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INNER OBSTACLE PROBLEM: CONVERGENCE OF THE SOLUTIONS FOR IMPEDIMENTS WITH VARYING DOMAINS

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Abstract: In this paper we consider the sequence of the inner obstacle problems. Assuming the convergence of impediments (in certain sense) we obtain strong convergence in $H_0^1(\Omega)$ of the solutions to the solution of the limit inner obstacle problem. It is worth pointing out that this result represents an extension of [6], where the sequence of global problems has been considered.

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1. Introduction

In the 1970's there was considerable interest in the analysis of obstacle problems. This was connected with the development of research on variational inequalities and has been studied by many authors (see [1], [9] and references therein).

Recently the interest in the analysis of the obstacle problems has increased. It is due to appearance of works on the inner problems (cf. [2], [4]). It seems that analysis of the inner obstacle problems is more complicated and interested than

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for the global ones. However the investigation of the inner problems requires a more detailed approach.

In this article we study the dependence of the solutions of the inner problems with respect to the behaviour of the impediments.

The concept of continuous dependence of the obstacles in the case of the global problem were considered by H. Toyoizumi [8], then the extensions for the double problems one can find in [6]. Our result presents the complement of those papers. We investigate the most general case: the impediments are defined only on the part of the domain Ω, which themselves are varying subsets of Ω.

The present paper is a part of the research program on free boundary problems.

2. Notation and Basic Definitions

Firstly, we introduce the following concept of "convergence" in L^2 of a sequence of functions defined on different domains.

Let A, B, $B_n \subset \mathbb{R}^k$ $(k, n \in \mathbb{N})$ be open, bounded sets, such that $B_n, B \subset A$.

Definition 1. Let $f_n : B_n \to \mathbb{R}$, $f : B \to \mathbb{R}$. We say that $f_n \to f$ provided (i) $\tilde{f}_n \to \tilde{f}$ in $L^2(A)$, where

$$\tilde{f}_n = \begin{cases} f_n & \text{in } B_n \\ 0 & \text{in } A \backslash B_n, \end{cases} \quad , \quad \tilde{f} = \begin{cases} f & \text{in } B \\ 0 & \text{in } A \backslash B, \end{cases}$$

Remark 2. Observe that the notion defined above does not determine the limit function uniquely. Moreover, that kind of convergence does not demand any dependence of corresponding domains.

Example 3. Consider the sequence (f_n) defined by:

$$\forall n \in \mathbb{N} \quad f_n : (-1 + \tfrac{1}{n}, 1 - \tfrac{1}{n}) \to \mathbb{R}, \;\; f_n(x) = \begin{cases} 1 & \text{for } x \in (-\frac{1}{3}, \frac{1}{3}), \\ 0 & \text{elsewhere.} \end{cases}$$

It easy to verify that we can put the limit function in the form $f : (-\frac{1}{3}, \frac{1}{3}) \to \mathbb{R}$, $f(x) = 1$, $x \in (-\frac{1}{3}, \frac{1}{3})$. However, (f_n) is also convergent to $g : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}$,

$$g(x) = \begin{cases} 1 & \text{for } x \in (-\frac{1}{3}, \frac{1}{3}), \\ 0 & \text{elsewhere.} \end{cases}$$

Despite the "not typical" behaviour of the described convergence it represents, under suitable assumptions, a useful tool to study convergence in H_0^1 of solutions of the inner obstacle problems.

In what follows we need the following assumptions:

(ii) $\Omega \subset \mathbb{R}^k$ is an open, bounded set with the boundary $\partial\Omega$ of $C^{1,1}$ class;

(iii) $\forall n \in \mathbb{N}$, E_n, E, are open, bounded sets with the boundaries ∂E_n, ∂E respectively, of $C^{1,1}$ class such that $E_n \subset \overline{E}_n \subset E \subset \overline{E} \subset \Omega$.

Definition 4. Let us consider a sequence (ϕ_n) elements of $L^2(E_n)$ such that $\forall n \in \mathbb{N} \max_{E_n} \phi_n > 0$ and function $\phi \in L^2(E)$. We say that $\phi_n \to \phi$ from below provided $\phi_n \to \phi$ in the sense of Definition 1 and $\forall n \in \mathbb{N}$ $\tilde{\phi}_n \leq \tilde{\phi}$ a.e. in Ω.

