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EXPLICIT GENERAL SERIES SOLUTION FOR EULER AND NAVIER-STOKES EQUATIONS

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Abstract: Explicit formulas for the coefficients of general series solutions of the Navier-Stokes and Euler equations in a general domain of R^3 (or R^n) are developed and analysed. The series are eigenvector expansions in spatial variables and orthogonal or Taylor series in time. While elaborate, the coefficient formulas offer a new avenue of analysis of the vector solutions of time-dependent motions of an incompressible fluid, viscous or not. Evaluation is made for a rectilinear filled domain, and a filled spherical domain. Approximations are developed for cubical and $\overline{C^\infty}$ Riemannian domains, operators, and solutions.

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1. Historical Background

Henri Poincaré remarked that "a problem is never solved. It is only more or less solved," an aphorism that encapsulates much of the development of mathematical analysis and applied mathematics. In the seventeenth century, Newton

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used power series to systematize his early studies of functions. In the eighteenth century, Euler carried the study of series to a peak of algebraic virtuosity. In the nineteenth century, Fourier began the study of trigonometric series which in the twentieth century was extended to very general types of expansions based on linear operators and linear analysis. As generality and the command of particular cases increased, so did the scope of explicit formulas, despite tendency to elaboration and lowered interest. Solutions of nonlinear problems, meanwhile, could be represented by explicit formulas only in fortunate particular instances, generally quite rare. Hence analysis turned toward soft or qualitative methods, asymptotics, or numerical analysis to understand the properties of solutions, while explicit methods were largely abandoned as having exhausted their scope. But explicit solution formulas remain no less desirable to the extent they can be constructed and used.

The equations of motion for a fluid and their associated initial and boundary value problems remain an outstanding challenge in this respect: to what extent, and in what degree of detail, can more or less explicit solutions of general flow problems be constructed? In this paper eigen expansions in space and orthogonal and power series in time will illuminate this question, and explicit formulas are put together for their coefficients, and for high-quality approximations.

2. Introduction

First we describe the eigenvalues and solenoidal eigenvectors of the Laplace operator with mixed boundary conditions on a certain region of R^3 (or R^n). We restrict consideration here to bounded regular regions, to ensure a discrete spectrum. The formalities and basic properties of the requisite eigenvector function expansions or generalized Fourier expansions will be introduced. Then the Navier-Stokes equations and Euler's equations will be examined and the appropriate formulas for eigenvector function expansion of their initial-boundary value problems derived. The system of ordinary differential equations in the time variable t for the Fourier coefficients is quadratic and of infinite order with coefficients having symmetries that are shown to give rise to permanent bounds on the solutions in problems of finite initial energy.

The first problem studied in detail is the construction of the series solution for Euler's equations, giving rise to successive convolutions of the coefficients. These are reduced to Fourier coefficient-type integrals, but with increasing numbers of eigenvector component factors. The magnitudes of these resulting series coefficients are studied as well as several special cases of small finite systems.

The source-free case for the Navier-Stokes equations is brought into the quadratic rubric by introducing a dummy zero order coefficient. Explicit formulas for the Navier-Stokes coefficients, which necessarily involve the eigenvalues, are worked out up to the fourth order, and by recursion to all finite orders.

The most complicated case is that of the Navier-Stokes equations with a general vector source. This is reduced to a series of terms, each similar to the preceding solutions, by the device of differentiation with respect to an embedding parameter, multiplying the source term.

The resulting solution formulas are assembled and discussed. Some estimates of the magnitude and convergence properties of coefficients and series involved are made. We consider the particular case of a rectangular box domain, for which the eigenvector expansions are combinations of multiple Fourier series, and for spherical domains. Some detailed calculations for these cases are undertaken. These constructions bring the solutions of Euler and Navier-Stokes initial value problems within a framework which is new for these nonlinear problems, but also, in many respects, is familiar for the classical linear theories of mathematical physics. Approximation systems for the domains, coefficients, and equations are discussed. The intention is to make the calculation of useful properties of these solutions available through further steps of synthesis and evaluation within that framework. The very existence of this framework in such general cases is also of interest in itself, while the study of the quadratically nonlinear system for the Fourier coefficients is possible in many ways not considered here.

3. Eigenvalues and Eigenvectors

For a given bounded, simply connected, regular region Ω in R^n, the Fourier expansions in spatial variables $x_1, \dots, x_n$ we shall use are based on the solenoidal (divergence free or incompressible), eigenvectors satisfying the boundary condition of vanishing tangential component. Thus for vector functions $u_i(x, t)$, $i = 1, \dots, n$ we have

$$\nabla^2 u_i^k + \lambda_k u_i^k = 0\,, \quad i = 1, \dots, n;\ k = 1, 2, \dots\,, \tag{1}$$

while $u_{i,i} \equiv \nabla \cdot \underline{u} = 0$. For a motionless boundary with local tangent plane $x_n = \text{const}$, the $n-1$ no-slip conditions

$$u_i(x, t) = 0\,, \quad i = 1, \dots, n-1 \tag{2}$$

are assumed. The incompressible condition $\nabla \cdot \underline{u} = 0$ supplies the last relation, namely

$$\frac{\partial u_n}{\partial x_n} = -\sum_{i \neq n} \frac{\partial u_i}{\partial x_i} \tag{3}$$

within and on the boundary. These are known as mixed boundary conditions overall [15]. The positive constant η denotes viscosity.

The existence of eigenvalues and the corresponding eigenvector functions can be shown variationally in a manner similar to that used for scalar eigenvalue problems in Courant-Hilbert. Such questions are a special case $p = 1$ of the more general theory of $p-$vector differential forms on a Riemann manifold treated in Duff [14]. As in the scalar case, the eigenvector functions are mutually orthogonal, and can be taken as normalized:

$$\int_\Omega u_i^k(x) u_i^\ell(x) dx = \delta_\ell^k = \begin{cases} 1 & k = \ell\,, \\ 0 & k \neq \ell\,. \end{cases} \tag{4}$$

The basic Fourier expansion coefficient formulas are

$$C_k = \int_\Omega u_i^k(x) u_i^k(x) dx\,, \tag{5}$$

by which an $L^2(\Omega)$ vector function $u_i(x)$ can be represented as

$$u_i(x) = \sum_{k=1}^{\infty} C_k u_i^k(x)\,, \tag{6}$$

the series having $L^2(\Omega)$ and other convergence properties as in the scalar problem of the same number n of space dimensions. The Parseval formula

$$\sum_{k=1}^{\infty} C_k^2 = \int_\Omega \sum_{i=1}^{n} u_i^2(x) dx \tag{7}$$

holds, together with the associated Bessel inequality, because these eigenvector functions are known to be complete in the linear space of L^2 solenoidal vector functions on Ω which vanish at the boundary $\partial\Omega$, provided that Ω is simply connected.

We also employ the representation of the vector delta function

$$\delta(x-y)\delta_j^i = \sum_{k=1}^{\infty} u_i^k(x) u_j^k(y)\,, \tag{8}$$

which is a formal consequence of the preceding formulas using a distributional interpretation. The domain of this delta function is the linear space of solenoidal vector functions with vanishing tangential component on the boundary; that is, the space spanned by the $u_i^k(x)$, $k = 1, 2, \ldots$.

4. The Euler and Navier-Stokes Equations

An incompressible viscous fluid motion represented by a vector function $u_i = u_i(x, t)$ in a domain $\Omega \subset R^n$ satisfies the system of equations of motion or Navier-Stokes equations

$$\frac{\partial v_i}{\partial t} + \sum_{k=1}^{n} v_k \frac{\partial v_i}{\partial x_k} = -\frac{\partial p}{\rho \partial x_i} + \eta \nabla^2 v_i + f_i(x, t), \quad i = 1, \ldots, n, \tag{9}$$

where also $p = p(x, t)$ denotes pressure, ρ density, η viscosity, and $f_i = f_i(x, t)$ denotes the external or body force vector. The incompressibility condition is

$$\sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0 \tag{10}$$

and the boundary conditions are

$$v_i(x, t) = 0, \quad x \in \partial\Omega, \, i = 1, \ldots, n. \tag{11}$$

We assume an initial condition of finite energy: $v_i(x, 0) = v_{i0}(x)$, where

$$\int_\Omega \sum v_{i0}^2(x) dx \equiv \|v_0\|^2 < \infty, \tag{12}$$

and a similar condition

$$\int_\Omega \sum f_i^2(x, t) dx \equiv \|f\|^2 (t) < \infty \tag{13}$$

for all times considered.

We also employ the Einstein summation convention for repeated spatial vector indices: $\sum_{i=1}^n u_i(x) v_i(x)$ is written as $u_i(x) v_i(x)$. Also, differentiation with respect to x_i will be denoted by a comma followed by subscript i, so that $\partial u / \partial x_i = u_{,i}$. We also write $u^2 \equiv u_i u_i$.

The Navier-Stokes equations are now written

$$v_{i,t} + v_k v_{i,k} + \frac{p_{,i}}{\rho} = \eta \nabla^2 v_i + f_i, \quad v_{i,i} = 0, \tag{14}$$

and the Euler equations (with $\eta = 0$) as

$$v_{i,t} + v_k v_{i,k} + \frac{p_{,i}}{\rho} = f_i\,, \quad v_{i,i} = 0\,.$$

In Hilbert space notation, $\|\underline{u}\|$ is the L_2 norm of the real vector function $\underline{u}$ on Ω, and $\underline{u}\cdot\underline{v}$ is the inner product $\int_\Omega u_i v_i dw$, with volume measure w, $\|\underline{u}\|^2 = \underline{u}\cdot\underline{u}$ and $L^2 = L^2(\Omega) = \{\underline{u} : \|\underline{u}\| < \infty\}$, $\nabla\cdot\underline{u} = u_{i,i}$. Further Sobolev norms are introduced later.

If S is the subspace of solenoidal functions vanishing on the boundary of Ω; then if $S = \{\underline{v} : \underline{v}_{i,i} = \underline{v}(\partial\Omega) = 0,\ \|\underline{v}\| < \infty\}$, then $\underline{v} \in S$ and $\underline{\nabla}\phi \in L^2$ imply $\underline{v}\cdot\underline{\nabla}\phi = 0$. By Gauss' Integral Theorem and $v_{i,i} = 0$,

$$\int_\Omega \phi_{,i}\, v_i dw = \int_\Omega (\phi u_i)_{,i}\, dw = \int_{\partial\Omega} \phi u_n dS = 0\,. \tag{15}$$

Applying Gauss' Theorem to the Navier-Stokes equations, we multiply (14) by $v_i(x,\,t)$ and integrate over Ω, obtaining the classical first energy integral bound. Thus for $\underline{u} \in S$, $\int_\Omega u_i \nabla^2 u_i dw = -\int_\Omega \nabla u_i \nabla u_i dw \equiv -\int_\Omega (\nabla\cdot\underline{u})^2 dw$ (the inequality sign may be needed if $\underline{v}$ is a weak solution). Thus

$$\frac{1}{2}\frac{d}{dt}\|\underline{v}\|^2 + \eta\,\|\nabla\cdot\underline{v}\|^2 \le \left|\underline{v}\cdot\underline{f}\right| \le \|\underline{v}\|\,\left\|\underline{f}\right\| \tag{16}$$

by Schwartz' Inequality.

On a bounded domain Ω, every $L^2(\Omega)$ function (or vector), satisfies a Poincaré Inequality, [36]

$$\|\underline{u}\| \le C\,\|\nabla\cdot\underline{u}\|\ , \tag{17}$$

so that

$$\frac{1}{2}\frac{d}{dt}\|\underline{v}\|^2 + \frac{\eta}{C^2}\|\underline{v}\|^2 \ \le\ \frac{1}{2}\frac{d}{dt}\|\underline{v}\|^2 + \eta\,\|\nabla\cdot\underline{v}\|^2 \ \le\ \|\underline{v}\|\,\left\|\underline{f}\right\|\,. \tag{18}$$

Removing a common factor $\|\underline{v}\|$ from the extremes of (18) we find the differential inequality

$$\frac{d}{dt}\|\underline{v}\| + \frac{\eta}{C^2}\|\underline{v}\| \ \le\ \left\|\underline{f}\right\|\,. \tag{19}$$

On integration over $0 \le t \le T$,

$$\|\underline{v}\|(T) \le \|\underline{v}\|(0)\exp\left(-\eta T/C^2\right) + \int_0^T \exp\left(-\eta(T-\tau)/C^2\right)\left\|\underline{f}\right\| d\tau\,. \tag{20}$$

This bound shows that when body forces are bounded in norm, the energy is bounded by $\frac{C^2}{\eta}\left\|\underline{f}\right\|$ plus an exponentially decreasing multiple of $\|\underline{v}(0)\|$.

To expand the N-S equations in a Helmholtz series, we multiply (14) by $u_i^{(k)}$, contract over i, and integrate over Ω to obtain driven linear-quadratic form dynamics. The quadratic form, whose terms are products of three integrals, is obtained by inserting two "delta-clusters" $u_p^{(m)}(\underline{x})u_p^{(m)}(\underline{y})$ and $u_q^{(\ell)}(\underline{x})u_q^{(\ell)}(\underline{z})$ into $\int_\Omega v_j(\underline{x},\, t)v_{i},_j\,(\underline{x},\, t)u_i^{(k)}(\underline{x})dw(\underline{x})$ integrating the result over Ω and summing over i and j, and using integration by parts repeatedly to dispose of known vanishing terms. The other terms result from obvious interchanges of operators and substitutions ($\nabla^2 u_i^{(k)} = -\lambda_k u_i^{(k)}$). If

$$c_k(t) = \int_\Omega v_i(\underline{x},\, t)u_i^{(k)}(\underline{x})dw(\underline{x})\,, \tag{21}$$

the derived equations are

$$\frac{dc_k}{dt} = -\eta\lambda_k c_k + \sum_{\ell,\, m} a_{k\ell m}c_\ell c_m + f_k(t)\,, \tag{22}$$

where

$$f_k(t) = f \cdot u^k, \quad a_{k\ell m} = \int_\Omega u_i^{(k)}u_j^{(\ell)}u_{i,j}^{(m)}dw\,. \tag{23}$$

Also, integrating by parts in (23) yields the double sums

$$a_{k\ell m} = -\int_\Omega u_{i,j}^{(k)}u_j^{(\ell)}u_i^{(m)}dw = -a_{m\ell k} \tag{24}$$

after removing vanishing combinations. Such skew-symmetry has the physical interpretation that energy shifted from mode k to mode m through any intermediate mode ℓ is balanced by removal from m back into k through ℓ. Also $a_{k\ell k} = -a_{k\ell k}$, so there is no self energy linked through any mode.

Equations like (22) appear in Volterra-Lotka population dynamics, but usually without the skew-symmetry of the coefficients.

Here also

$$f_k(t) = \int_\Omega f_i(\underline{x},\, t)u_i^{(k)}(\underline{x})dw\,, \tag{25}$$

are the expansion coefficients of the body forces. With initial conditions

$$c_k(0) = c_{k0} = \int_\Omega u_i^{(k)}(\underline{x})v_i(\underline{x},\, 0)dw\,, \tag{26}$$

for which $\sum_{k=1}^\infty c_{k0}^2 = \|u_0\|^2$, the coefficients $c_k(t)$ are well defined as solutions of the system (22). Our primary problem now is finding the solutions $c_k(t)$of this driven Riccati system, (22).

In (22), it is desirable (necessary) to have symmetrized coefficients:

$$b_{k\ell m} = \frac{1}{2}(a_{k\ell m} + a_{m\ell k})\,. \tag{27}$$

Then a cyclic identity holds due to skew symmetry:

$$b_{k\ell m} + b_{\ell mk} + b_{mk\ell} = \frac{1}{2}(a_{k\ell m} + a_{km\ell} + a_{\ell mk} + a_{\ell km} + a_{mk\ell} + a_{m\ell k}) = 0\,. \tag{28}$$

The cyclic identity (28) expresses the conservation of energy among the three modes k, ℓ and m.

The system (22) now has the form

$$\frac{dc_k(t)}{dt} = -\eta\lambda_k c_k(t) + \sum_{\ell,\,m} b_{k\ell m} c_\ell(t) c_m(t) + f_k(t)\,. \tag{29}$$

The overall energy relation is obtained from (29) by multiplying the k-th equation by $c_k(t)$ and summing over k. In view of (28) the resulting cubic form on the right side vanishes identically. Thus

$$\frac{d}{dt}\sum_{k=1}^{\infty} c_k^2(t) = -\eta\sum_{k=1}^{\infty}\lambda_k c_k^2(t) + \sum_{k=1}^{\infty} c_k(t) f_k(t)\,. \tag{30}$$

That is, the rate of change of overall energy is determined by the viscous losses due to the individual modes and the losses or the gains due to coupling modes with the imposed body forces (as represented by their Fourier-Helmholtz components $f_k(t)$).

