamma_i}{4k^2} \int_0^\pi f_i''(x) \cos 2kx \, dx = -\frac{1}{4} \frac{1}{4k^2} f_i'''(x) \cos 2k|_0^\pi

- \frac{1}{4k^2} \int_0^\pi f_i^{IV}(x) \cos 2kx \, dx - \frac{\gamma_i}{4k^2} \int_0^\pi f_i''(x) \cos 2kx \, dx

= \frac{1}{16k^2} \int_0^\pi f_i^{IV}(x) \cos 2kx \, dx - \frac{\gamma_i}{4k^2} \int_0^\pi f_i''(x) \cos 2kx \, dx.

Hence we have

\sum_{k=1}^\infty \sum_{i=1}^\infty |(k^2 + \gamma_i) \int_0^\pi f_i(x) \cos 2kx| dx

\leq \sum_{k=1}^\infty \sum_{i=1}^\infty [\frac{1}{16k^2} \int_0^\pi |f_i^{IV}(x)| \, dx + |\gamma_i| \int_0^\pi |f_i''(x)| \, dx]

\leq [\sum_{k=1}^\infty \int_0^\pi |f_i^{IV}(x)| \, dx + \sum_{i=1}^\infty \int_0^\pi \gamma_i |f_i''(x)| \, dx] \sum_{k=1}^\infty k^{-2}. \quad (2.7)

Furthermore

\sum_{i=1}^\infty \int_0^\pi |f_i^{IV}(x)| \, dx = \lim_{n \to \infty} \int_0^\pi \sum_{i=1}^n |f_i^{IV}(x)| \, dx

\leq \int_0^\pi \sum_{i=1}^\infty |f_i^{IV}(x)| \, dx = \int_0^\pi \sum_{i=1}^\infty |(Q^{IV}(x) \varphi_i, \varphi_i)| \, dx, \quad (2.8)

\sum_{i=1}^\infty \int_0^\pi \gamma_i |f_i''(x)| \, dx

\leq \int_0^\pi \sum_{i=1}^\infty \gamma_i |f_i''(x)| \, dx = \int_0^\pi \sum_{i=1}^\infty |(AQ''(x) \varphi_i, \varphi_i)| \, dx. \quad (2.9)

Since for every \( x \in [0, \pi] \) \( Q^{IV}(x) \in \sigma_1(H) \) and \( AQ''(x) \in \sigma_1(H) \), the inequalities

\sum_{i=1}^\infty |(Q^{IV}(x) \varphi_i, \varphi_i)| \leq \|Q^{IV}(x)\|_{\sigma_1(H)}, \quad (2.10)

\sum_{i=1}^\infty |(AQ''(x) \varphi_i, \varphi_i)| \leq \|AQ''(x)\|_{\sigma_1(H)} \quad (2.11)
are satisfied. By (2.7), (2.8), (2.9), (2.10) and (2.11) we obtain:

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |(k^2 + \gamma_i) \int_0^{\pi} \cos 2kx.f_i(x)dx| < c_0 \left[ \int_0^{\pi} \|QIV(x)\|_{\sigma_1(H)}dx + \int_0^{\pi} \|AQ''(x)\|_{\sigma_1(H)}dx \right],$$

here $c_0 = \sum_{k=1}^{\infty} k^{-2}$.

Since the functions $\|AQ''(x)\|_{\sigma_1(H)}, \|QIV(x)\|_{\sigma_1(H)}$ are bounded and measurable in the interval $x \in [0, \pi]$, by the last inequality

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |(k^2 + \gamma_i) \int_0^{\pi} \cos 2kx.(Q(x)\varphi_i,\varphi_i)d\lambda| < \infty$$

is obtained. \hfill \Box

Let $\{\psi_q\}_{1}^{\infty}$ be the orthonormal eigenvectors system corresponding to the eigenvalues $\{\mu_q\}_{1}^{\infty}$ of the operator $L_0$. Since the orthonormal eigenvectors according to the eigenvalues $k^2 + \gamma_j$ ($k = 0, 1, 2; j = 1, 2, ...$) of the operator $L_0$ are $M_k \cos kx.\varphi_j$ ($k = 0, 1, 2; j = 1, 2, ...$) respectively then

$$\psi_q = M_{k_q} \cos k_q x.\varphi_j \quad (q = 1, 2, ...). \quad (2.12)$$

### 3. Calculating of the Second Regularized Trace

From (2.3) and (2.1) we have

$$\sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \sum_{j=1}^{p} D_{mj} + D_{m}^{(p)}. \quad (3.1)$$