Now, let us consider a'priori given function $\phi \in L^2(E)$ such that E are satisfying (iii). We define so-called admissible set

$$K_\phi \quad = \quad \{v \in H_0^1(\Omega) : v \geq \phi \text{ a.e. in } \overline{E}\}. \tag{1}$$

Throughout this paper we agree that we are given an operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ which is Lipschitz i.e.

$$\exists\, \gamma > 0 \;\; \forall\, u, v \in H_0^1(\Omega) \qquad \|Au - Av\| \leq \gamma \|u - v\|\,, \tag{2}$$

and coercive, i.e.

$$\exists\, \alpha > 0 \;\; \forall\, u, v \in H_0^1(\Omega) \qquad \langle Au - Av, u - v\rangle \geq \alpha \|u - v\|^2. \tag{3}$$

Let us consider the following inner obstacle problem P:
For given $\phi \in L^2(E)$ find $u \in K_\phi$ such that

$$\langle Au, v - u\rangle \geq \langle f, v - u\rangle\,, \quad \forall\, v \in K_\phi, \tag{4}$$

where $f \in H^{-1}(\Omega)$.

Our paper is devoted to the investigation of the convergence of the solutions of the following inner problems P_n:
For given $\phi_n \in L^2(E_n)$ find $u_n \in K_{\phi_n}$ such that

$$\langle Au_n, v - u_n\rangle \geq \langle f, v - u_n\rangle\,, \quad \forall\, v \in K_{\phi_n}, \tag{5}$$

where $f \in H^{-1}(\Omega)$, to the solution u of the problem P.

Note that there exists unique solutions $u_n \in K_{\phi_n}$, $u \in K_\phi$ of the problems (4), (5) provided the admissible set K_{ϕ_n}, K_ϕ are nonempty (see [5]).

3. Continuity Results

In this section we study continuous dependence of solutions to inner obstacle problems on impediments. In the following theorem we obtain convergence of the sequence of solutions (u_n) of the problems P_n to the solution u of the limit problem P.

The proof is based on properties of $H_0^1(\Omega)$ functions taking advantage of Sobolev's Embedding Theorem and Minty's Lemma.

Theorem 5. *Let the sets E, E_n ($\forall n \in \mathbb{N}$) satisfy the assumption (iii) as well as the functions $\phi \in L^2(E)$, $\phi_n \in L^2(E_n)$ satisfy the assumption (i). For an operator A satisfying (2), (3) and $f \in H^{-1}(\Omega)$, if we assume that the sequence (ϕ_n) converges to ϕ from below and that the corresponding admissible sets K_{ϕ_n}, K_ϕ are nonempty then the solutions u_n of the inner problems P_n converge strongly in $H_0^1(\Omega)$ to the solution u of the inner problem P.*

Proof. Firstly, we will show adapting ideas from [8] that if there exists $v_0 \in \bigcap_{n\geq 1} K_{\phi_n}$ then there exists $C > 0$ such that the following estimate holds:

$$\|u_n - v_0\| \leq C\left(\|f\| + \|v_0\|\right).$$

Indeed, if we substitute $v = v_0$ into every problem P_n we get

$$\langle Au_n, v_0 - u_n\rangle \geq \langle f, v_0 - u_n\rangle. \tag{6}$$

Due to the fact that A is coercive and Lipschitz and $f \in H^{-1}(\Omega)$ we obtain

$$\begin{aligned} \alpha\|u_n - v_0\|^2 \leq& \langle Au_n - Av_0, u_n - v_0\rangle = \langle Au_n, u_n - v_0\rangle - \langle Av_0, u_n - v_0\rangle \\ &\leq \langle f, u_n - v_0\rangle - \langle Av_0, u_n - v_0\rangle \leq \|f\|\,\|u_n - v_0\| + \gamma\|v_0\|\,\|u_n - v_0\|. \end{aligned} \tag{7}$$