5. Euler's Equations in the Homogeneous Case

When $\eta = 0$ and $f_i(z,\,t) = 0$ the basic Riccati system (22) reduces to homogeneous forms

$$\frac{dc_k(t)}{dt} = \sum_{\ell,\,m} b_{k\ell m} c_\ell(t) c_m(t)\,. \tag{31}$$

With $c_k(t)$ having assigned values $c_k(0)$ initially at $t = 0$, we note the energy conservation relation $\sum_k c_k(t) = \sum_k c_k(0) = \|\underline{u}(x,\,0)\|^2$. Now we construct the formal power series solution in powers of t. Since $b_{k\ell m}$ is symmetric in ℓ, m, we may write

$$\frac{d^2c_k}{dt^2} = 2\sum_{\ell,m} b_{k\ell m}c_\ell(t)\frac{dc_m(t)}{dt} = 2\sum_{\ell,m} b_{k\ell m}c_\ell(t)\sum_{p,q} b_{kpq}c_p(t)c_q(t)$$
$$= 2\sum_{\ell,p,q} d_{k\ell pq}c_\ell(t)c_p(t)c_q(t)\,, \quad (32)$$

where
$$d_{k\ell pq} = \sum_m b_{k\ell m}b_{mpq} \quad (33)$$
$$= \frac{1}{4}\sum_m (a_{k\ell m}a_{mpq} - a_{m\ell k}a_{mpq} + a_{k\ell m}a_{mqp} - a_{m\ell k}a_{mqp})\,.$$

The first of these four terms contains the sum
$$\sum_m a_{k\ell m}a_{mpq} \quad (34)$$
$$= \int_\Omega u_{i_1}^{(k)}(\underline{x})u_{j_1}^{(\ell)}(\underline{x})\sum_m u_{i_1,j_1}^{(m)}(\underline{x})dw(\underline{x})\int_\Omega u_{i_2}^{(m)}(\underline{y})u_{j_2}^{(p)}(\underline{y})u_{i_2,j_2}^{(q)}(\underline{y})dw(\underline{y})$$
$$= \int_\Omega\int_\Omega u_{i_1}^{(k)}(\underline{x})u_{j_1}^{(\ell)}(\underline{x})u_{j_2}^{(p)}(\underline{y})u_{i_2,j_2}^{(q)}(\underline{y})\sum_m u_{i_1,j_1}^{(m)}(\underline{x})u_{i_2}^{(m)}(\underline{y})dw(\underline{x})dw(\underline{y})\,.$$

The inner sum over m yields by (8) the derivative of a delta function, so the expression becomes
$$\int_\Omega\int_\Omega u_{i_1}^{(k)}(\underline{x})u_{j_1}^{(\ell)}(\underline{x})u_{j_2}^{(p)}(\underline{y})u_{i_2,j_2}^{(q)}(\underline{y})\frac{\partial}{\partial x_{i_1}}\delta(x-y)\delta_{i_2}^{i_1}dw(\underline{x})dw(\underline{y}) \quad (35)$$
$$= -\int_\Omega u_{i_1,j_1}^{(k)}(\underline{x})u_{j_1}^{(\ell)}(\underline{x})u_{j_2}^{(p)}(\underline{x})u_{i_2,j_2}^{(q)}(\underline{x})\,.dw(\underline{x})\,.$$

The third term of (33) is obtained from the first term of (33) by interchanging p and q. Skew symmetry allows $a_{km\ell} = -a_{k\ell m}$. When applying this distributional interpolation to the sums over m, it is advisable to remove the differentiation with respect to x_{j_1} by integration by parts first, so that the summation and resulting delta function operate within the inner space of solenoidal vector functions satisfying the mixed boundary condition, within which the bilinear eigenvector sum (8) defines the vector delta function. This has been indicated symbolically, thus introducing the negative sign and derivative in the first factor of the integrand. Because of the solenoidal property of the second factor, the second integral term in (33) similarly yields
$$-\sum_m a_{m\ell k}a_{mpq} \quad (36)$$

$$= \int_\Omega u_{i_1}^{(m)}(\underline{x}) u_{j_1}^{(\ell)}(\underline{x}) u_{i_1,j_1}^{(k)}(\underline{x}) dw(\underline{x}) \cdot \int_\Omega u_{i_2}^{(m)}(\underline{y}) u_{j_2}^{(p)}(\underline{y}) u_{i_2,j_2}^{(q)}(\underline{y}) dw(\underline{y})$$

$$= \int_\Omega u_{i_1}^{(k)}(\underline{x}) u_{i_1,j_2}^{(q)}(\underline{x}) u_{j_2}^{p}(\underline{x}) u_{i_1,j_1}^{(\ell)}(\underline{x}) dw(\underline{x})$$

and the fourth term likewise, with p and ℓ interchanged.

In preparation for systematic further steps of this calculation, we adopt second order subscripts for indices p, q, enumerating eigenvalues by p_1, p_2, . . . and q_1, q_2, . . . We emerge with the formula:

$$d_{kp_1p_2q_1} = \int_\Omega \left[u_{i_1}^{(k)} u_{i_1,j_1}^{(p_1)} - u_{i_1,j_1}^{(k)} u_{j_1}^{(p_1)} \right] \left[u_{j_2}^{(p_2)} u_{i_1,j_2}^{(q_1)} + u_{j_2}^{(q_1)} u_{i_1,j_2}^{(p_2)} \right] dw \,. \tag{37}$$

This is symmetric in q_1 and p_2 but not in k and p_1 or in p_1 and p_2. The index k is fixed. Thus to simplify second derivatives, we define

$$b_{kp_1p_2q} = \frac{1}{3}(d_{kp_1p_2q} + d_{kp_2qp_1} + d_{kqp_1p_2}) \,, \tag{38}$$

so that by tedious calculations of the preceding type,

$$\frac{d^2 c_k(t)}{dt^2} = 2 \sum_{p_1, p_2, q} b_{kp_1p_2q} c_{p_1}(t) c_{p_2}(t) c_q(t) \,. \tag{39}$$

The leading index k is not summed or symmetrized.

Again, differentiating with respect to t we have

$$\frac{d^3 c_k(t)}{dt^3} = 3! \sum_{p_1, p_2, q_1} b_{kp_1p_2q_1} c_{p_1}(t) c_{p_2}(t) \frac{dc_q(t)}{dt} \tag{40}$$

$$= 3! \sum_{p_1, p_2, q_1} b_{kp_1p_2q_1} c_{p_1}(t) c_{p_2}(t) \sum_{p_2, q_2} b_{q_1p_3q_2} c_{p_3}(t) c_{q_2}(t)$$

$$= 3! \sum_{p_1, p_2, p_3, q_2} d_{kp_1p_2p_3q_2} c_{p_1}(t) c_{p_2}(t) c_{p_3}(t) c_{q_2}(t) \,,$$

where

$$d_{kp_1p_2p_3q_2} = \sum_q b_{kp_1p_2q} b_{qp_3q_2} \,.$$

Substituting integral expressions in this sum, as in (33)-(37), we have 24 terms of which the first is

$$- \sum_q \int_\Omega u_{i_1,j_1}^{(k)}(\underline{x}) u_{j_1}^{(p_1)}(\underline{x}) u_{j_2}^{(p_2)}(\underline{x}) u_{i_1,j_2}^{(q)}(\underline{x}) dw(\underline{x}) \tag{41}$$

$$
\begin{aligned}
&\cdot \int_\Omega u_{i_3}^{(q)}(\underline{y}) u_{j_3}^{(p_3)}(\underline{y}) u_{i_3,j_3}^{(q_2)}(\underline{y}) dw(\underline{y}) \\
= &- \int_\Omega \int_\Omega u_{i_1,j_1}^{(k)}(\underline{x}) u_{j_1}^{(p_1)}(\underline{x}) u_{j_2}^{(p_2)}(\underline{x}) u_{j_3}^{(p_3)}(\underline{y}) u_{i_3,j_3}^{(q_2)}(\underline{y}) \\
&\cdot \sum_q \frac{\partial}{\partial x_{j_2}} u_{i_1}^{(q)}(\underline{x}) u_{i_3}^{(q)}(\underline{y}) dw(\underline{x}) dw(\underline{y}) \\
= &\int_\Omega \int_\Omega \left[u_{i_1,j_1,j_2}^{(k)}(\underline{x}) u_{j_1}^{(p_1)}(\underline{x}) + u_{i_1,j_1}^{(k)}(\underline{x}) u_{j_1,j_2}^{(p_1)}(\underline{x}) \right] \\
&\cdot u_{j_2}^{(p_2)}(\underline{x}) u_{j_3}^{(p_3)}(\underline{y}) u_{i_3,j_3}^{(q_2)}(\underline{y}) \delta(x-y) \delta_{i_3}^{i_1} dw(\underline{x}) dw(\underline{y}) \\
= &\int_\Omega \left[u_{i_1,j_1,j_2}^{(k)}(\underline{x}) u_{j_1}^{(p_1)}(\underline{x}) + u_{i_1,j_1}^{(k)}(\underline{x}) u_{j_1,j_2}^{(p_1)}(\underline{x}) \right] \\
&\cdot u_{j_2}^{(p_2)}(\underline{x}) u_{j_3}^{(p_3)}(\underline{x}) u_{i_1,j_3}^{(q_2)}(\underline{x}) dw(\underline{x}) \,.
\end{aligned}
$$

After 23 repetitions of this analysis of terms in $d_{kp_1p_2p_3q_2}$, we have in each case a linear combination of products of single integrals over Ω. Each product has five factors, of which at least two are eigenvector components, the others being derivatives of eigenvector components. As there is already symmetry with respect to the pairs p_1, p_2 and p_3, q_2, pairwise symmetrization with respect to p_1, p_2, p_3 and q_2 together now requires an average over ${}_4C_2 = 6$ such linear combinations, to define $b_{kp_1p_2p_3q_2}$ as an average over 4 slightly different d_{kpqrs} integral terms:

$$b_{kp_1p_2p_3q_2} = \frac{1}{4}(d_{kp_1p_2p_3q_2} + d_{kp_2p_3q_2p_1} + d_{kp_3q_2p_1p_2} + d_{kq_2p_1p_2p_3}) \,. \tag{42}$$

After differentiation and substitution, we find

$$\frac{d^3 c_k(t)}{dt^3} = 3! \sum_{p_1,p_2,p_3,q_2} b_{kp_1p_2p_3q_2} c_{p_1}(t) c_{p_2}(t) c_{p_3}(t) c_{q_2}(t) \,. \tag{43}$$

We can now differentiate this quartic expression (43), in the $c_p(t)$, obtaining

$$\frac{d^4 c_k(t)}{dt^4} = 4! \sum_{p_1,p_2,p_3,q_2} b_{kp_1p_2p_3q_2} c_{p_1}(t) c_{p_2}(t) c_{p_3}(t) \frac{dc_{q_2}(t)}{dt} \,. \tag{44}$$

This leads, on substitution of the first derivative from the original differential equation, to a fifth degree expression in the $c_p(t)$'s. Continuing in this way, alternating between products and symmetrization, an induction on n shows that

$$\frac{d^n c_k(t)}{dt^n} = n! \sum_{p_1, p_2 \dots p_n, q_2} b_{kp_1 p_2 \dots p_n q_2} c_{p_1}(t) c_{p_2}(t) \dots c_{p_n}(t) c_{q_2}(t) \,, \tag{45}$$

where $b_{kp_1p_2\dots p_nq_2}$ is symmetric (except in k). More specifically, $b_{kp_1p_2\dots p_nq_2}$ is a symmetrized linear combination of single integrals over Ω, with $n+2$ integrand factors, each of which is an eigenvector component or a derivative (of order at most $n-1$) of such a component.

Finally, Taylor's formula

$$c_k(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n c_k(0)}{dt^n} t^n \tag{46}$$

yields the following theorem.

Theorem. *The solution of system (31) for the Fourier coefficients of a solution of Euler's equations with initial values*

$$u_i(\underline{x},\, 0) = \sum_{k=1}^{\infty} c_k(0) u_i^{(k)}(\underline{x}) \,,$$

in the force-free case where $\nabla^2 u_i^{(k)} = -\lambda_k u_i^{(k)}$, $u_{i,i}^{(k)} = 0$ *in* Ω, *and* $(\underline{n} \cdot \nabla)\underline{u}^k = 0$ *on* Ω, *is*

$$c_k(t) - c_k(0) = \sum_{n=1}^{\infty} \sum_{p_1 \dots p_n q = 1}^{\infty} b_{kp_1p_2\dots p_n q} c_{p_1}(0) c_{p_2}(0) \dots c_{p_n}(0) c_q(0) t^n \,. \tag{47}$$

Here the $b_{kp_1p_2\dots p_nq_2}$ are the successively symmetrized triple and multiple coefficients of the eigenvector sequence $u_i^{(k)}(\underline{x})$, based on the nonlinear convective terms $u_j(\underline{x},\, t)$, $u_{i,j}(\underline{x},\, t)$ of Euler's equations. They have the recursive form

$$b_{kp_1p_2\dots p_n q} = \sum_{q_1 \cdots q_{n-1}}{}' \, b_{kp_1q_1} b_{q_1p_2q_2} b_{q_2p_3q_3} \dots b_{q_{n-1}p_nq} \,, \tag{48}$$

where the prime on the summation indicates symmetrization over $p_1, p_2, \dots,$ p_n, q.

This is analogous to the recursive analysis of Leipnik [28] using scalar and vector potentials to solve the Navier-Stokes equation. It will be extended to that problem in Section 6, following an example of the Euler equation and showing a connection between the two problems.

6. Special Examples

A) If all N-S eigenvalues λ_i are supposed equal, the N-S system becomes

$$\frac{dc_k(t)}{dt} = -\eta\lambda c_k(t) + \sum_{\ell,m} b_{k\ell m} c_\ell(t) c_m(t) + f_k(t)\,. \tag{49}$$

A common integrating factor $e^{\lambda' t}$ exists, where $\lambda' = \eta\lambda$, and leads to

$$\frac{d}{dt}\left[e^{\lambda' t} c_k(t)\right] = e^{-\lambda' t} \sum_{\ell,m} b_{k\ell m} \left[e^{\lambda' t} c_\ell(t)\right]\left[e^{\lambda' t} c_m(t)\right] + e^{\lambda' t} f_k(t)\,. \tag{50}$$

Multiplying again by $e^{\lambda' t}$ and introducing as new variable

$$\tau = \int_0^t e^{-\lambda' t'} dt' = \frac{1 - e^{-\lambda' t}}{\lambda'}\,, \quad t = \frac{-\ln(1-\lambda'\tau)}{\lambda'}\,, \quad e^{\lambda' t} = \frac{1}{1-\lambda'\tau}\,, \tag{51}$$

where $0 < t < \infty$ maps onto $0 < \tau < \frac{1}{\lambda'}$, we have the Euler system

$$\frac{d}{d\tau} C_k(\tau) = \sum_{\ell,m} b_{k\ell m} C_\ell(\tau) C_m(\tau) + \frac{f_k(-\lambda'^{-1}\ln(1-\lambda'\tau))}{(1-\lambda'\tau)^2}\,, \tag{52}$$

with $\quad C_k(\tau) = e^{\lambda' t} c_k(t) = c_k\left(-\lambda'^{-1}\ln(1-\lambda'\tau)\right)\Big/(1-\lambda'\tau).$

B) It is amusing if not useful to study low-dimensional systems that mimic or approximate the infinite dimensional continuous systems representing fluid flows. An Eulerian system in two variables can be integrated explicitly.

The special two dimensional Eulerian system is

$$\frac{dc_1}{dt} = -b_1 c_1 c_2 + b_2 c_2^2\,; \quad \frac{dc_2}{dt} = -b_2 c_1 c_2 + b_1 c_1^2\,, \tag{53}$$

where

$$b_1 = b_{211} = -2b_{112} = -b_{112} - b_{121}\,; \tag{54}$$

$$b_2 = b_{122} = -2b_{221} = -b_{221} - b_{212}\,.$$

Multiplying the two derivatives by c_1, c_2 respectively, we find $\frac{d}{dt}\left(c_1^2 + c_2^2\right) = 0$; a first integral is $c_1^2 + c_2^2 = c^2 =$ constant. Let $c_1 = c\cos\theta$, $c_2 = c\sin\theta$; the first equation becomes

$$\frac{dc_1}{dt} = -c\sin\theta\frac{d\theta}{dt} = -b_1 c^2 \sin\theta\cos\theta + b_2 c^2 \sin^2\theta\,, \tag{55}$$

from which $\sin\theta = 0$, or

$$\frac{d\theta}{dt} = b_1 c\cos\theta - b_2 c\sin\theta = bc\cos(\theta+\alpha)\,,$$

where $b_1^2 + b_2^2 = b^2$, $b_1 = b\cos\alpha$, $b_2 = b\sin\alpha$. The second equation gives $\cos\theta = 0$ or $\frac{d\theta}{dt} = bc\cos(\theta+\alpha)$.

The elementary secant integral is

$$\sec(\theta+\alpha) + \tan(\theta+\alpha) = e^{bc(t-t_0)}. \tag{56}$$

The singular solution is c_1, c_2 constant, $b_1c_1 = b_2c_2$, and the general relation is

$$c_1 = c\cdot\frac{\cos\alpha \pm \sinh[bc(t-t_0)]\sin\alpha}{\cosh[bc(t-t_0)]}\,, \tag{57}$$

$$c_2 = c\cdot\frac{\pm\sinh[bc(t-t_0)]\cos\alpha - \sin\alpha}{\cosh[bc(t-t_0)]}\,.$$

Values of c, t_0, and $\pm$ will depend on initial conditions.