Here

$$D_{mj} = \frac{(-1)^{\bar{j}}}{\pi i j} \int_{|\lambda|=b_m} \lambda \text{tr} [(QR_0^\lambda)^j]d\lambda, \quad (3.2)$$

$$D_{m}^{(p)} = \frac{(-1)^{p}}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \text{tr} [R_\lambda(QR_0^\lambda)^{p+1}]d\lambda. \quad (3.3)$$

By using the equality
\[ D_{mj} = \frac{2(-1)^j}{j} \frac{1}{2\pi i} \int_{|\lambda|=b_m} \text{tr} \left[ \lambda (QR^0_\lambda)^j \right] d\lambda \]

\[ = 2(-1)^j j^{-1} \sum_{k=1}^{n_m} \text{Res}_{\lambda=\mu_k} \text{tr} \left[ \lambda (QR^0_\lambda)^j \right], \]

the formula (3.1) can be written in the following form:

\[ \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) - 2 \sum_{j=2}^{p} (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} \text{tr} \left[ \lambda (QR^0_\lambda)^j \right] = D_{m1} + D_m^{(p)}. \tag{3.4} \]

**Theorem 3.1.** If \( \gamma_j \sim a j^\alpha \) \((a > 0, \alpha > 2)\) as \( j \to \infty \) and \( Q(x) \) satisfies the conditions 1), 2), 3), 4) then

\[ \lim_{m \to \infty} D_{m1} = \frac{1}{2} [\text{tr} AQ(0) + \text{tr} AQ(\pi)] - \frac{1}{8} [\text{tr} Q''(0) + \text{tr} Q''(\pi)]. \]

**Proof.** According the formula (3.2)

\[ D_{m1} = \frac{-1}{\pi i} \int_{|\lambda|=b_m} \lambda \text{tr} (QR^0_\lambda) d\lambda. \tag{3.5} \]

Since \( QR^0_\lambda \) is a kernel operator for every \( \lambda \in \rho(L_0) \) and \( \{\psi_n\}_1^\infty \) is an orthonormal basis in the space \( H_1 \), we have

\[ \text{tr} (QR^0_\lambda) = \sum_{q=1}^{\infty} (QR^0_\lambda \psi_q, \psi_q)_1. \]

If this equality is written into the equality (3.5) and the equality

\[ R^0_\lambda \psi_q = (L_0 - \lambda I)^{-1} \psi_q = (\mu_q - \lambda I)^{-1} \psi_q \]

is considered, then we obtain

\[ D_{m1} = \frac{-1}{\pi i} \int_{|\lambda|=b_m} \lambda \sum_{q=1}^{\infty} (QR^0_\lambda \psi_q, \psi_q)_{H_1} d\lambda \]

\[ = \frac{-1}{\pi i} \int_{|\lambda|=b_m} \lambda \sum_{q=1}^{\infty} \frac{1}{\mu_q - \lambda} (Q \psi_q, \psi_q)_{H_1} d\lambda \]

\[ = \frac{1}{\pi i} \sum_{q=1}^{\infty} (Q \psi_q, \psi_q)_{H_1} \int_{|\lambda|=b_m} \frac{\lambda}{\mu_q - \lambda} d\lambda. \]
By using the formula
\[ \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{\lambda}{\lambda - \mu_q} d\lambda = \begin{cases} \mu_q, & \text{if } q \leq n_m, \\ 0, & \text{if } q > n_m, \end{cases} \]
and the equality (2.12), we obtain

\[ D_{m1} = 2 \sum_{q=1}^{n_m} \mu_q \langle Q \psi_q, \psi_q \rangle H_1 = 2 \sum_{q=1}^{n_m} \mu_q \int_0^\pi (Q(x)\psi_q(x), \psi_q(x)) \, dx \]
\[ = 2 \sum_{q=1}^{n_m} \mu_q \int_0^\pi (Q(x)M_{k_q} \cos k_q x. \varphi_{j_q}, M_{k_q} \cos k_q x. \varphi_{j_q}) dx \]
\[ = 2 \sum_{q=1}^{n_m} M_{k_q}^2 \mu_q \int_0^\pi \cos^2 k_q x (Q(x)\varphi_{j_q}, \varphi_{j_q}) dx \]
\[ = \sum_{q=1}^{n_m} M_{k_q}^2 \mu_q \int_0^\pi (1 + \cos 2k_q x). (Q(x)\varphi_{j_q}, \varphi_{j_q}) \, dx. \]