Hence

$$\|u_n - v_0\| \leq \frac{1}{\alpha}\left(\|f\| + \gamma\|v_0\|\right).$$

Since v_0 is apriori known we get that (u_n) is bounded. This implies that there exists a subsequence

$$u_{n_k} \to u^* \ \text{ weakly in } H_0^1(\Omega);$$

Applying the standard procedure (for details see [3]) we get

$$u_n \to u^* \ \text{ strongly in } L^2(\Omega), \quad u_n \to u^* \ \text{ a.e. in } \Omega, \tag{8}$$

for $u^* \in H_0^1(\Omega)$. Since $\phi_n \leq u_n$ a.e. in E_n if we let $n \to \infty$ we obtain that $\phi \leq u^*$ a.e. in E. Therefore $u^* \in K_\phi$.

Now we show that $\bigcap_{n\geq 1} K_{\phi_n}$ is nonempty. Indeed, let us take arbitrary element $v \in K_\phi$. Then we have: $v \geq \phi$ a.e. in $\bar{E}$ in particular $v \geq \tilde{\phi}$ a.e. in $\bar{E}$ and $v \geq \tilde{\phi} \geq \tilde{\phi}_n$ a.e. in $\bar{E}$ since ϕ_n approaches ϕ from below and $\forall n \in \mathbb{N}\ E_n \subset E$. Thus $v \geq \tilde{\phi} \geq \tilde{\phi}_n = \phi_n$ a.e. in $\bar{E}_n$. Finally, we get that

$$\forall n \in \mathbb{N}, \quad v \in K_{\phi_n}. \tag{9}$$

By Minty's Lemma (see [5]) we can identify the problem P_n with the following one:

Find $u_n \in K_{\phi_n}$ such that

$$\langle Av, v - u_n \rangle \geq \langle f, v - u_n \rangle, \quad \forall\, v \in K_{\phi_n}.$$

Having (9) we may replace K_{ϕ_n} by K_ϕ and get

$$\langle Av, v - u_n \rangle \geq \langle f, v - u_n \rangle, \quad \forall\, v \in K_\phi.$$

If we let $n \to \infty$ we arrive at the following

$$u^* \in K_\phi : \langle Av, v - u^* \rangle \geq \langle f, v - u^* \rangle, \quad \forall\, v \in K_\phi.$$

Applying Minty's Lemma to the last problem we get

$$u^* \in K_\phi : \langle Au^*, v - u^* \rangle \geq \langle f, v - u^* \rangle, \quad \forall\, v \in K_\phi.$$

Uniqueness of the solution to the problem P allows us to state that $u^* = u$. So we have shown that $u_n \to u$ weakly in $H_0^1(\Omega)$.

In order to show strong convergence we observe that $u \in \bigcap\limits_{n\geq 1} K_{\phi_n}$ so we can put $v_0 = u$ in (7). Hence $\alpha \cdot \|u_n - u\|^2 \leq \langle f - Au, u_n - u \rangle$. If we let $n \to \infty$ we obtain that $u_n \to u$ strongly in $H_0^1(\Omega)$ which finishes the proof. □

Remark 6. The same result would be obtained if we considered the sequence of double inner obstacle problems with the admissible set $K_{\phi_n}^{\psi_n} = \{v \in H_0^1(\Omega) : v \geq \phi_n$ a.e. in $\overline{E_n}$ and $v \leq \psi_n$ a.e. in $\overline{F_n}\}$. In this case we assume that for F_n (iii) holds and the sequence of obstacles (ψ_n) behaves reversely to (ϕ_n).

Remark 7. In order to guarantee the nonemptyness of the admissible sets one can assume (following Stampacchia's pioneering paper [7]) that the obstacles can be extended in H^1 way onto the whole domain Ω, having nonpositive values at the boundary. It is also possible to follow the ideas presented in [8] assuming that the obstacles admit extensions being the elements of $L_C^2(\Omega)$ space.

Remark 8. One can notice that it is necessary to assume the special behaviour of the converging obstacles. In [6] we present an illustrative example (for the fixed domains) where the impediments are converging from above and the sequence of the solutions does not approach to the solution of the limit problem.

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