This is a rotation in α-space with Lorentz variables, since

$$\cosh^2[bc(t-t_0)] - \sinh^2[bc(t-t_0)] = 1.$$

C) A classic integrable case in three variables that can be integrated explicitly is the rigid rotator:

$$\frac{dc_i}{dt} = 2b_{ijk}c_jc_k\,; \quad i \neq j \neq k \text{ not summed} \tag{58}$$

(all b_{ijj} vanish). The substitution $c_i = A_i\omega_i$ transforms this system into the well known system of Euler's equations for a rigid body with moments of inertia A_i and angular velocity ω_i, freely rotating about a fixed centre. The two first integrals in this case are $c_1^2 + c_2^2 + c_3^2 =$ const. and $(b_{231} - b_{321})c_1^2 + (b_{321} - b_{123})c_2^2 + (b_{123} - b_{231})c_3^2 =$ const. As shown in classical mechanics texts, they permit separation of variables and integration to Jacobian elliptic functions for c_1, c_2, c_3.

7. Navier-Stokes Initial Value Problem – Force-Free Case

The Euler equations assume zero viscosity, and in Section 5 did not include external force. In this section, viscosity is included (indirectly) by a linear extension of the Euler equations, but not yet external force. Surprisingly, viscosity greatly complicates the results, though a simple device yields equations of Euler type on a slightly extended index domain. These force-free Navier-Stokes equations take the form of an infinite Riccati system:

$$\frac{dc_k^*}{dt} = -\eta\lambda_k c_k^* + \sum_{\ell,\, m=1}^{\infty} b_{k\ell m} c_\ell^* c_m^*\,, \quad k = 1,\, 2, \ldots\,, \tag{59}$$

where the $\{b_{k\ell m}\}$ are the same as in Section 5, and η is the viscosity. Finite Riccati systems were solved in several places, such as Leipnik [27].

The $\{\lambda_k\}$ are the eigenvalues of the operator $-\nabla^2$, for eigenfunction vectors $\left\{u_j^{(k)}\right\}$ on Ω which are taken as orthogonal and with vanishing tangential components on the boundary $\partial\Omega$, assumed to be piecewise smooth, permitting in particular rectangular boxes for Ω. Thus

$$\nabla^2 u_j^{(k)} = -\lambda_k u_j^{(k)}\,; \quad u_i^{(k)} u_i^{(\ell)} = \delta_k^\ell \text{ on } \Omega\,, \tag{60}$$
$$(\underline{n} \cdot \nabla)\underline{u}^{(k)} = 0 \text{ on } \partial\Omega\,,$$

where $\underline{n}$ is perpendicular to $\partial\Omega$, except at edges of Ω.

These equations can be written as Euler equations of the extended type:

$$\frac{dc_k^*}{dt} = \sum_{\ell,\, m=0}^{\infty} b_{k\ell m} c_\ell^* c_m^*\,, \quad k = 0,\, 1,\, 2, \ldots\,, \tag{61a}$$

with

$$c_0^*(t) = 1\,, \quad b_{0\ell m} = 0\,, \quad \text{for } \ell,\, m = 0,\, 1,\, 2, \ldots\,, \qquad 61b$$
$$b_{kk0} = b_{k0k} = (-1/2)\eta\lambda_k\,, \quad k = 1,\, 2, \ldots$$

and $b_{k\ell m}$ is unchanged for $k > 0$, ℓ, $m = 1, 2, \ldots$ Then the Euler solutions of Section 5 are used, with the extended indices. Formally this is satisfactory, but the physical behavior is concealed. To reveal it, by showing the effect of the viscosity and the shape of Ω, requires considerable analytic and algebraic effort. The external forces can be included later, by use of an embedding pseudo-time s.

Indicative of the Taylor approach to (59) is the once-differentiated equation (59)

$$\frac{d^2 c_k^*}{dt^2} = (-\eta\lambda_k)^2 c_k^* - \eta \sum_{\ell,\, m=1}^{\infty} b_{k\ell m}(\lambda_k + \lambda_\ell + \lambda_m) c_\ell^* c_m^* \tag{62}$$
$$+2 \sum_{p_1 p_2 q} b_{kp_1p_2q} c_{p_1}^* c_{p_2}^* c_q^*\,.$$

The successive derivatives will clearly take the form

$$\frac{d^n c_k^*}{dt^n} = (-\eta\lambda_k)^n c_k^* \tag{63}$$

$$+\sum_{\ell_1 q} A_{k,\ell_1 q}^{(1,n)} c_{\ell_1}^* c_q^* + \sum_{\ell_1\ell_2 q} A_{k,\ell_1\ell_2 q}^{(2,n)} c_{\ell_1}^* c_{\ell_2}^* c_q^*$$

$$+\cdots+\sum_{\ell_1\ell_2\ldots\ell_{n-1}q} A_{k,\ell_1\ell_2\ldots\ell_{n-1}q}^{(n-1,n)} c_{\ell_1}^* c_{\ell_2}^* \cdots c_{\ell_{n-1}}^* c_q^*$$

$$+n!\sum_{\ell_1\ldots\ell_n q} b_{k,\ell_1\ldots\ell_n q} c_{\ell_1}^* \cdots c_{\ell_n}^* c_q^*\,, \quad n = 2,3,4,\ldots\,.$$

For $n = 2$,

$$A_{k,\ell_1 q}^{(1,2)} = -\eta(\lambda_{\ell_1} + \lambda_q + \lambda_k) b_{k,\ell_1 q} \tag{64}$$

satisfies (63). For $n = 3$, $A^{(1,3)}$ and $A^{(2,3)}$ remain to be identified. Now

$$\frac{d^3 c_k^*}{dt^3} = (-\eta\lambda_k)^3 c_k^* x + \sum_{\ell_1 q} A_{k,\ell_1 q}^{(1,3)} c_{\ell_1}^* c_q^* \tag{65}$$

$$+\sum_{\ell_1\ell_2 q} A_{k,\ell_1\ell_2 q}^{(2,3)} c_{\ell_1}^* c_{\ell_2}^* c_q^* \;+\; 6\sum_{\ell_1\ell_2\ell_3 q} b_{k,\ell_1\ell_2\ell_3 q} c_{\ell_1}^* c_{\ell_2}^* c_{\ell_3}^* c_q^*$$

$$= \frac{d}{dt}\left[(-\eta\lambda_k)^2 c_k^* - \eta\sum_{\ell,\, m=1}^{\infty} b_{k\ell m}(\lambda_k + \lambda_\ell + \lambda_m) c_\ell^* c_m^* \right.$$

$$\left. +2\sum_{p_1 p_2 q} b_{k p_1 p_2 q} c_{p_1}^* c_{p_2}^* c_q^*\right],$$

so

$$\frac{d^3 c_k^*}{dt^3} = (-\eta\lambda_k)^2 \left[-\eta\lambda_k c_k^* + \sum_{ru} b_{kru} c_r^* c_u^*\right] \tag{66}$$

$$-\eta\sum_{\ell m} b_{k\ell m}(\lambda_k + \lambda_\ell + \lambda_m)\left[c_\ell^*\left(-\eta\lambda_m c_m^* + \sum_{r_1 u_1} b_{m r_1 u_1} c_{r_1}^* c_{u_1}^*\right)\right.$$

$$\left. +c_m^*\left(-\eta\lambda_\ell c_\ell^* + \sum_{r_2 u_2} b_{\ell r_2 u_2} c_{r_2}^* c_{u_2}^*\right)\right]$$

$$+2\sum_{p_1p_2q} b_{kp_1p_2q}\left\{c^*_{p_1}c^*_{p_2}\left(-\eta\lambda_q c^*_q + \sum_{r_3u_3} b_{qr_3u_3}c^*_{r_3}c^*_{u_3}\right)\right.$$
$$+c^*_{p_1}c^*_q\left(-\eta\lambda_{p_2}c^*_{p_2} + \sum_{r_4u_4} b_{p_2r_4u_4}c^*_{r_4}c^*_{u_4}\right)$$
$$\left.+c^*_{p_2}c^*_q\left(-\eta\lambda_{p_1}c^*_{p_1} + \sum_{r_5u_5} b_{p_1r_5u_5}c^*_{r_5}c^*_{u_5}\right)\right\}.$$

This rearranges to

$$(-\eta\lambda_k)^3c^*_k + \left[(-\eta\lambda_k)^2\sum_{ru} b_{kru}c^*_rc^*_u \right. \tag{67}$$
$$\left. +\eta^2\sum_{\ell m} b_{k\ell m}(\lambda_k+\lambda_\ell+\lambda_m)(\lambda_m c^*_\ell c^*_m + \lambda_\ell c^*_\ell c^*_m)\right]$$
$$-\eta\sum_{\ell m} b_{k\ell m}(\lambda_k+\lambda_\ell+\lambda_m)\left\{\sum_{r_1u_1} c^*_\ell c^*_{r_1}c^*_{u_1}b_{mr_1u_1} + \sum_{r_2u_2} c^*_m c^*_{r_2}c^*_{u_2}b_{\ell r_2u_2}\right\}$$
$$-2\eta\sum_{p_1p_2q} b_{kp_1p_2q}\left\{(\lambda_{p_1}+\lambda_{p_2}+\lambda_q)c^*_{p_1}c^*_{p_2}c^*_q\right\}$$
$$+2\sum_{p_1p_2q} b_{kp_1p_2q}\left\{\sum_{r_3u_3} b_{qr_3u_3}c^*_{p_1}c^*_{p_2}c^*_{r_3}c^*_{u_3}\right.$$
$$\left.+\sum_{r_4u_4} b_{p_2r_4u_4}c^*_{p_1}c^*_qc^*_{r_4}c^*_{u_4} + \sum_{r_5u_5} b_{p_1r_5u_5}c^*_{p_2}c^*_qc^*_{r_5}c^*_{u_5}\right\}.$$

The first term is obvious; the last term is Eulerian. Matching intermediate expressions produces

$$\eta^2 b_{k,\ell_1q}\left[\lambda_k^2 + (\lambda_k+\lambda_{\ell_1}+\lambda_q)(\lambda_{\ell_1}+\lambda_q)\right] = A^{(1,3)}_{k,\ell_1q}. \tag{68}$$

This leaves only $A^{(2,3)}_{k,\ell_1\ell_2q}$. From (67),

$$-\eta\sum_{\ell_1\ell_2} b_{k\ell_1\ell_2}(\lambda_k+\lambda_{\ell_1}+\lambda_{\ell_2})\sum_{r_1q_1} b_{\ell_2r_1q_1}c^*_{\ell_1}c^*_{r_1}c^*_{q_1} \tag{69}$$
$$-\eta\sum_{\ell_1\ell_2} b_{k\ell_1\ell_2}(\lambda_k+\lambda_{\ell_1}+\lambda_{\ell_2})\sum_{r_2q_2} b_{\ell_1r_2q_2}c^*_{\ell_2}c^*_{r_2}c^*_{q_2}$$

$$-2\eta \sum_{\ell_1\ell_2 q} b_{k\ell_1\ell_2 q}(\lambda_{\ell_1} + \lambda_{\ell_2} + \lambda_q) c^*_{\ell_1} c^*_{\ell_2} c^*_q$$
$$= \sum_{\ell_1\ell_2 q} A^{(2,3)}_{k,\ell_1\ell_2 q} c^*_{\ell_1} c^*_{\ell_2} c^*_q \,,$$

where the u_1 and u_2 indices in (67) are replaced by q.

Detachment from c^*_q in (69) is feasible, leaving, by symmetry, the quadratic forms

$$(-\eta)\left[\sum_{\ell_2} b_{k\ell_1\ell_2}\Lambda_{k\ell_1\ell_2}\left(\sum_{r_1} b_{\ell_2 r_1 q} c^*_{\ell_1} c^*_{r_1} + \sum_{r_1} b_{\ell_1 r_1 q} c^*_{\ell_2} c^*_{r_1}\right) \ell_1\right. \tag{70a}$$

$$\left. +2\sum_{\ell_1\ell_2} b_{k\ell_1\ell_2 q}\Lambda_{q\ell_1\ell_2} c^*_{\ell_1} c^*_{\ell_2}\right] = \sum_{m_1 m_2} A^{(2,3)}_{k,m_1 m_2 q} c^*_{m_1} c^*_{m_2} \,,$$

where

$$\Lambda_{k\ell_1\ell_2} = \lambda_k + \lambda_{\ell_1} + \lambda_{\ell_2} \,. \tag{70b}$$

Now if

$$e_{k\ell m} = b_{k\ell m}\Lambda_{k\ell m} \tag{71a}$$

and

$$\widetilde{e}_{k\ell_1\ell_2 q} = \sum_r (e_{k\ell_1 r} b_{r\ell_2 q} + e_{k\ell_2 r} b_{r\ell_1 q}) \,, \tag{71b}$$

then the second term of (70a) yields

$$-\eta\left(\widetilde{e}_{km_1 m_2 q} + 2e_{m_1 m_2 q}\right) = A^{(2,3)}_{k,m_1 m_2 q} \tag{72}$$

completing (63) for $n = 3$. This is fully symmetrical in m_1, m_2, q.

The general inductive step proceeds similarly. For $n \geq 2$, calculate

$$\frac{d^{n+1} c^*_k}{dt^{n+1}} = (-\eta\lambda_k)^n \dot{c}^*_k + \sum_{\ell q} A^{(1,n)}_{k\ell q}\left(\dot{c}^*_\ell c^*_q + c^*_\ell \dot{c}^*_q\right) \tag{73}$$

$$+\sum_{r=2}^{n-1}\sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q}\left[\dot{c}^*_{\ell_1} c^*_{\ell_2}\cdots c^*_q\right.$$

$$\left. +\sum_{s=2}^{r-1} c^*_{\ell_1}\cdots\dot{c}^*_{\ell_s}\cdots c^*_{\ell_r} c^*_q + c^*_{\ell_1}\cdots c^*_{\ell_{r-1}}\dot{c}^*_{\ell_r} c^*_q + c^*_{\ell_1}\cdots c^*_{\ell_r}\dot{c}^*_q\right]$$

$$+n!\sum_{\ell_1\ldots\ell_n q} b_{k\ell_1\ldots\ell_n q}\left[\dot{c}^*_{\ell_1}c^*_{\ell_2}\cdots c^*_{\ell_n}c^*_q\right.$$
$$\left.+\sum_{s=2}^{n-1} c^*_{\ell_1}\cdots\dot{c}^*_{\ell_s}\cdots c^*_{\ell_n}c^*_q + c^*_{\ell_1}\cdots\dot{c}^*_{\ell_n}c^*_q + c^*_{\ell_1}\cdots c^*_{\ell_n}\dot{c}^*_q\right].$$

This can be viewed as 11 slightly different batches of terms. Substitution of the derivative formula (29) leads to 22 batches of terms of different orders and sizes:

$$\frac{d^{n+1}c^*_k}{dt^{n+1}} = (-\eta\lambda_k)^n\left[-\eta\lambda_k c^*_k + \sum_{u_1v_1} b_{ku_1v_1}c^*_{u_1}c^*_{v_1}\right] \tag{74}$$
$$\sum_{\ell q} A^{(1,n)}_{k\ell q}\left[\left(-\eta\lambda_\ell c^*_\ell + \sum_{u_2v_2} b_{\ell u_2v_2}c^*_{u_2}c^*_{v_2}\right)c^*_q\right.$$
$$\left.+c^*_\ell\left(-\eta\lambda_q c^*_q + \sum_{u_3v_3} b_{qu_3v_3}c^*_{u_3}c^*_{v_3}\right)\right]$$
$$+\sum_{r=2}^{n-1}\sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q}\left[\left(-\eta\lambda_{\ell_1}c^*_{\ell_1} + \sum_{u_4v_4} b_{\ell_1u_4v_4}c^*_{u_4}c^*_{v_4}\right)c^*_{\ell_2}\cdots c^*_{\ell_r}c^*_q\right.$$
$$+\sum_{s=2}^{r-1} c^*_{\ell_1}\cdots c^*_{\ell_{s-1}}\left(-\eta\lambda_{\ell_s}c^*_{\ell_s} + \sum_{u_5v_5} b_{\ell_su_5v_5}c^*_{u_5}c^*_{v_5}\right)c^*_{\ell_{s+1}}\cdots c^*_{\ell_r}c^*_q$$
$$+c^*_{\ell_1}\cdots c^*_{\ell_{r-1}}\left(-\eta\lambda_{\ell_r}c^*_{\ell_r} + \sum_{u_6v_6} b_{\ell_ru_6v_6}c^*_{u_6}c^*_{v_6}\right)c^*_q$$
$$\left.+c^*_{\ell_1}\cdots c^*_{\ell_r}\left(-\eta\lambda_q c^*_q + \sum_{u_7v_7} b_{qu_7v_7}c^*_{u_7}c^*_{v_7}\right)\right]$$
$$+n!\sum_{\ell_1\ldots\ell_n q} b_{k\ell_1\ldots\ell_n q}\left[\left(-\eta\lambda_{\ell_1}c^*_{\ell_1} + \sum_{u_8v_8} b_{\ell_1u_8v_8}c^*_{u_8}c^*_{v_8}\right)c^*_{\ell_2}\cdots c^*_{\ell_n}c^*_q\right.$$
$$+\sum_{s=2}^{n-1} c^*_{\ell_1}\cdots c^*_{\ell_{s-1}}\left(-\eta\lambda_{\ell_s}c^*_{\ell_s} + \sum_{u_9v_9} b_{\ell_su_9v_9}c^*_{u_9}c^*_{v_9}\right)c^*_{\ell_{s+1}}\cdots c^*_{\ell_n}c^*_q$$
$$+c^*_{\ell_1}\cdots c^*_{\ell_{n-1}}\left(-\eta\lambda_{\ell_n}c^*_{\ell_n} + \sum_{u_{10}v_{10}} b_{\ell_nu_{10}v_{10}}c^*_{u_{10}}c^*_{v_{10}}\right)c^*_q$$

$$+c^*_{\ell_1}\cdots c^*_{\ell_n}\left(-\eta\lambda_q c^*_q + \sum_{u_{11}v_{11}} b_{qu_{11}v_{11}} c^*_{u_{11}} c^*_{v_{11}}\right)\Bigg] .$$