Since \( Q(x) \) satisfies the condition 4) and \( M_k = \sqrt{2\pi^{-1}} \) for \( k = 1, 2, \ldots \), by the last equality we have

\[ D_{m1} = 2 \sum_{q=1}^{n_m} \mu_q \int_0^\pi \cos 2k_q x. (Q(x)\varphi_{j_q}, \varphi_{j_q}) dx. \quad (3.6) \]

In accordance with Theorem 2.1, the multiple series
\[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k^2 + \gamma_j) \int_0^\pi \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx \]
is absolute convergent. In this case as known

\[ \lim_{j \to \infty} \sum_{q=1}^{n_m} (k^2 + \gamma_{j_q}) \int_0^\pi \cos 2k_q x. (Q(x)\varphi_{j_q}, \varphi_{j_q}) dx \]
\[ = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k^2 + \gamma_j) \int_0^\pi \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx. \]

By using (3.6) and the last equality we obtain

\[ \lim_{m \to \infty} D_{m1} = 2 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k^2 + \gamma_j) \int_0^\pi \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx, \]
or

$$\lim_{m \to \infty} D_{m1} = \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} k^2 \int_{0}^{\pi} \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx$$

$$+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \gamma_j \int_{0}^{\pi} \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx.$$  

If we use the equality

$$\int_{0}^{\pi} \cos 2kx. (Q(x)\varphi_j, \varphi_j) dx = -\frac{1}{4k^2} \int_{0}^{\pi} \cos 2kx. (Q''(x)\varphi_j, \varphi_j) dx,$$

then we have

$$\lim_{m \to \infty} D_{m1} = -\frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx. (Q''(x)\varphi_j, \varphi_j) dx$$

$$+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx. (Q(x)\varphi_j, \gamma_j\varphi_j) dx.$$  

Hence, we obtain

$$\lim_{m \to \infty} D_{m1} = -\frac{1}{4\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ \int_{0}^{\pi} \cos kx. (Q''(x)\varphi_j, \varphi_j) dx \right]$$

$$+ (-1)^k \int_{0}^{\pi} \cos kx. (Q''(x)\varphi_j, \varphi_j) dx]$$

$$+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx. (Q(x)\varphi_j, A\varphi_j) dx$$

$$= -\frac{1}{8} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ 2 \int_{0}^{\pi} \cos kx. (Q''(x)\varphi_j, \varphi_j) dx \cos k0 \right]$$

$$+ \sum_{k=1}^{\infty} \left[ 2 \int_{0}^{\pi} \cos kx. (Q''(x)\varphi_j, \varphi_j) dx \cos k\pi \right]$$

$$+ \frac{2}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{0}^{\pi} \cos 2kx. (AQ(x)\varphi_j, \varphi_j) dx$$

$$= -\frac{1}{8} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ 2 \int_{0}^{\pi} \cos kx. (Q''(x)\varphi_j, \varphi_j) dx \cos k0 \right].$$
\[
+ \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_0^\pi \cos kx. (Q''(x)\varphi_j, \varphi_j) dx \right) \cos k\pi \\
+ \frac{1}{\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ \int_0^\pi \cos kx. (AQ(x)\varphi_j, \varphi_j) dx \right] \\
+ (-1)^k \int_0^\pi \cos kx. (AQ(x)\varphi_j, \varphi_j) dx \\
= -\frac{1}{8} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_0^\pi \cos kx. (Q''(x)\varphi_j, \varphi_j) dx \right) \cos k0 \\
+ \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_0^\pi \cos kx. (Q''(x)\varphi_j, \varphi_j) dx \right) \cos k\pi \\
+ \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{2}{\pi} \int_0^\pi \cos kx. (AQ(x)\varphi_j, \varphi_j) dx \right) \cos k0 \\
+ \sum_{k=1}^{\infty} \frac{2}{\pi} \left( \int_0^\pi \cos kx. (AQ(x)\varphi_j, \varphi_j) dx \right) \cos k\pi.
\]