Direct formation of the $(n+1)$-st derivative yields the shorter result:

$$\frac{d^{n+1}c^*_k}{dt^{n+1}} = (-\eta\lambda_k)^{n+1}c^*_k + \sum_{\ell q} A^{(1,n+1)}_{k\ell q} c^*_\ell c^*_q \tag{75}$$

$$+\sum_{r=2}^{n}\sum_{\ell_1\ldots\ell_r q} A^{(r,n+1)}_{k\ell_1\ldots\ell_r q} c^*_{\ell_1}\cdots c^*_{\ell_r} c^*_q + (n+1)! \sum_{\ell_1\ldots\ell_{n+1}q} b_{k\ell_1\ldots\ell_{n+1}q} c^*_{\ell_1}\cdots c^*_{\ell_n} c^*_{\ell_{n+1}} c^*_q .$$

When (74) and (75) are compared, $A^{(r,\,n+1)}$ will depend linearly on $A^{(p,\,n)}$ for $p \le r$ and on the b's and λ's as coefficients. Formula (74) can be separated into 22 parts.

$$B^{(n)}_{11} = (-\eta\lambda_k)^{n+1} c^*_k\,, \tag{76.1}$$

$$B^{(n)}_{12} = (-\eta\lambda_k)^{n} \sum_{u_1 v_1} b_{ku_1v_1} c^*_{u_1} c^*_{v_1}\,, \tag{76.2}$$

$$B^{(n)}_{22} = -\eta \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \lambda_{\ell_1} c^*_{\ell_1} c^*_q\,, \tag{76.3}$$

$$B^{(n)}_{23} = \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \sum_{u_2 v_2} b_{\ell_1 u_2 v_2} c^*_{u_2} c^*_{v_2} c^*_q\,, \tag{76.4}$$

$$B^{(n)}_{32} = -\eta \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \lambda_q c^*_{\ell_1} c^*_q\,, \tag{76.5}$$

$$B^{(n)}_{33} = \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \sum_{u_3 v_3} b_{qu_3 v_3} c^*_{\ell_1} c^*_{u_3} c^*_{v_3}\,, \tag{76.6}$$

$$B^{(n)}_{4,\,r+1} = -\eta \sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q} \lambda_{\ell_1} c^*_{\ell_1} c^*_{\ell_2}\cdots c^*_{\ell_r} c^*_q \tag{76.7}$$

(summed over $r = 2$ to $n-1$)

$${B'_{4,\,r+2}}^{(n)} = \sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q} \sum_{u_4 v_4} b_{\ell_1 u_4 v_4} c^*_{\ell_2}\cdots c^*_{\ell_r} c^*_{u_4} c^*_{v_4} c^*_q \tag{76.8}$$

(summed over $r = 2$ to $n-1$)

$$B^{(s,\,n)}_{5,\,r+1} = -\eta \sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q} \lambda_{\ell_s} c^*_{\ell_1}\cdots c^*_{\ell_r} c^*_q \tag{76.9}$$

(summed over $s = 2, 3, \dots r-1$ and then over $r = 3$ to $n-1$)

$$B'^{(s,n)}_{5,\,r+2} = \sum_{\ell_1\dots\ell_r q} A^{(r,n)}_{k\ell_1\dots\ell_r q} \sum_{u_5 v_5} b_{\ell_s u_5 v_5} c^*_{\ell_1} \cdots c^*_{\ell_{s-1}} c^*_{\ell_{s+1}} \cdots c^*_{\ell_r} c^*_{u_5} c^*_{v_5} c^*_q \quad (76.10)$$

(summed over $s = 2, 3, \dots r-1$ and then over $r = 3$ to $n-1$)

$$B^{(n)}_{6,r+1} = -\eta \sum_{\ell_1\dots\ell_r q} A^{(r,n)}_{k\ell_1\dots\ell_r q} \lambda_{\ell_r} c^*_{\ell_1} \cdots c^*_{\ell_r} c^*_q \quad (76.11)$$

(summed over $r = 2$ to $n-1$)

$$B^{(n)}_{6,\,r+2} = \sum_{\ell_1\dots\ell_r q} A^{(r,n)}_{k\ell_1\dots\ell_r q} \sum_{u_6 v_6} b_{\ell_r u_6 v_6} c^*_{\ell_1} \cdots c^*_{\ell_{r-1}} c^*_{u_6} c^*_{v_6} c^*_q \quad (76.12)$$

(summed over $r = 2$ to $n-1$)

$$B^{(n)}_{7,\,r+1} = -\eta \sum_{\ell_1\dots\ell_r q} A^{(r,n)}_{k\ell_1\dots\ell_r q} \lambda_q c^*_{\ell_1} \cdots c^*_{\ell_r} c^*_q \quad (76.13)$$

(summed over $r = 2$ to $n-1$)

$$B'^{(n)}_{7,\,r+2} = \sum_{\ell_1\dots\ell_r q} A^{(r,n)}_{k\ell_1\dots\ell_r q} \sum_{u_7 v_7} b_{q u_7 v_7} c^*_{\ell_1} \cdots c^*_{\ell_r} c^*_{u_7} c^*_{v_7} \quad (76.14)$$

(summed over $r = 2$ to $n-1$)

The remaining B's do not involve A's.

$$B^{(n)}_{8,\,n+1} = -\eta n! \sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q} \lambda_{\ell_1} c^*_{\ell_1} c^*_{\ell_2} \cdots c^*_{\ell_n} c^*_q \quad (76.15)$$

$$B'^{(n)}_{8,\,n+2} = n! \sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q} \sum_{u_8 v_8} b_{\ell_1 u_8 v_8} c^*_{\ell_2} \cdots c^*_{\ell_n} c^*_{u_8} c^*_{v_8} c^*_q \quad (76.16)$$

$$B^{(s,n)}_{9,\,n+1} = -\eta n! \sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q} \lambda_{\ell_s} c^*_{\ell_1} \cdots c^*_{\ell_n} c^*_q \quad (76.17)$$

(summed over $s = 2, 3, \dots n-1$)

$$B'^{(s,n)}_{9,\,n+2} = n! \sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q} \sum_{u_9 v_9} b_{\ell_s u_9 v_9} c^*_{\ell_1} \cdots c^*_{\ell_{s-1}} c^*_{\ell_{s+1}} \cdots c^*_{\ell_n} c^*_{u_9} c^*_{v_9} c^*_q \quad (76.18)$$

(summed over $s = 2, 3, \dots n-1$)

$$B^{(n)}_{10,\,n+1} = -\eta n! \sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q} \lambda_{\ell_n} c^*_{\ell_1} \cdots c^*_{\ell_n} c^*_q \quad (76.19)$$

$$B'^{(n)}_{10,\,n+2} = n! \sum_{\ell_1 \dots \ell_n q} b_{k\ell_1 \dots \ell_n q} \sum_{u_{10} v_{10}} b_{\ell_n u_{10} v_{10}} c^*_{\ell_1} \cdots c^*_{\ell_{n-1}} c^*_{u_{10}} c^*_{v_{10}} c^*_q \qquad (76.20)$$

$$B^{(n)}_{11,\,n+1} = -\eta n! \sum_{\ell_1 \dots \ell_n q} b_{k\ell_1 \dots \ell_n q} \lambda_q c^*_{\ell_1} \cdots c^*_{\ell_n} c^*_q \qquad (76.21)$$

$$B'^{(n)}_{11,\,n+2} = n! \sum_{\ell_1 \dots \ell_n q} b_{k\ell_1 \dots \ell_n q} \sum_{u_{11} v_{11}} b_{q u_{11} v_{11}} c^*_{\ell_1} \cdots c^*_{\ell_n} c^*_{u_{11}} c^*_{v_{11}} \,. \qquad (76.22)$$

The agreement between (74) and (75), needed to define the coefficients, can be obtained in six stages. The first order trivially equates

$$(-\eta\lambda_k)^{n+1} c^*_k \text{ and } (-\eta\lambda_k)^n (-\eta\lambda_k c^*_k)\,.$$

The second order compares (76.2)

$$B^{(n)}_{12} + B^{(n)}_{22} + B^{(n)}_{32} = (-\eta\lambda_k)^n \sum_{u_1 v_1} b_{k u_1 v_1} c^*_{u_1} c^*_{v_1} \qquad (77)$$

$$-\eta \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \lambda_{\ell_1} c^*_{\ell_1} c^*_q - \eta \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \lambda_q c^*_{\ell_1} c^*_q \,,$$

with

$$\sum_{\ell q} A^{(1,n+1)}_{k\ell q} c^*_\ell c^*_q \,.$$

Thus

$$(-\eta\lambda_k)^n b_{k\ell q} - \eta A^{(1,\,n)}_{k\ell q} (\lambda_\ell + \lambda_q) = A^{(1,\,n+1)}_{k\ell q} \,, \quad \text{for} \quad n \geq 2\,. \qquad (78)$$

The third order involves five batches:

$$B^{(n)}_{23} + B^{(n)}_{33} + B^{(n)}_{4,3} + B^{(n)}_{6,3} + B^{(n)}_{7,3} = \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \sum_{\ell_2 v_2} b_{\ell_1 \ell_2 v_2} c^*_{\ell_2} c^*_{v_2} c^*_q \qquad (79)$$

$$+ \sum_{\ell_1 q} A^{(1,n)}_{k\ell_1 q} \sum_{\ell_2 v_3} b_{q \ell_2 v_3} c^*_{\ell_1} c^*_{\ell_2} c^*_{v_3} - \eta \sum_{\ell_1 \ell_2 q} \Lambda_{\ell_1 \ell_2 q} A^{(2,n)}_{k\ell_1 \ell_2 q} c^*_{\ell_1} c^*_{\ell_2} c^*_q$$

compared with

$$\sum_{\ell_1 \ell_2 q} A^{(2,n+1)}_{k\ell_1 \ell_2 q} c^*_{\ell_1} c^*_{\ell_2} c^*_q \,.$$

Extracting the components and interchanging subscripts several times finally yields

$$2 \sum_v A^{(1,\,n)}_{k v \ell_1} b_{v \ell_2 q} = A^{(2,\,n+1)}_{k\ell_1 \ell_2 q} + \eta A^{(2,\,n)}_{k\ell_1 \ell_2 q} \Lambda_{\ell_1 \ell_2 q} \qquad (80)$$

for $n \geq 3$, and its dual when ℓ_1 and ℓ_2 are exchanged. This recursion is started by $A^{(2,3)}_{k\ell_1\ell_2 q}$, which is given in (72), and of course $A^{(1,n)}_{k\ell q}$ has been determined in (78) and previously. It is expected that $A^{(r,\,n+1)}$ will depend on $A^{(r,\,n)}$ and $A^{(r-1,\,n)}$ as a weighted version of $\begin{pmatrix} n+1 \\ r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} + \begin{pmatrix} n \\ r-1 \end{pmatrix}$ for binomial coefficients. Thus (80) is analogous to $\begin{pmatrix} n+1 \\ 2 \end{pmatrix} = \begin{pmatrix} n \\ 2 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix}$.

The fourth order is the first one with a full set of relations, continuing to the $(n-2)$nd. So take r between 3 and $n-1$, and consider the terms in (74) and (75). The eight batches of (76) involved are

$$B^{(n)}_{4,\,r+1}\,,\; {B'_{4,\,r}}^{(n)}\,,\; B^{(s,\,n)}_{5,\,r+1}\,,\; {B'_{5,\,r+2}}^{(s,\,n)} \;\; \text{summed over} \;\; s = 2, \ldots r-1\,,$$

$$B^{(n)}_{6,\,r+1}\,,\; {B'_{6,\,r}}^{(n)}\,,\; B^{(n)}_{7,\,r+1}\,,\; {B'_{7,\,r+2}}^{(n)}\,,$$

so their sum is

$$\begin{aligned}
&-\eta \sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q} \lambda_{\ell_1} c^*_{\ell_1} \cdots c^*_{\ell_r} c^*_q \\
&+ \sum_{\ell_1\ldots\ell_{r-1} q} A^{(r-1,n)}_{k\ell_1\ldots\ell_{r-1} q} \sum_{u_4 v_4} b_{\ell_1 u_4 v_4} c^*_{\ell_2} \cdots c^*_{\ell_{r-1}} c^*_{u_4} c^*_{v_4} c^*_q \\
&-\eta \left(\sum_{s=2}^{r-1} \lambda_{\ell_s}\right) \sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q} c^*_{\ell_1} \cdots c^*_{\ell_r} c^*_q \\
&+\sum_{s=2}^{r-1} \sum_{\ell_1\ldots\ell_{r-1} q} A^{(r-1,n)}_{k\ell_1\ldots\ell_{r-1} q} \sum_{u_5 v_5} b_{\ell_s u_5 v_5} c^*_{\ell_1} \cdots c^*_{\ell_{s-1}} c^*_{\ell_{s+1}} \cdots c^*_{\ell_{r-1}} c^*_{u_5} c^*_{v_5} c^*_q \\
&-\eta \sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q} \lambda_{\ell_r} c^*_{\ell_1} \cdots c^*_{\ell_r} c^*_q \\
&+ \sum_{\ell_1\ldots\ell_{r-1} q} A^{(r-1,n)}_{k\ell_1\ldots\ell_{r-1} q} \sum_{u_6 v_6} b_{\ell_{r-1} u_6 v_6} c^*_{\ell_1} \cdots c^*_{\ell_{r-2}} c^*_{u_6} c^*_{v_6} c^*_q \\
&-\eta \sum_{\ell_1\ldots\ell_r q} A^{(r,n)}_{k\ell_1\ldots\ell_r q} \lambda_q c^*_{\ell_1} \cdots c^*_{\ell_r} c^*_q \\
&+ \sum_{\ell_1\ldots\ell_{r-1} q} A^{(r-1,n)}_{k\ell_1\ldots\ell_{r-1} q} \sum_{u_7 v_7} b_{q u_7 v_7} c^*_{\ell_1} \cdots c^*_{\ell_{r-1}} c^*_{u_7} c^*_{v_7}\,.
\end{aligned} \tag{81}$$

These 8 multiple sums in (81) should add to

$$\sum_{\ell_1\dots\ell_r q} A^{(r,n+1)}_{k\ell_1\dots\ell_r q} c^*_{\ell_1}\cdots c^*_{\ell_r} c^*_q\,,\ \text{for } 3\le r\le n-1.$$

The coefficient recursion that emerges is

$$A^{(r,\,n+1)}_{k\ell_1\ell_1\dots\ell_r q} + \eta A^{(r,\,n)}_{k\ell_1\dots\ell_r q}\Lambda_{\ell_1\dots\ell_r q} = r\sum_v A^{(r-1,\,n)}_{kv\ell_1\dots\ell_r} b_{v\ell_r q}\,,\ \text{for}\ \ r\le n-1\,, \tag{82a}$$

where

$$\Lambda_{\ell_1\ell_2\dots\ell_r q} = \lambda_{\ell_1}+\lambda_{\ell_2}+\cdots+\lambda_{\ell_r}+\lambda_q\,. \tag{82b}$$

Also valid are the $r!-1$ duals of (82a) under permutations of $\ell_1,\ \ell_2,\ \dots\ell_r$. Verbally, the first, third, fifth, and seventh of the preceding form the terms in (82a) containing $-\eta$. The second, fourth, sixth, and eighth correspond to the terms in $A^{(r-1,\,n)}$. The fixed placement of v is arbitrary, but does not matter because the A's are to be independent of the order of the subscripts.

The sixth stage is given by terms not related to viscosity, that is, the Eulerian case, which repeats recursions previously established. However, the fifth stage remains:

$$B'_{4,\,n+1}{}^{(n)} = \sum_{\ell_1\dots\ell_{n-1}q} A^{(n-1,n)}_{k\ell_1\dots\ell_{n-1}q}\sum_{u_4v_4} b_{\ell_1u_4v_4}c^*_{\ell_2}\cdots c^*_{\ell_{n-1}}c^*_{u_4}c^*_{v_4}c^*_q\,, \tag{83}$$

$$B'_{5,\,n+1}{}^{(s,\,n)} = \sum_{\ell_1\dots\ell_{n-1}q} A^{(n-1,n)}_{k\ell_1\dots\ell_{n-1}q}\sum_{u_5v_5} b_{\ell_s u_5v_5}c^*_{\ell_1}\cdots c^*_{\ell_{s-1}}c^*_{\ell_{s+1}}\cdots c^*_{\ell_{n-1}}c^*_{u_5}c^*_{v_5}c^*_q$$

(summed over $s=2,\,3,\dots n-1$),

$$B'_{6,\,n+1}{}^{(n)} = \sum_{\ell_1\dots\ell_{n-1}q} A^{(n-1,n)}_{k\ell_1\dots\ell_{n-1}q}\sum_{u_6v_6} b_{\ell_{n-1}u_6v_6}c^*_{\ell_1}\cdots c^*_{\ell_{n-2}}c^*_{u_6}c^*_{v_6}c^*_q\,,$$

$$B'_{7,\,n+1}{}^{(n)} = \sum_{\ell_1\dots\ell_{n-1}q} A^{(n-1,n)}_{k\ell_1\dots\ell_{n-1}q}\sum_{u_7v_7} b_{qu_7v_7}c^*_{\ell_1}\cdots c^*_{\ell_{n-1}}c^*_{u_7}c^*_{v_7}\,,$$

$$B^{(n)}_{8,\,n+1} = -\eta n!\sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q}\lambda_{\ell_1}c^*_{\ell_1}c^*_{\ell_2}\cdots c^*_{\ell_n}c^*_q\,,$$

$$B^{(s,\,n)}_{9,\,n+1} = -\eta n!\sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q}\lambda_{\ell_s}c^*_{\ell_1}\cdots c^*_{\ell_n}c^*_q$$

(summed over $s=2,\,3,\dots n-1$),

$$B^{(n)}_{10,\,n+1} = -\eta n!\sum_{\ell_1\dots\ell_n q} b_{k\ell_1\dots\ell_n q}\lambda_{\ell_n}c^*_{\ell_1}\cdots c^*_{\ell_n}c^*_q\,,$$

$$B^{(n)}_{11,\,n+1} = -\eta n! \sum_{\ell_1 \ldots \ell_n q} b_{k\ell_1 \ldots \ell_n q} \lambda_q c^*_{\ell_1} \cdots c^*_{\ell_n} c^*_q \,.$$

This again covers multiple summations, 4 without η and 4 with η. Comparison with the two types of terms in the fourth stage shows that

$$A^{(n,\,n+1)}_{k\ell_1\ell_1 \ldots \ell_n q} + \eta n! \Lambda_{\ell_1 \ldots \ell_n q} b_{k\ell_1 \ldots \ell_n q} = n \sum_v A^{(n-1,\,n)}_{kv\ell_1 \ldots \ell_{n-1}} b_{v\ell_n q} \,. \tag{84}$$

This is slightly simpler than (82a), for any r.

We can now state the major result for force-free Navier-Stokes (Fourier-Helmholtz) time modes.

Theorem. *The solution of system (59) is*

$$c^*_k(t) - c^*_k(0) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \Bigg\{ (-\eta\lambda_k)^n \, c^*_k(0) \tag{85}$$

$$+ \sum_{r=1}^{n-1} \sum_{\ell_1 \ldots \ell_r q=1}^{\infty} A^{(r,\,n)}_{k\ell_1\ell_2 \ldots \ell_r q} \left[\prod_{m=1}^{r} c^*_m(0) c^*_q(0) \right]$$

$$+ n! \sum_{\ell_1 \ldots \ell_r q=1}^{\infty} b_{k\ell_1 \ldots \ell_r q} \left[\prod_{m=1}^{n} c^*_m(0) c^*_q(0) \right] \Bigg\} \,,$$

where the A coefficients are defined in (78), (80), (82a), (84) and the b coefficients in (23), (27), and (48).

These time-modes yield for the Navier-Stokes velocity

$$u_i(\underline{x},\, t) = \sum_k u^{(k)}_i(\underline{x}) c^*_k(t) \,, \tag{86}$$

where $u^{(k)}_i(\underline{x})$ is the i-th component of the eigenvector $\underline{u}^{(k)}(\underline{x})$ satisfying $\nabla^2 u^{(k)}_i(\underline{x}) = -\lambda_k u^{(k)}_i(\underline{x})$.

8. Navier-Stokes Initial Value Problem – with External Force

When the system is subjected to an external force $\underline{f}(\underline{x},\, t)$, it is necessary, in the present approach, that the Fourier-Helmholtz components $f_j(t) = \int_\Omega f_i(\underline{x},\, t) u^{(j)}_i(\underline{x}) dw(x) = \underline{f} \cdot \underline{u}^{(j)}$ exist. This requires some spatial regularity in $\underline{f}(\underline{x},\, t)$. However, the time equations (22) and (29) also require some temporal regularity to permit solution. For example, $dc/dt = -\lambda c + f(t)$ is solvable for finite times if $f(t)$ is bounded and measurable, although slightly weaker conditions may suffice, such as finite or infinite spikes which are sufficiently sharp and infrequent. The fact that incompressibility has been assumed implies that shocks are efficiently propagated and reverberations from a rigid boundary can be expected. The mathematical treatment must cope with this disadvantage. Making the boundary flexible would have its own difficulties, of course.

We should point out that with density constant, the "force" contribution from the relative gradient of pressure $-\nabla p/\rho$ in the Navier-Stokes equation does not survive into the algebraisation (27) of the problem. In effect, the pressure gradient can then be calculated *after* the solution for the closed-domain velocity. Similarly, any forces which are of the form $f_1 = \nabla\phi$ within the fixed closed region Ω will also disappear from the treatment.

More generally, recall there is the Helmholtz decomposition of a vector $\underline{f}$ into $\underline{f}_1 + \underline{f}_2$, where $\nabla \times \underline{f}_1 = 0$ (irrotational) and $\nabla \cdot \underline{f}_2 = 0$ (solenoidal, like $\underline{v}$ itself). If the system is rotating with a fixed angular velocity $\underline{\omega}$ about an axis, then the "external force" in the fluid corresponding is $\underline{\omega} \times \underline{v}$. Now $\nabla \times (\underline{\omega} \times \underline{v}) = \underline{v}\,(\nabla \cdot \underline{\omega}) = 0$ since $\underline{\omega}$ is constant, so $\underline{f}_3 = \underline{\omega} \times \underline{v}$ is irrotational. Writing the N-S equation (9) as

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\,\underline{v} = -\frac{\nabla p}{\rho} + \eta \nabla^2 \underline{v} + \underline{f}_1 + \underline{f}_2 + \underline{f}_3 \tag{p1}$$

apply the operator $\nabla\cdot$ to obtain

$$\frac{\partial}{\partial t}(\nabla \cdot \underline{v}) = -\nabla\left[(\underline{v} \cdot \nabla)\,\underline{v}\right] - \frac{1}{\rho}\nabla^2 p + \eta \nabla \cdot \nabla^2 \underline{v} + \nabla \cdot \left(\underline{f}_1 + \underline{f}_3\right). \tag{p2}$$

Although logically $\nabla \cdot \nabla^2 \underline{v} = \nabla^2 (\nabla \cdot \underline{v}) = 0$, this third order derivative is difficult to calculate in an approximation scheme and a trace of it can be retained, to avoid unphysical oscillations. Thus

$$\nabla^2 p = +\rho\left[\eta \nabla \cdot \left(\nabla^2 \underline{v}\right) + \nabla \cdot \left(\underline{f}_1 + \underline{f}_3\right) - \nabla\,(\underline{v} \cdot \nabla)\,\underline{v}\right] \tag{p3}$$

is a Poisson equation for p when $\underline{v}$ is known.

Boundary conditions emerge from evaluation of the projection of the dynamics on a normal $\underline{n}$ at the surface $\partial\Omega$, to obtain

$$(\underline{n} \cdot \nabla)\,p|_{\partial\Omega} = \rho\underline{n} \cdot \left[-\frac{\partial \underline{v}}{\partial t} - (\underline{v} \cdot \nabla)\,\underline{v} + \eta \nabla^2 \underline{v} + \underline{f}_1 + \underline{f}_2 + \underline{f}_3\right]\Bigg|_{\partial\Omega}. \tag{p4}$$

From no-slip $\underline{n}\cdot(\underline{v}\cdot\nabla)\,\underline{v}|_{\partial\Omega}=0$ and from no penetration $\underline{n}\cdot\left.\frac{\partial \underline{v}}{\partial t}\right|_{\partial\Omega}=0$, so

$$\left(\underline{n}\cdot\nabla\right)p|_{\partial\Omega}=\rho\underline{n}\cdot\left[\eta\nabla^2\underline{v}+\underline{f}_1+\underline{f}_2+\underline{f}_3\right]\Big|_{\partial\Omega}. \tag{p5}$$

Compatibility entails

$$\int_{\partial\Omega}\left(\underline{n}\cdot\nabla\right)pd^2x=\int_{\partial\Omega}\nabla^2pd^3x \tag{p6}$$

when expressed in terms of $\underline{v}$. These 6 pressure equations are separately numbered, as they interrupt the general development.

The problem with flow in a region which has an internal cavity or one or more holes in the containing surface is that the pressure (or other conservative external force) will not then disappear from the projection onto the eigenfunctions of the closed domain Ω, unless $p=0$ is assumed on these regions, which is dubious, if there is an atmosphere. Also, the role of surface tension on a free surface, as in fluid transfer, must be admitted. Another possibility is to incorporate the pressure, etc. into a revised projection problem. This can easily be done in the context of the extended Euler version (61a) of the Navier-Stokes equation, in the special case when the pressure (or other external force) is both conservative and time-independent.

For a small region of fluid, gravitation in particular may be considered as time-independent. The modification of (61b) consists of setting $b_{k00}=f_k$ and so introducing $c_0^2f_k$ into the system. Since $c_0=1$, this is acceptable. If $f_k(t)$ is slowly varying, this causes problems in higher-order terms in the Euler-type calculations of the $b_{k\ell_1\ell_2\ldots q}$ in (48).

Since our system is quadratic and infinite-dimensional, we may expect that linear infinite-dimensional systems with time-dependent coefficients and forces will be relevant. This brings up the ingenious product-integral and product-derivative of Peano [35], used by Birkhoff [4], Volterra [44], and Lappo-Danilevsky, referenced for example in Gantmacher [20]. Since it is seldom mentioned in the literature of fluid mechanics, some explanations are desirable. First, although our equations have fixed coefficients, the Taylor series approach used above, when extended to the forced case, entails a type of successive linearisation about the force-free case where the coefficients depend on the time-dependent $c_k^*(t)$, and a new pseudo-time is used to parametrise the calculations for series-expansion purposes. This $c_k^*(t)$ is extended to $c_k(t,\,s)$ so that $c_k^*(t)=c_k(t,\,0)$ and the partial derivatives $\left(\frac{\partial}{\partial s}\right)^v c_k(t,\,s)$ and $c_{k0}^{(v)}(t)=\left(\frac{\partial}{\partial s}\right)^v c_k(t,\,s)\Big|_{s=0}$ and series expansion $c_k(t,\,s)=\sum_j s^jc_k^j(t)/j!$ are required. This would involve an infinite number of infinite-dimensional forced linear problems. Practically, the truncation used for calculation of the individual linear problems would be supplemented by a truncation of the order used in the assembly of these truncated problems. That understood, the details of the pseudo-time expansion can now be discussed. Specifically, the technique of equations of variation, which goes back to Lagrange, enables the forced non-linearities of systems to be reduced to forced linearities about a solved non-linearity. In recent years, a complete exposition of the method is found in the outstanding work of J. Kurzweil [25, Chapters 13, 14, Appendices]. The equations of interest:

$$\dot{c}_k(t) = -\eta\lambda_k c_k + \sum_{\ell,\, m} b_{k\ell m} c_\ell c_m + f_k(t) \tag{87}$$

are extended to the "embedded" form

$$\frac{\partial c_k}{\partial t}(t,\, s) = -\eta\lambda_k c_k(t,\, s) + \sum_{\ell,\, m} b_{k\ell m} c_\ell(t,\, s) c_m(t,\, s) + s f_k(t)\,, \tag{88}$$

which reduces to (87) for $s = 1$ and to (59) (solved previously) for $s = 0$. A Taylor series

$$c_k(t,\, s) = c_k(0,\, 0) + \sum_{n,\, m} \frac{t^n}{n!}\frac{s^m}{m!}\left.\frac{\partial^{n+m} c_k(t,\, s)}{\partial t^n \partial s^m}\right|_{s=t=0}\,, \tag{89}$$

which converges in a rectangle $0 < t < t_0$, $0 < s < 1 + \delta$, is desired. Thus s is not a conventional small parameter. On the other hand, the input "perturbation" $s f_k(t)$ is independent of the c_k. The variational equations referred to are obtained by differentiation of order v of (88) with respect to s, followed by evaluation at $s = 0$: notation,

$$c_{k0}^{(v)}(t) = \left(\frac{\partial}{\partial s}\right)^v c_k(t,\, s)\bigg|_{s=0}\,, \text{ for } v = 1,\, 2,\, 3, \ldots\,. \tag{90a}$$

Note that

$$\left(\frac{\partial}{\partial s}\right)^v [c_\ell(t,\, s) c_m(t,\, s)] \tag{90b}$$

$$= \sum_{j=0}^{v} \begin{pmatrix} v \\ j \end{pmatrix} \left[\left(\frac{\partial}{\partial s}\right)^j c_\ell(t,\, s)\right]\left[\left(\frac{\partial}{\partial s}\right)^{v-j} c_m(t,\, s)\right].$$

The cross-differentiability

$$\frac{\partial^i}{\partial t^i}\frac{\partial^\ell}{\partial s^\ell} c_k(t,\, s) = \frac{\partial^\ell}{\partial s^\ell}\frac{\partial^i}{\partial t^i} c_k(t,\, s)$$

and continuity at $s = 0$ are shown in Kurzweil [25] (because of symmetry, the number of distinct terms above is actually $\left[\frac{v+1}{2}\right]$ of which all but 1 are doubled, instead of v). Also

$$(c_{\ell 0}(t)c_{m0}(t))^{(v)} = \sum_{j=0}^{v} \binom{v}{j} c_{\ell 0}^{(j)}(t)c_{m0}^{(v-j)}(t) \tag{90c}$$

$$\text{and } c_{k0}(t) = c_k^*(t)$$

as studied previously.

Before we enter the details of the applications of product integrals, some background is needed. The set of differential equations

$$dx_{j\ell}/dt = \sum_{k=1}^{n} p_{jk}(t)x_{k\ell}(t)\,, \quad j = 1,\, 2, \dots n\,,\; \ell = 1,\, 2, \dots m\,, \tag{91a}$$

is conventionally written in matrix form

$$dX/dt = PX\,. \tag{91b}$$

If P is constant, $X(t) = \exp(Pt)X_0$. But if P is dependent on t, the solution for a finite square matrix P replaces $\exp(Pt)$ by the matricant series

$$M_{t_0}^{t}(P) = I + \sum_{m=1}^{\infty} J_m(P)\,, \tag{92a}$$

where

$$J_1(P) = \int_{t_0}^{t} P(\tau)d\tau\,, \tag{92b}$$

$$(J_{m+1}(P))\,(t) = \int_{t_0}^{t} P(\tau)J_m(P)(\tau)d\tau\,; \quad m = 1,\, 2, \dots .$$

If $M_{t_0}^{t}(P)$ is differentiable term-by-term, then

$$(d/dt)M_{t_0}^{t}(P) = P(t)M_{t_0}^{t}(P)\,. \tag{93}$$

Assume that $g(t) = \max\{|p_{jk}(t)|\,,\; j,\, k \le n\}$ and $h(t) = \int_{t_0}^{t} g(\tau)d\tau < \infty$. Then the partial sums of $M_{t_0}^{t}(P)$ have every element bounded by $\exp(nh(t))$. Thus the series (92a) is uniformly absolutely convergent, and differentiable if the functions $p_{jk}(t)$ are absolutely integrable on $[0,\, \infty]$ (Graves [21]).

Two properties of the matricant are relevant:

$$M_{t_0}^{t}(P) = M_{t_1}^{t}(P)M_{t_0}^{t_1}(P) \tag{94a}$$

and

$$M_{t_0}^t(P+Q) = M_{t_0}^t(P) \cdot M_{t_0}^t\left(M_{t_0}^t(P)^{-1} Q M_{t_0}^t(P)\right) . \tag{94b}$$

For approximation purposes, the fact that $|P(t)| \leq |Q(t)|$ implies

$$\left|M_{t_0}^t(P) - I - \sum_{n=1}^{s} J_n(P)\right| \leq \left|M_{t_0}^t(Q) - I - \sum_{n=1}^{s} J_n(Q)\right| , \tag{95a}$$

shows how to estimate matricants. Another approximate result is that if $\overline{I}$ is the $n \times n$ matrix of all $1'$s, and both $|Q| \leq q\overline{I}$ and $|P-Q| \leq d\overline{I}$, then

$$\left|M_{t_0}^t(P) - M_{t_0}^t(Q)\right| \tag{95b}$$

$$\leq \frac{1}{n} \exp\left(nq(t-t_0)\right) \cdot \exp\left(nd(t-t_0) - 1\right) \overline{I} .$$

When we pass from finite matrices to infinite-dimensional matrices, the concept of product integrals, symbolized by $\prod \int$, becomes appropriate. Consider an infinite sequence of disjoint adjacent open intervals Δt_k of length $t_{k+1} - t_k$, with internal variables $\tau_1$in Δt_1, $\tau_2$in $\Delta t_2, ...,$ where the union of the intervals $\bigcup_k (\Delta t_k)$ has $[t_0,\, t]$ as its closure and let

$$\left(\prod \int\right)_{t_0}^{t} P(\tau) d\tau = \lim_{|\Delta t| \to 0} \prod_{m=1}^{n} \left(I + P(\tau_m) \Delta t_m\right) , \tag{96a}$$

where

$$|\Delta t| = \max_{m \leq n} |\Delta t_m| \quad \text{and } n \to \infty . \tag{96b}$$

If P is $n \times n$, then $\left(\prod \int\right)_{t_0}^{t} P(\tau) d\tau = M_{t_0}^t(P)$, the ordinary matricant defined in (92).A suggestive alternate form, based on the idea that the higher order terms in the exponentials will disappear in the limit, is

$$\prod \int_{t_0}^{t} P(\tau) d\tau = \lim_{|\Delta t| \to 0} \left[\int_{\Delta t_1} \exp P(\tau_1) d\tau_1 \cdot \int_{\Delta t_2} \exp P(\tau_2) d\tau_2 \cdot \cdot\cdot\right] , \tag{97}$$

where $P(\tau_1)$ is an infinite matrix, such that $\exp P(\tau_k)$ is a convergent series for fixed τ_k, $\int_{\Delta t_k} \exp P(\tau_k) d\tau_k$ is a convergent integral, and the infinite product limit exists.