If we consider that \( Q(x) \) satisfies the conditions 1) and 4), then the sums according to \( k \) on the right hand side of the last relation are the values at 0 and \( \pi \) of the Fourier series of the functions \( (Q''(x)\varphi_j, \varphi_j) \) and \( (AQ(x)\varphi_j, \varphi_j) \) according to the functions \( \{ \cos kx \}_{k=0}^{\infty} \) in the interval \([0, \pi]\) respectively. Therefore

\[
\lim_{m \to \infty} D_{m1} = -\frac{1}{8} \sum_{j=1}^{\infty} [(Q''(0)\varphi_j, \varphi_j) + (Q''(\pi)\varphi_j, \varphi_j)] \\
+ \frac{1}{2} \sum_{j=1}^{\infty} [(AQ(0)\varphi_j, \varphi_j) + (AQ(\pi)\varphi_j, \varphi_j)],
\]

or

\[
\lim_{m \to \infty} D_{m1} = \frac{1}{2} [\text{tr} \ AQ(0) + \text{tr} \ AQ(\pi)] - \frac{1}{8} [\text{tr} \ Q''(0) + \text{tr} \ Q''(\pi)]
\]

is obtained.

**Theorem 3.2.** If \( \gamma_j \sim a_j^\alpha \ (0 < a < \infty, 2 < \alpha < \infty) \) as \( j \to \infty \) and \( Q(x) \) satisfies the conditions 1), 2), 3), 4) then

\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) - 2 \sum_{j=2}^{p} (-1)^j j^{-1} \text{Res} \left[ \text{tr} \left[ \lambda (QR_j)^j \right] \right]
\]
Here \( p = [\frac{5\alpha + \delta}{\alpha - 2}] + 1 \).

**Proof.** By using the formula (3.3) we have

\[
|D_m^{(p)}| \leq \int_{|\lambda|=b_m} |\lambda|^2 |\lambda| \, |\text{tr} \left[ R_\lambda (QR_\lambda^0)^{p+1} \right]| \, d\lambda
\]

\[
\leq b_m^2 \int_{|\lambda|=b_m} \|R_\lambda (QR_\lambda^0)^{p+1} \|_{\sigma_1(H_1)} \, d\lambda
\]

\[
\leq b_m^2 \int_{|\lambda|=b_m} \|R_\lambda \|_1 \| (QR_\lambda^0)^{p} \|_{\sigma_1(H_1)} \, d\lambda
\]

\[
\leq b_m^2 \int_{|\lambda|=b_m} \|R_\lambda \|_1 \| (QR_\lambda^0)^{p} \|_1 \| (QR_\lambda^0)^{p} \|_1 \, d\lambda
\]  

(3.7)

Using the inequalities

\[
\|R_\lambda\|_{\sigma_1(H_1)} \leq \text{const} n_m^{1-\delta}, \quad \|R_\lambda\|_1 \leq \text{const} n_m^{-\delta} \quad \left( \delta = \frac{\alpha - 2}{\alpha + 2} \right)
\]

(see [3]), and the inequality (3.7), we obtain

\[
|D_m^{(p)}| \leq \text{const} b_m^3 n_m^{-(1+p)\delta} n_m^{1-\delta} \quad \text{or}
\]

\[
|D_m^{(p)}| \leq \text{const} n_m^{3\delta} n_m^{-(1+p)\delta} n_m^{1-\delta} = \text{const} n_m^{4-(p-1)\delta},
\]

when \( b_m \leq \text{const} n_m^{1+\delta} \).

Therefore, if \( p = [\frac{\alpha}{\delta} + 1]] + 1 \) or \( p = [\frac{5\alpha + \delta}{\alpha - 2}] + 1 \) then we obtain

\[
\lim_{m \to \infty} D_m^{(p)} = 0.
\]

(3.8)

By Theorem 3.1 and the formulas (3.4) and (3.8), the formula in the form

\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) - 2 \sum_{j=2}^{p} (-1)^j j^{-1} \text{Res} \left. \lambda (QR_\lambda^0)^j \right|_{\lambda = \mu_k}
\]

\[
= \frac{1}{2} \left[ \text{tr} AQ(0) + \text{tr} AQ(\pi) \right] - \frac{1}{8} \left[ \text{tr} Q''(0) + \text{tr} Q''(\pi) \right]
\]

is obtained for the second regularized trace of the operator \( L \).  

\( \square \)
References