This bold idea of an infinite-dimensional matricant is implicit in the early work of Lappo-Danilevskii [26] (most of which was published posthumously) and has been used in Kolmogorov-Wiener-Doob theory of stochastic processes as well as theoretical work. It does not seem to have been used previously in deterministic fluid mechanics.

The remaining theoretical question is the solution of

$$\underline{\dot{X}} = P(t)\underline{X} + \underline{f}(t)\,. \tag{98a}$$

If $\dot{X}_1 = P(t)X_1$ is an equation whose solution is an invertible infinite matrix X_1, then the Lagrange form

$$X = X_1 W \tag{98b}$$

is a proposed solution of (98a). Now $\dot{X} = \dot{X}_1 W + X_1 \dot{W} = P(t)X_1 W + X_1 \dot{W} = P(t)X_1 W + \underline{f}(t)$, so $X_1 \dot{W} = \underline{f}(t)$, $\dot{W} = X_1^{-1}\underline{f}(t)$ and

$$X_1 W = X_1 \int_{t_0}^{t} X_1^{-1} \underline{f}(\tau)d\tau + X_1 W_0 \tag{98c}$$

is the general solution. Thus $X(t_0) = X_1(t_0)W_0$, which suggests the choice

$$W_0 = X_1^{-1}(t_0)X(t_0)\,. \tag{98d}$$

Since

$$X_1(t) = \prod \int_{t_0}^{t} (I + P(\tau)d\tau)\ , \tag{99a}$$

it follows that

$$X_1^{-1}(\tau) = \prod \int_{\tau}^{t_0} (I + P(\sigma)d\sigma) \tag{99b}$$

in product-integral notation. The above mixture of integral and product-integral notation can be rendered completely in product-integral notation.

Returning to the application (88), a single differentiation with respect to s yields

$$\frac{\partial}{\partial s}\frac{\partial c_k(t,\,s)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial c_k(t,\,s)}{\partial s} \tag{100a}$$

$$= -\eta\lambda_k \frac{\partial c_k(t,\,s)}{\partial s} + \sum_{\ell,\,m} b_{k\ell m}\left(c_\ell \frac{\partial c_m}{\partial s} + \frac{\partial c_\ell}{\partial s} c_m\right) + f_k(t)\,.$$

Evaluation of (100a) at $s = 0$ produces the first variation equation

$$\frac{d}{dt} c_{k0}^{(1)} = -\eta\lambda_k c_{k0}^{(1)} + \sum_{\ell,\,m} b_{k\ell m}\left(c_\ell^*(t)c_{m0}^{(1)}(t) + c_m^*(t)c_{\ell 0}^{(1)}(t)\right) + f_k(t)\,. \tag{100b}$$

This is clearly a driven infinite linear system for the $c_{k0}^{(1)}$, with time-dependent coefficients. Direct calculation using $b_{k\ell m} = b_{km\ell}$ shows that the rectangular double infinite sum in (100b) can be replaced by the triangular double infinite sum

$$2\sum_m \left(\sum_{\ell \le m} b_{k\ell m} c_\ell^*\right) c_{m0}^{(1)} . \tag{100c}$$

Thus the matrix $P^{(0)}(t)$ of interest satisfies

$$P_{km}^{(0)}(t) = -\eta\lambda_k\delta_{km} + 2\sum_{\ell\le m} b_{k\ell m} c_\ell^*(t) \tag{101a}$$

and $\underline{c}_0^{(1)} = \frac{\partial \underline{c}(t,s)}{\partial s}\Big|_{s=0}$ satisfies

$$\frac{d}{dt}\underline{c}_0^{(1)} = P^{(0)}(t)\underline{c}_0^{(1)} + \underline{f}^{(0)} , \tag{101b}$$

where $f_j^{(0)} = f_j$ and $P^{(0)}(t)$ is a bilinear expression in $\underline{c}^*$, b, and $\underline{\lambda}$.

Setting $\underline{f} = \underline{f}^{(0)}$ and $P = P^{(0)}$ in (98abcd) yields the desired solution $X = \underline{c}_0^{(1)}$ of (101b). With some complications of domain-generated "reverberant" forces $\underline{f}^{(v-1)}$, the above results hold for all $\underline{c}_0^{(v)}$, $v = 2, 3, \ldots$.

Before assembling the $\underline{c}_0^{(v)}$, some properties of the product integral should be mentioned.

$$\prod\int_{t_0}^{t} (I + Pd\tau) = \prod\int_{t_1}^{t} (I + Pd\tau) \cdot \prod\int_{t_0}^{t_1} (I + Pd\tau) , \tag{102a}$$

$$\left[\prod\int_{t_0}^{t} (I + Pd\tau)\right]^{-1} = \prod\int_{t}^{t_0} (I + Pd\tau) , \tag{102b}$$

$$\prod\int_{t_0}^{t} \left(I + CPC^{-1}d\tau\right) = C\prod\int_{t_0}^{t} (I + Pd\tau)\, C^{-1}, \tag{102c}$$

for constant C.

If $\left(\prod D\right)_t X = (D_t X)\, X^{-1} = \dot{X}X^{-1}$ is the product derivative, then

$$\prod\int_{t_0}^{t} \left[I + \left(R + \prod D_\tau X\right) d\tau\right] \tag{102d}$$

$$= X(t)\left[\prod\int_{t_0}^{t} \left(I + X^{-1}RX d\tau\right)\right] X^{-1}(t_0) .$$

As an application of (102d), let $\dot{X} = QX = XQ$, where Q is diagonal and constant, and $X(0) = I$. Then $(\prod D)_t X = Q$, $X(t) = \exp(tQ)$, and $X^{-1}(t) = \exp(-tQ)$ are diagonal and they commute with $P(t)$ for all t. Thus

$$\left(\prod\int\right)_{t_0}^{t} [I + (P+Q)\,d\tau] = e^{(t-t_0)Q}\left(\prod\int\right)_{t_0}^{t} (I + P d\tau)\,. \tag{102e}$$

Adding a constant diagonal Q to P has a simple effect on its product integral! From (102c), a linear transformation of coordinates on a matrix has a covariant effect on its product integral. While property (102a) is basic, we actually use (102bcd) in the present application.

The w-th derivative of (88) with respect to s is

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\right)^{w} c_k \;=\; \left(\frac{\partial}{\partial s}\right)^{w}\frac{\partial c_k}{\partial t} \tag{103a}$$

$$= -\eta\lambda_k\left(\frac{\partial}{\partial s}\right)^{w} c_k + 2\sum_m\sum_{\ell\le m} b_{k\ell m}c_\ell(t,\,s)\left(\frac{\partial}{\partial s}\right)^{w} c_m(t,\,s) + f_{\Omega k}^{(w-1)}(t,\,s)\,,$$

where

$$f_{\Omega k}^{(w-1)}(t,\,s) = 2\sum_m\sum_{\ell\le m}\sum_{v=1}^{w-1}\begin{pmatrix} w\\ v\end{pmatrix} b_{k\ell m}\left(\frac{\partial}{\partial s}\right)^{v} c_\ell(t,\,s)\left(\frac{\partial}{\partial s}\right)^{w-v} c_m(t,\,s) \tag{103b}$$

is the $(w-1)$th reverberation of the flow. Evaluation of (103ab) at $s = 0$ leads for $w \ge 2$ to the system equations

$$\frac{d}{dt}\underline{c}_0^{(w)} = \left[P^{(0)}(t)\right]\underline{c}_0^{(w)} + \underline{f}_\Omega^{(w-1)}(t)\,, \tag{104a}$$

where

$$f_{\Omega k}^{(w-1)}(t) = 2\sum_m\sum_{\ell\le m}\sum_{v=1}^{w-1}\begin{pmatrix} w\\ v\end{pmatrix} b_{k\ell m}c_{\ell 0}^{(v)}c_{m0}^{(w-v)}\,. \tag{104b}$$

We already have $\underline{c}^*$ (85) and an equation (101a) for $\underline{c}_0^{(1)}$ and for $P^{(0)}$. Clearly (104ab) is a recursion for all other $\underline{c}_0^{(w)}$. The solutions are obtained by inserting $P^{(0)}$ for P and successively $\underline{f}_\Omega^{(w-1)}$ for $\underline{f}$ in the solution (101b).

The connection between these quantities is in the definition of $b_{k\ell m}$ and λ_k in terms of the eigenfunctions $\underline{u}_k(\underline{x})$ which are solenoidal ($\nabla\cdot\underline{u}_k = 0$). These eigenfunctions are known completely for a few three-dimensional regions: boxes, ellipsoids, elliptic cylinders, toruses, elliptic cones. Suitably oriented difference sets, such as hollow boxes, are also viable. Uniform rotation, uniform gravity, and conforming eigenvector forces are preferred. For regions of unsymmetrical or complicated form, typically finite element, finite difference, or Galerkin bases (other than the above pure Helmholtzian) are used in a very extensive "CFD" literature. This is discused in Section 10, after a brief treatment of boxes and spheres in Section 9.

From the $\underline{c}_0^{(w)}$ of all orders, the $\underline{c}^*$and the initial values of either the velocities or their Fourier-Helmholtz initial coefficient values, the velocities can be constructed by the formulas (21) and (22).

9. Cubes and Spheres – Exact Solenoidal Bases

In the orthogonal system (q_1, q_2, q_3) whose line element is $ds^2 = \sum_{j=1}^3 h_j^2 (dq_j)^2$, the vector system $\nabla^2 \underline{u} = -\lambda \underline{u}$ has families $(\underline{u}, \lambda)$ of solution inside Ω_1, outside Ω_2, and in the shell $\Omega_1 - \Omega_2$ between two conforming domains $\Omega_1$and $\Omega_2 \subset \Omega_1$. In the discrete case, the set of eigenvalues is countable. Each family of solutions relates to A) the Dirichlet, Neumann, or Robin boundary behavior on the surfaces $\partial\Omega_1$, $\partial\Omega_2$, and $\partial\Omega_1 \cup \partial\Omega_2$, and to B) the preferred or polarization direction vector of the solution with respect to the surfaces $q_1 = \text{const}$, $q_2 = \text{const}$, $q_3 = \text{const}$. The B options can be automatically generated from the Debye potential representations below. This generates 27 possibilities, for location, behavior on the boundary, and polarization. Many of these possibilities have not been fully explored, even with rectangular boxes, in which the three polarizations are effectively the same. With a fixed domain, and a viscous fluid, the incompressible case yields zero velocities on the boundary and solenoidal behavior in the interior, which reduces the number of possibilities. Only two domain types are here explored, cubes and spheres, but the general solenoidal context will be reviewed, since the PDE literature is focused on scalar equations, and the Debye representations are usually presented in works on magnetic (TM) waves. Courant-Hilbert (volume 2) discusses the Helmholtz system aspect quite generally but avoids many details important for TM vibrations (their term-waves have second time derivatives and flows have first time derivatives).

If $\sigma = (j, k, \ell)$ is a cyclic permutation of (1, 2, 3), the Debye polarization w_σ associated with $\nabla^2 \underline{u} = -\lambda \underline{u}$ is the solution of the scalar equation

$$\frac{\partial}{\partial q_j}\left(\frac{h_k}{h_j}\frac{\partial w_\sigma}{\partial q_j}\right) + \frac{\partial}{\partial q_k}\left(\frac{h_j}{h_k}\frac{\partial w_\sigma}{\partial q_k}\right) + h_j h_k \left(\frac{\partial^2 w_\sigma}{\partial q_\ell^2} + \lambda_\sigma w_\sigma\right) = 0\,. \tag{106a}$$

The solutions $(w_\sigma, \lambda_\sigma)$ for $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (2, 3, 1)$, $\sigma_3 = (3, 1, 2)$ then generate the vectors $\underline{u}_\sigma = (u_{1\sigma},\, u_{2\sigma},\, u_{3\sigma})$ according to the rules

$$u_{1\sigma} = \frac{1}{h_j}\frac{\partial^2 w_\sigma}{\partial q_j \partial q_\ell}, \quad u_{2\sigma} = \frac{1}{h_k}\frac{\partial^2 w_\sigma}{\partial q_k \partial q_\ell}, \quad u_{3\sigma} = \frac{\partial^2 w_\sigma}{\partial q_\ell^2} + \lambda_\sigma w_\sigma\,. \tag{106b}$$

The $\underline{u}_\sigma$ are the solenoidal vectors, λ_σ the eigenvalues.

In the rectangular case, $h_j = h_k = h_\ell = 1$ so the three polarizations all yield the same scalar equations for all w_σ:

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \lambda\right) w_\sigma = 0 \tag{107a}$$

and the representations reduce to

$$u_1 = \frac{\partial^2 w}{\partial x_1 \partial x_3}, \quad u_2 = \frac{\partial^2 w}{\partial x_2 \partial x_3}, \quad u_3 = \frac{\partial^2 w}{\partial x_3^2} + \lambda w\,. \tag{107b}$$

The boundary conditions for Ω_1 will be zero on the six faces of a box Ω_1:

$$x_1 = \pm b_1\,,\ |x_2| \le b_2\,,\ |x_3| \le b_3\,, \tag{108}$$

$$x_2 = \pm b_2\,,\ |x_1| \le b_1\,,\ |x_3| \le b_3\,, \quad \text{and } x_3 = \pm b_3\,,\ |x_1| \le b_1\,,\ |x_2| \le b_2\,.$$

Similarly, for a box Ω_2 with $(b_1,\, b_2,\, b_3)$ replaced by $(c_1,\, c_2,\, c_3)$ and a "shell" $\Omega_1 - \Omega_2$ with (outside) faces of Ω_2 and (inside) faces of Ω_1 the expression

$$w = \sum_{jk\ell} A_{jk\ell} \exp\left(i \sum_{m=1}^{3} \alpha_{jk\ell m} x_m\right) = \sum_{jk\ell} A_{jk\ell}\psi_{jk\ell}(\underline{x}) \tag{109}$$

is sufficient, where $\sum_{m=1}^{3} \alpha_{jk\ell m} = \lambda$ for each $j,\, k,\, \ell$. Now

$$u_1 = -\sum_{jk\ell} A_{jk\ell}\psi_{jk\ell} \cdot \alpha_{jk\ell 1}\alpha_{jk\ell 3},$$

$$u_2 = -\sum_{jk\ell} A_{jk\ell}\psi_{jk\ell} \cdot \alpha_{jk\ell 2}\alpha_{jk\ell 3}, \quad u_3 = -\sum_{jk\ell} A_{jk\ell}\psi_{jk\ell} \cdot \left(\alpha_{jk\ell 3}^2 - \lambda\right)$$

must all vanish on the six planes of Ω_1, Ω_2 or twelve planes of $\Omega_1 - \Omega_2$. These triple Fourier series then enter the formulas (47) for the c_k^*, etc. as the Helmholtzian basis in rectangulars, or, as we shall see, in sphericals. To realize these constraints on one of these faces, multiply u_j by $\exp(ig_2x_2 + ig_3x_3)$, set $x_1 = b_1$, and integrate with respect to x_2 over $(-b_2,\, b_2)$ and with respect to x_3 over $(-b_3,\, b_3)$. Repeating over the other 5 (or 11) faces results in a $6 \times 3 =$

18-fold (or 36-fold) set of equations. But between *two* independent sets of frequencies and the set of amplitudes, there is a sufficient multiplicity, to satisfy the equations.

According to Fredholm's theory of integral equations, into which the Helmholtz equation can be converted, a set Ω which, at each boundary point, can have a cone attached locally which contains none of the complement of Ω, is smooth enough to have a discrete set of eigenvalues (this permits corners). These can be approached by minimizing a quadratic integral expression, in principle. By minimizing it computationally (as in the Bellman method of invariant embedding sequences) a numerical realization of the first few eigenvalues and eigenvectors can be reached. Since our method rests on a Galerkin expansion in Helmholtz eigenfunctions, its practicality depends for non-traditional Ω on approximation at the basis stage as well as other stages. For cubes and spheres, however, the solenoidal eigenvalues and eigenvectors are known exactly, and such approximations are unnecessary.

In spherical coordinates, appropriate to Ω_2 (exterior of a sphere), Ω_1 (interior of a sphere), $\Omega_1 - \Omega_2$ (the shell between two concentric spheres), the arc weights are

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta \,. \tag{110}$$

The three Debye potentials for the various polarizations are:

radial:

$$\left[r^2 \sin\theta \frac{\partial^2}{\partial r^2} + \sin\theta \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} + \cos\theta\frac{\partial}{\partial\theta} + \lambda r^2 \sin\theta\right] w_{\mathrm{rad}} = 0\,, \tag{111a}$$

latitude:

$$\left[r \sin\theta \frac{\partial^2}{\partial\theta^2} + r\sin\theta \frac{\partial^2}{\partial r^2} + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi^2} + \sin\theta\frac{\partial}{\partial r} + \lambda r \sin\theta\right] w_{\mathrm{lat}} = 0\,, \tag{111b}$$

longitude:

$$\left[r \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial^2}{\partial\theta^2} + r\frac{\partial^2}{\partial\phi^2} + \frac{\partial}{\partial r} + \lambda r\right] w_{\mathrm{lon}} = 0\,. \tag{111c}$$

They are all separable, and in each case, the longitudinal factors have simple sinusoidal modes: $\Phi_n''(\phi) = -n^2\Phi_n(\phi)$, and so describable without further comment.

Thus

$$w_{\mathrm{rad}} = R_{\mathrm{rad}}\Theta_{\mathrm{rad}}\Phi_n\,, \quad w_{\mathrm{lat}} = R_{\mathrm{lat}}\Theta_{\mathrm{lat}}\Phi_n\,, \tag{112}$$
$$w_{\mathrm{lon}} = R_{\mathrm{lon}}\Theta_{\mathrm{lon}}\Phi_n\,.$$

Then

$$\left[\frac{d^2}{dr^2} + \left(\lambda_{\text{rad}} - \frac{c}{r^2}\right)\right] R_{\text{rad}} = 0\,, \tag{113a}$$

$$\left[\frac{d^2}{d\theta^2} + \cot\theta \frac{d}{d\theta} + c - \frac{n^2}{\sin^2\theta}\right] \Theta_{\text{rad}} = 0\,, \tag{113b}$$

where c is a latitude eigenvalue in the radial equation.

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k}{r^2}\right] R_{\text{lat}} = 0\,, \tag{114a}$$

$$R_{\text{lat}} = r^{\pm\alpha},\ \alpha^2 = k \neq 0\,; \quad R_{\text{lat}} = \ln(r/r_0),\ k = 0\,,$$

$$\left[\frac{d^2}{d\theta^2} + \left(k + \lambda_{\text{lat}} - \frac{n^2}{\sin^2\theta}\right)\right] \Theta_{\text{lat}} = 0\,, \tag{114b}$$

where k is a radial eigenvalue in the latitude equation.

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \lambda_{\text{lon}} - n^2 - \frac{m^2}{r^2}\right] R_{\text{lon}} = 0\,, \qquad 115a$$

$$\Theta''_{\text{lon}} = -m^2 \Theta_{\text{lon}}\,. \qquad 115b$$

The radial factor in the latitude equation and the latitude factor in the longitude equation are quite simple. The other factors are not. The radial factor in the radial polarization is confluent hypergeometric (Polyanin and Zaitsev [37] - 2.1.2, no. 110), while in the latitude and longitude polarization, is a Bessel function. The latitude factor in the radial polarization is hypergeometric if $\Theta_{\text{rad}} \to \widetilde{\Theta} = \Theta\,|\sin\theta|^{-n/4}$, and $\theta \to \widetilde{\theta} = \frac{1-\cos\theta}{2}$ (Polyanin and Zaitsev [37] - 2.1.2, no. 212), and in the latitude polarization is a Legendre polynomial. All these details are necessary for obtaining a full set of solenoidal eigenfunctions and eigenvalues, which does not seem to have been noticed by spherical fluid numerical analysts, who have concentrated on Bessel and Legendre solutions for spherical spectral information. It is a natural assumption, because the scalar spherical operator

$$\nabla^2 + \lambda = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \lambda \tag{116}$$

(except for $\left(\frac{1}{r} \to \frac{2}{r}\right)$) combines the Θ_{rad} and the R_{lon} factors, when separated.

10. Numerical and Comparative

Temam [43] in particular has attempted to combine sophisticated numerical methods with advanced mathematical technique. This becomes more relevant in our explicit treatment of the Navier-Stokes equation, so improved numerical integration should be balanced with the improved numerical differentiation and polytope Morse theory now available. Clearly, improvement overall requires that the three legs of numerical geometry, differentiation, and integration be carried to comparable levels of accuracy. Numerical integration goes back to Simpson's rule. A modern version, due to Alpert (1999) [1], uses three pairs of end weights to correct a simple middle. Thus

$$\int_a^b f(x)dx \sim \mathrm{Alp}_3(f)$$

$$= \frac{b-a}{n}\left[\frac{6075}{11431}\left(f(x_0)+f(x_n)\right)+\frac{823}{852}\left(f(x_1)+f(x_{n-1})\right)\right.$$

$$\left.+\frac{1937}{1932}\left(f(x_2)+f(x_{n-2})\right)+\sum_{j=3}^{n-3} f(x_n)\right]$$

for mostly equidistant $(x_0, \ldots x_n)$, with the end-setting $x_0 = a + \frac{19}{90}\Delta x$, $x_j = a + j\Delta x$, $x_n = b - \frac{19}{90}\Delta x$, which is accurate to

$$\mathrm{O}\left[-\left(\frac{b-a}{n}\right)^5 \frac{91}{259200}\left(f^{(4)}(a)+f^{(4)}(b)\right)\right].$$

Alpert's original $\mathrm{Alp}_2(f)$ used $\frac{1}{5}$ instead of $\frac{19}{90}$ and 2 pairs with weights $\left(\frac{25}{48}, \frac{47}{48}\right)$, but is much less accurate.

Numerical differentiation goes back to Euler. A formula of comparable accuracy to Alpert's is due to Strand and his Swedish associates, and uses a middle stencil of

$$u' = \frac{u_{j+3} - u_{j-3} - 9\left(u_{j+2} - u_{j-2}\right) + 45\left(u_{j+1} - u_{j-1}\right)}{60h}$$

for $j = 4, \ldots n-3$, with end stencils ($j = (1, n)$, $j = (2, n-1)$, $j = (3, n-2)$) designed (following Kreiss) to conserve "summation by parts" formulas of the type $(u'v + uv')]_0^n = u_n v_n - u_0 v_0$. The details are quite complicated for the end stencils (Strand, 1994, [42]).

Some polyhedral details have been considered by S. Wille [45] in Norway and by Sherwin and Karniadakis in the U.S. Spatial elements arise in two ways; an original subdivision, and further binary or ternary "hereditary" refinements, which hopefully produce monotone approximation sequences. The Voronoi "balloon" approach to the original subdivision of Ω is to choose a number of points in each of the connected components of Ω. These are distributed more densely toward the boundaries and especially near its edges and corners. Then consider these points as centers of small balloons which are to be uniformly inflated until Ω is filled completely. Since the pressure is balanced, the balloon-balloon boundaries will be portions of planes, and the resulting interior surfaces will define the Voronoi "polyhedra" associated with Ω and the selection of centers. The exterior surfaces of these "polyhedra" will conform to $\partial\Omega$, hence will not usually be polyhedral. Those internal faces which touch $\partial\Omega$ do so at a finite number of points $V_1, V_2, \ldots, V_n$. By considering the $\binom{n}{3}$ triangles determined by these points, a preliminary polyhedron can be formed from the interior planes, capped by these triangles. If the corresponding regions of Ω are convex, this produces the desired polyhedron. If not, and any face protrudes out of Ω, the preliminary polyhedron can be retracted along each line between the point V_j and the interior polygon generated by the centers of the Voronoi regions which contributed to V_j, until it does not pass through any part of the complement of Ω. If W_j is the point where the contraction can be stopped, then the lines joining the W_j's to each other will be some of the edges of the final polyhedron. That can then be dissected into tetrahedra, while avoiding "slivers" by minimizing the maximum ratio of surface area to volume (which may result in many small, but well-shaped, tetrahedra).

To notate binary hereditary refinements of any of the selected tetrahedra, consider the 14 Wille parameters:

a) - 1 for the generation;

b) - 4 for the corner locations (which will be 12 numbers in Cartesians);

c) - 8 for the first generation of binary offspring generated by; connecting bisectors of the parent edges, and

d) - 1 for attaching a congruent neighbor tetrahedron, if subdivision seems excessive.

To locate a point, a 2-stage search is used, the first stage compares $P = (x, y, z)$ with the 4 vertices (x_k, y_k, z_k) of the tetrahedra by the simple "box tests": $\min_k x_k \leq x \leq \max_k x_k$ (8 comparisons), and 16 more for y and z. If P passes the box tests, then there will be line elements $\underline{L} = (L_1, L_2, L_3, L_4)$ such that the matrix

$$\begin{bmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \\ \underline{1} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix},$$

where $\underline{x} = (x_1, x_2, x_3)$, etc. If $0 \leq L_k \leq 1$ for $k = 1, 2, 3, 4$ then $\underline{x}$ is in the $(\underline{x}_k, \underline{y}_k, \underline{z}_k)$ tetrahedron. If not, then $\underline{x}$ will be in another tetrahedron, but still in the box. The box steps are very rapid, and save time overall, according to Wille.

There are 2 modes of failure: if the line is "parallel" to a boundary element, the corresponding diagonal is (within machine ϵ) equal to zero, and a random direction is assigned. If a line hits an edge or a vertex, one or more of the L_k is "equal to 0", and a random direction is chosen. If the subdivision is too much refined in one direction, rebalancing by division into 2 or 4 tetrahedra in the deficient directions can be employed. An algorithm for this is in Wille [45], pp. 235-236. Division into cubes is simpler, but the number of faces involved is much greater. Wille applies this directly to the Navier-Stokes equation.

To dissect a box into tetrahedra, start with a $s \times s \times \ell$ box which has two square faces of area s^2 and four rectangular faces of area $s\ell$. Then number the vertices on one square face as 1 2 3 4 and the remaining box vertices (on the same sides of the rectangles) as 5 6 7 8. Then cut off four tetrahedra by leaving intact vertex 1 but cutting at vertices 2, 4, 5; leaving 3 but cutting at vertices 2, 4, 7; leaving 6 but cutting at vertices 5, 7, 2; leaving 8 but cutting at vertices 5, 7, 4. Each "outside" tetrahedron has one triple right angle and volume $\frac{1}{6}s^2\ell$. Thus the remaining inside tetrahedron has corners at 2, 4, 5, 7 and volume $s^2\ell - \frac{2}{3}s^2\ell = \frac{1}{3}s^2\ell$. Cut in half, it produces 2 tetrahedra of volume $\frac{1}{6}s^2\ell$. Thus 4 singles and 1 double, or 6 singles of equal volume are produced. Of course, a cubical box or a box of arbitrary sides will allow similar dissections.

The spatial approach is through the Helmholtz equation $\nabla^2 \underline{f} + \lambda \underline{f} = 0$ for $\underline{f} = \underline{f}(\underline{x})$ over a region Ω. Two types of region are considered here: connected discrete regions in rectangular coordinates, and more mathematical, connected $\overline{C}_\infty$ regions in Riemannain C_∞ coordinates. For connected discrete, there are three isomorphic types. The prototype is x_1-oriented: a square grid of elementary area a^2 is placed in the x_2x_3 plane. Vertical lines (spikes) are placed passsing through the centers of a finite number of adjacent squares of the grid. On each spike is placed a finite number of contiguous parallel cubes (tower), which may extend below the x_2x_3 plane. Any two towers on adjacent spikes will be assumed to have at least one cube in each tower in full contact with a cube on the adjacent tower, so the towers are connected, forming together an

x_1-connected discrete region. Similarly, define x_2-connected and x_3-connected discrete regions. Two such regions are called connected non-overlapping if their interiors are disjoint, even though they must have some cube faces in common. Luckily, the intersection of two x_j-oriented regions is an x_j-oriented region. Also, the intersection of an x_j-oriented region with an x_k-oriented region is a non-overlapping union of doubly-oriented regions. A finite union of non-overlapping x_1-, x_2-, and x_3- discrete connected regions is called a discrete connected region. A union of such regions is then considered. Notations for this are not difficult to arrange. Given two such unions U_1, U_2 and a linear functional F defined on them, the set difference $U_1 - (U_1 \cap U_2)$ extends F via the agreement $F(U_1 - (U_1 \cap U_2)) = F(U_1) - F(U_1 \cap U_2)$. This allows for dealing with multiply connected discrete sets, like discrete toroids.

In dealing with convergence proofs, norms for error are essential. Such norms were originally devised for continuous problems with integral equations, and stated in integral form by Hilbert. The so-called Hilbert norm of a square-integrable function is $\|f\|_{2,1} = \left(\int |f|^2\, dx\right)^{1/2}$. For a function of 3 variables $(x_1, x_2, x_3) = \underline{x}$ this becomes $\|f\|_{2,3} = \left(\int\int\int |f|^2\, dx_1 dx_2 dx_3\right)^{1/2} = \left(\int_{R_3} |f|^2\, dw\right)^{1/2}$, where w is the 3-dimensional Lebesgue measure. This can be restricted to any underlying set S in R_3 over which f is defined.

When differential equations of the kth order are considered, the related Sobolev norms (H_k-norms) are defined for a single variable by $\|f\|_{H_k,1} = \|f\|_{2,1} + \|Df\|_{2,1} + \cdots + \|D^k f\|_{2,1}$. For functions of more variables, the appropriate norms are

$$\|f\|_{H_1,2} = \|f\|_{2,2} + \left\|\frac{\partial f}{\partial x_1}\right\|_{2,2} + \left\|\frac{\partial f}{\partial x_2}\right\|_{2,2},$$

$$\|f\|_{H_2,2} = \|f\|_{H_1,2} + \left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_{2,2} + \left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|_{2,2} + \left\|\frac{\partial^2 f}{\partial x_2^2}\right\|_{2,2},$$

$$\|f\|_{H_1,3} = \|f\|_{2,3} + \left\|\frac{\partial f}{\partial x_1}\right\|_{2,3} + \left\|\frac{\partial f}{\partial x_2}\right\|_{2,3} + \left\|\frac{\partial f}{\partial x_3}\right\|_{2,3},$$

$$\|f\|_{H_2,3} = \|f\|_{H_1,3} + \sum_{j \le k} \left\|\frac{\partial^2 f}{\partial x_j \partial x_k}\right\|_{2,3}.$$

Generally the second subscript (referring to the underlying dimension) is omitted from the norm notation, since it is the same throughout a particular discussion. Here it will be 3 dimensions and so the notation will read $\|\quad\|_2$ for the Hilbert norm and $\| \quad \|_{H_1}$, $\| \quad \|_{H_2}$ for the two Sobolev norms needed, the three dimensions being implied. Thus $\|f\|_{H_2}$ has 10 components.

In the difference equations and sum equations obtained by discretizing the domain and differential equations, the analogous discretized norms $\|s\|_{2,1} = \left(\sum_n |s_n|^2\right)^{1/2}$, $\|s\|_{h_1,1} = \|s\|_{2,1} + \|\Delta s\|_{2,1}$, $\|s\|_{h_2,1} = \|s\|_{h_1,1} + \left\|\Delta^2 s\right\|_{2,1}$, where Δs is whichever of the versions of the "first difference" is being used (we refer to the discussion earlier of difference formulas). Similarly in three-dimensional difference equations obtained from differential equations, the notations are $s_{\underline{n}} = s_{n_1,n_2,n_3}$,

$$\|s\|_{2,3} = \left(\sum_{n_1,n_2,n_3} |s_{n_1,n_2,n_3}|^2\right)^{1/2} = \left(\sum_{\underline{n}} |s_{\underline{n}}|^2\right)^{1/2},$$

$$\|s\|_{h_1,3} = \|s\|_{2,3} + \|\Delta_1 s\|_{2,3} + \|\Delta_2 s\|_{2,3} + \|\Delta_3 s\|_{2,3}$$

and

$$\|s\|_{h_2,3} = \|s\|_{h_1,3} + \sum_{j\le k} \|\Delta_{jk} s\|_{2,3} ,$$

where $\Delta_j s$ is whichever version of the first difference in the j-th coordinate is used, and similarly for $\Delta_{jk} s$, the second difference for $j = k$ and the double first difference for $j \neq k$.

For technical reasons, although the N-S equations only involve second-order partial derivatives, elimination of pressure seems to require the existence of third-order derivatives, so that the third-order Sobolev norms, $\|f\|_{H_3}$, defined in the obvious way, could be required also. Note further that any particular scalar norm $\|f\|_s$ can be extended to a vector norm $\left\|\underline{f}\right\|_v$ by $\left\|\underline{f}\right\|_v^2 = \sum_{j=1}^m \|f_j\|_s^2$ when $\underline{f} = (f_1, f_2, \ldots f_m)$. Similarly, the discrete scalar norms are extendable to discrete vector norms.

In the Fourier-Helmholtz-Galerkin approach followed here, the space and time aspects are, in theory, separated. to be united after the temporal equations for the components are solved. At the level of computing, however, the temporal equations are solved only to a particular order of n^{-1} when the nth approximations to the Helmholtz solutions are used. Thus the discrete temporal solutions must also approximate the exact temporal solutions for the exact spatial solutions. More precisely, a solution scheme for the temporal equations must maintain a type of stability with respect to: (a) the coefficients; (b) the discretization of the operators; and (c) to the spatially discretized driving functions.

Returning now to the Helmholtz problem, in a continuum version, natural because fluid flows are basically continuous when molecular aspects (such as Brownian motion due to temperature effects) and surface-tensioned droplets are ignored. The pre-modern definition of a Riemannian manifold M is a set of points $\underline{x} \in R_3$ and metric tensor $G = [g_{ij}(\underline{x})]$ which is once-continuously differentiable in the interior of M, equipped with a metric σ derived from G. More precisely, if the origin $\underline{0} \in M$, and $\underline{x} = \underline{x}(t)$ is any differentiable curve C in M joining points $\underline{p}, \underline{q} \in M$, and $ds^2 = \sum_{i,j=1}^{3} g_{ij}(\underline{v}) dx^i dx^j$ defines the differential arc-length, then $L(C) = \int_{\underline{p}}^{q} ds(\underline{x})$ is the length of C between $\underline{p}$ and $\underline{q}$, finite if C is of bounded variation. The metric $\sigma(\underline{p}, \underline{q}) = \inf_C L(C)$, and $\sigma(\underline{0}, \underline{p}) = \|\underline{p}\|_\sigma$ is a norm on M.

The topology on M induced by the above metric makes (M, G) into a Riemannian manifold. Differential geometers usually restrict themselves to this type of domain. However, limits of sequences also are permitted. For example, a single cube can be reached by rounding off the edges and corners by three-dimensional "partitions of unity" whose rounding radii tend to zero and whose coordinate functions are C^∞, that is, infinitely differentiable with all derivatives equal to zero along the lines where the rounding begins (a standard device in the Schwarz-Sobolev distribution theory). The classic objects in differential geometry are three tensors; torsion, curvature, and Ricci-Einstein. All are defined in terms of the Christoffel connections:

$$\Gamma^i_{jk} = \frac{1}{2} \sum_i g^{i\ell} \left[\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right] ,$$

where $\sum_\ell g^{i\ell} g_{\ell j} = \delta^i_j$, which are not quite tensors. The torsions are differences of connections. The curvature tensor is

$$R^\ell_{kij} = \frac{\partial \Gamma^\ell_{jk}}{\partial x^i} - \frac{\partial \Gamma^\ell_{ik}}{\partial x^j} + \sum_m \left(\Gamma^\ell_{im} \Gamma^m_{jk} - \Gamma^\ell_{jm} \Gamma^m_{ik} \right) .$$

The Ricci-Einstein reduced curvature tensor is $R_{ij} = \sum_k R^k_{kij}$. Its "contraction" $R = \sum_{ij} R_{ij} g^{ij}$ is called the scalar curvature. Facility with these non-linear constructs is helpful in theoretical fluid mechanics. The element of volume in Riemannian geometry is $\sqrt{g}\, dx^1 dx^2 dx^3 = dw$, where $g = \det[g_{ij}]$. The Laplacian is

$$\nabla^2_g f = \sum_{ij} \frac{\partial}{\partial x^i} \left(g_{ij} \frac{\partial f}{\partial x^j} \right) ,$$

a self-adjoint operator.

A result of Aubin is that if M is compact in the g-based topology, then there are an infinity of "eigenvectors" f_n which are C^2 on M, such that $\nabla_g^2 f_n + \lambda f_n = 0$, where $\lambda_0 = 0$ and $\{\lambda_n\}$ is an increasing sequence tending to ∞, and so the Helmholtz scalar problem is solved.

The differential equations of geodesic (minimal length) curves on a Riemannian manifold M are quadratic second-order equations

$$\frac{d^2 u_i}{ds^2} + \sum_{jk} \Gamma^i_{jk} \frac{du^j}{ds} \frac{du^k}{ds} = 0 \,,$$

where s is a parameter in $u^i = u^i(s)$. If s is considered as time (or an increasing function of time), there is a conjecture that the streamlines in a fluid flow will be related to geodesics associated with the connection Γ^i_{jk} on the boundary $\partial\Omega$, when it is a Riemannian manifold.

A little geometric measure theory, needed in combining topology and differential equations, is due to H.M. Morse, A.P. Morse, and A. Sard.

The first problem is singularities ("critical points"), in functions $\phi : M \to M'$ between two C^∞ manifolds. If ϕ is a C^1 function, and $\underline{p}$ is a point in M, and $T_{\underline{p}}$, $T'_{\phi(\underline{p})}$ are the tangent spaces of M, M' at $\underline{p}$ and $\phi(\underline{p})$ respectively, then $\psi_{\underline{p}} : T_{\underline{p}} \to T'_{\phi(\underline{p})}$ defines a linear mapping conventionally written as $(d\phi)_{\underline{p}}$. If $(d\phi)_{\underline{p}}$ is surjective (range is all of $T'_{\phi(\underline{p})}$) then $\underline{p}$ is called a ϕ-regular point of M. If not, then $\underline{p}$ is called a ϕ-critical point.

One hopes that critical points are rare, for example, when ϕ is a C^2 function. However, all that is certain now is that if ϕ itself is C^∞, then (A.P. Morse-Sard), the set of ϕ-critical points has Lebesgue measure zero, and thus can be disregarded in continuum models.

The (H.M. Morse) theory (of interest here) essentially deals with approximations of manifolds by polytopes (polyhedral complexes that are not necessarily closed). N-S fluid mechanics involves Newtonian fluids and physical space and thus has a substrate of differential geometry of curves, surfaces, and the 3-dimensional continua known as bodies. The discipline of differential geometry includes these objects and their set-theoretic generalizations such as Hausdorff bodies, whose dimensions need not be integers. It is no accident that the Gauss-Green-Stokes geometric integral machinery is a key element in fluid mechanics calculations. The calculations in Sections 6, 7, and 8 are quite similar to basic identities in modern differential geometry. Other non-linear differential equations, such as the second order static Monge-Ampere equations, which ultimately involve exponentials, also rely heavily on differential-geometric thinking and results. This usually proceeds through approximation of general continua by a class of simpler continua, for which the desired results can be exactly obtained, together with inequalities which allow limits to be inferred. There are five stages involved. First, identification of a class of simple continua. Second, relevant calculations based on this class. Third, showing that these calculations are suited to the target class of continua. Fourth, noticing or deriving the inequalities which bridge the gap between the simple continua and the target continua, and lastly, applying these inequalities in a total context. In many cases, a monotone sequence can be arranged, so that boundedness guarantees that a limit exists, and it is only the boundedness which requires the inequalities.

Obviously, the field of differential geometry is extremely wide and deep, and the theorems are numerous. Since fluid mechanics ultimately involves differential and integral geometry, sometimes called "geometric measure theory" (Federer [18]), they are a major component. The phrase "analysis on manifolds" is also used, to indicate both differential and integral geometry.

Some of the basic concepts, inequalities, and theorems of this subject are listed below. The books Aubin [2], Bishop and Crittenden [5], de Rham [13], Federer [18], Krasnoselskii [24], Lions and Magenes [29], Milnor [30], M. Morse [32], Palais [34] are helpful.

Twelve inequalities are of some use in manifold analysis, called Aubin, Bliss, Calderon-Zygmund, Cauchy (Schwartz), Clarkson, Hölder, index, isodiamatic, isoperimetric, Gagliardo (Nirenberg), Poincaré, Sobolev. Some of these generalize to bodies of variable density.

Because it is elegant, implies many other inequalities, and has a nice proof, we mention the inequality due to Bliss, an American mathematician of the 1920's.

Theorem. *If $g(r)$ is a decreasing function tending to zero at infinity, and absolutely continuous, then*

$$I(g)=\int_0^\infty |g(r)|^p\, r^{n-1}dr \le \left\{K(n,\,q)\left[\int_0^\infty \left|g'(r)\right|^q r^{n-1}dr\right]^{\frac{1}{q}}\right\}^p .$$

Proof. Maximize $I(g)$ given

$$J(g)=\int_0^\infty \left|g'(r)\right|^q r^{n-1}dr=\text{const}\,.$$

The Euler equation is

$$\left(\left|g'(r)\right|^q r^{n-1}\right)'=Kg^{p-1}r^{n-1}\,.$$

Let

$$g(r) = f\left(r^{\frac{q}{q-1}}\right) .$$

Then

$$f(x) = (\lambda + x)^{1-\frac{n}{q}} \quad \text{for } \lambda > 0$$

is the solution family and provides the absolute minimum. □

While improving the accuracy of the work, algebraic manifold theory somewhat obscures the meaning of the processes. The theory has two related aspects, unfortunately with similar names: homology and homotopy. Homotopy is more related to topological and group properties (including the properties of homology groups), of geometric objects. Homology is more quantitative. A basic element in homology is polyhedral complexes and how they combine and possibly converge. A knowledge of tensor algebra is essential. An excellent introduction is found in Article 417 of EDM, Vol. II, 2nd ed [17], pp. 1571-1577. Likewise, a cogent summary of homology theory is available in Article 201 of EDM, Vol. I, 2nd ed [17], pp. 761-774. A more expansive account of homology theory with an emphasis on the measure-theoretic aspects is contained in Federer [18], especially Section 3.4 (pp. 310-340); Chapter 4 (pp. 341-512) - differentials; and Chapter 5 (pp. 513-604) - integrals. A thorough account of the Gauss-Green theorem is available there on pp. 479-505. Morse theory, which focuses on singularities in homotopy theory, is well summarized in Article 279 of EDM, Vol. I, 2nd ed [17], pp. 1049-1059. The major positive results of Morse theory are: (1) that every compact manifold arises topologically by smoothly attaching toroidal handles to a sphere; (2) inequalities for alternating combinations of critical point indices; (3) if a critical point $\underline{x}_0$ of $f(\underline{x}) \in C^2$ has a positive Hessian: $\det\left[\partial^2 f / (\partial x_i \partial x_j)\right]_0 \geq K > 0$, then there is an approximation ϕ_0 and a projection P_0 such that

$$f(\underline{y}) = f(\underline{x}) + \left\|P_0\phi_0(\underline{y})\right\|^2 - \left\|(I - P_0)\phi_0(\underline{y})\right\|^2 ,$$

for $|\underline{x} - \underline{x}_0|$ and $\left|\underline{y} - \underline{y}_0\right|$ sufficiently small.

This is a major improvement on Taylor's "Rectangular" Theorem – of second order – with remainder. The approximation ϕ_0 may be a polytope made of sufficiently small triangles (in 2-D) or tetrahedra (in 3-D). Its applications to N-S approximations are obvious.

A recent system of N-S computing on difficult 3-D domains is found in Evangelinos, Sherwin, and Karniadakis [16]. This emphasizes direct computation, as do we, but by somewhat less analytical methods, and was supported by grants from the US Dept. of Energy. Their target simulations were for

10^9 grid points on 10^{14}-flop machines, anticipating 10^{15}-flop machines. Their work is partly quasi-2-dimensional, in that one of the dimensions is exactly homogeneous, and also 3-dimensional. Either a pure strategy of all triangles (tetrahedra, or a mixed strategy of rectangles-triangles) a panoply of cubes, triangular prisms, pyramids, and tetrahedra was employed (p. 404). The associated solution basis functions $\phi_{pqr}(\underline{\xi})$ are: separable (cubes); 1 degree non-separable $\phi_{pqr} = \phi_p^a(\xi_1)\phi_q^b(\xi_2)\phi_{qr}^c(\xi_3)$ for prisms; transformed 1 degree non-separable $\phi_{pqr} = \phi_p^a(\overline{\eta_1})\phi_q^a(\eta_2)\phi_{pqr}^c(\xi_3)$ for pyramids; and 2 degree non-separable for tetrahedra, with $\phi_{pqr} = \phi_p^a(\eta_1)\phi_{pq}^b(\eta_2)\phi_{pqr}^c(\xi_3)$; where

$$\eta_1 = \frac{2\,(1+\xi_1)}{-\xi_2-\xi_3} - 1\,, \quad \overline{\eta_1} = \frac{2\,(1+\xi_1)}{1-\xi_3}\,, \quad \eta_2 = \frac{2\,(1+\xi_2)}{1-\xi_3} - 1\,,$$

$\underline{\xi}$ are Cartesian. The paired and tripled ϕ were taken as weighted Jacobi polynomials of reduced type:

$$\left(\frac{1-z}{2}\right)^i P_j^{2i+1,\,0}(z) \quad \text{and} \quad \left(\frac{1-z}{2}\right)^{i+j} P_j^{2i+2j+2,\,0}(z)\,.$$

A fuller exposition of their approach is in the fine book by Karniadakis and Sherwin (1999) [23] on spectral approximation in PDEs. These are slightly more general than the conventional spectral Legendre polynomials. Recall that for spheres, the solenoidal (Debye) functions include, in polarizations, Bessel, hypergeometric and confluent hypergeometric eigenfunctions, as well as Legendre polynomials. Obviously, some polarizations are favored by their using Jacobi polynomials in place of the correct polarized eigenfunctions, but their system is already rather complicated. However, the solenoidal properties are compromised by the frequent computational recourse to Chorin-type compressible softening, to preclude spurious oscillations. Continuity is enforced in this study by suitable decomposition into interior and boundary elements. The latter possess vertex, face, and edge contributions (p. 405). Since each cube yields 5 tetrahedra, one would expect $5N^3$ tetrahedra from N^3 cubes, prior to binary subdivisions to obtain convergence. In reality, $6N^3$ tetrahedrons are required, since some of the tetrahedral shapes are awkward and improve under a second division. A fuller description is found in the paper cited. Their time-integration scheme is slightly inferior to Alp_3, mentioned earlier. Interestingly, their general approach also relies on spatial expansion integration via Helmholtz functions, as does ours, but their usual assumption of homogeneity (periodicity) in one direction allows them to rely on 2 dimensional (or prismatic) Laplacians. They could then use the exact solutions based on the classical harmonic integrals of Schwarz-Christoffel (see Kantorovich and Krylov [22], Chapter VII, S.L. Sobolev (1936)[41]).

It is computationally significant that domain-subdivision methods (required for accuracy, and essential to constructing error estimates) should allow decompositions to proceed in parallel, thus permitting an efficient geometry-based parallel machine work distribution. This is a great advantage of the Evangelinos scheme. Also, the separation of boundary modes from interior modes allows geometrically disjoint interior modes to be decoupled (Smith and Gropp (1996) [40]) (Sherwin and Karniadakis (1995) [39]), reducing matrix sizes. This is only a small sample of the enormous computational efforts invested in modern fluid mechanics, whose successes have been well documented in the massive review by T.J. Chung (2002), [10]. The present work takes a rather different slant in that for most of the paper, the analytical side is kept uppermost. However, computational and experimental verification should always be considered as the ultimate authority in fluid dynamics.

A natural question in that context is the degree of accuracy to be expected from retaining a given number of eigenmodes of the Helmholtz equation. Although the general question is theoretically difficult, a somewhat similar spectral calculation (in a special problem) by Zhao and Ling (2003), [47] shows that their eigenspectrum plot verified that 20 modes were active. Also, the fourth pair of modes was the most active. When used to reconstruct the flow, these 20 modes produced excellent fits to the iso-vorticity surfaces over a wide geometrical range (p. 308). Other authors (cited in Chung (2002) [10]) claim that spectral approximations converge adequately after as few as 5 modes. For example, there is the earlier work by T.J. Chung [9] and K.T. Yoon (1996) [46] on compressible flows and the previous S.J. Sherwin and G.E. Karniadakis (1995) [39] paper on 2-D incompressible flows. Of course, we recommend avoiding over-dependence on simplified eigenvectors, such as Legendre, Chebyshev, or Laguerre-based eigenvectors. These would distort flows in a sphere, where the Debye eigenvectors of Section 9 suggest a more realistic direction, and by extension, flows in curved containers. The case of cylindrical flows, using these techniques, will be pursued in a later communication. The absence of polynomials among infinite cylinder eigenfunctions (and general eigenfunctions) causes technical difficulties.

A detailed account of the geometry involved in the various simple shapes in 2-D and 3-D and use in the Karniadakis-Sherwin method is found in their (1999) specialized book [23]. Also, a treatment of the Helmholtz equation by K-S methods is developed in their Chapter 7. For 2-D Helmholtz problems, a pre-conditioner scheme reduces overall computer time. A tri-diagonal Schur complement transformation was found to be effective in Chan and Goovaerts (1990) [6]. The 2-D problem for the first eigenvalue $\lambda = 1$, taking the nominal start eigenvector for a square $u_1 = \sin(\pi x_1)\cos(\pi x_2)$, was solved by K-S on a "fist" domain; the error was below 10^{-8} (at iteration 16) with a quasi-random triangulation. Work by Wille (1996) [45] suggests that a septi-diagonal preconditioner is satisfactory for triangular prisms. Both K-S and Wille aim to use ultra-fast parallel computing. The drawback in the Wille boundary treatment is that the inner-product used to produce optimal summation-by parts regularity is of the "weighted" type. Moreover, Wille shows that uniform weights on the inner product produce significant errors when coupled with his optimal interior difference algorithms. Unfortunately, few computer scientists are prepared to deal with irregular tetrahedral regions, so necessary for difficult 3-D calculations. Moreover, theoretical harmonic analysis on tetrahedra is also under-developed.

The history of this work may be of interest. Encouraged by the late Professor G.F.D. Duff, during a random encounter in Australia, Professor Leipnik began an attempt to extend the number of exact solutions of the N-S equations. This resulted in a 1996 Canadian publication, using a vector analog of Hopf series, structured in terms of scalar and vector potentials, to provide general explicit solutions. Professor Duff, a specialist in Riemannian differential geometry, and PDE, later interpreted and clarified this as infinite-dimensional quadratic algebra, reducing the problem by projection on Helmholtz functions, an accepted technique in computational fluid dynamics (spectral method). He also laid out the plan of this joint paper, and completed the Euler part before his death. Professor Leipnik produced the viscous extension (which takes up Sections 7, 8) and added the sections on box and sphere regions and computation (which he had previously experienced in plasma physics at the Michelson Laboratory of the US Navy). He wishes to thank that organization for its generous policy toward international cooperation, the University of California at Santa Barbara, which supported, in various ways, this 25 year effort, and the Universities of Adelaide and Toronto for being cooperative hosts during sabbaticals from UCSB, between 1980 and 2000.

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