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OSCILLATION AND COMPARISON THEOREMS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract: Oscillatory property of solutions of a class of second order differential equations are discussed and a comparison theorem is obtained.

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1. Introduction

We consider the following second order differential equation

$$\left(r(t)\,\psi(x)\,x'\right)' + p(t)\,f(x) = 0, \quad t \geq t_0 \geq 0, \tag{1}$$

where $r \in C^1([t_0,\infty),(0,\infty))$, $p \in C([t_0,\infty),(-\infty,\infty))$, $\psi \in C^1((-\infty,\infty),[0,\infty))$ and $f \in C^1((-\infty,\infty),(-\infty,\infty))$.

We will also assume throughout that $\psi(x) \neq 0$ and $xf(x) > 0$ for $x \neq 0$, $\inf_{x\neq 0}(f'(x)/\psi(x)) \geq \varepsilon^*$ for some constant $\varepsilon^* > 0$ and $F(x) = f'(x)/\psi(x)$ is nondecreasing on the interval $(0,\infty)$ and nonincreasing on interval $(-\infty,0)$.

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By a (continuable) solution of (1), we mean a continuously differentiable function on $[t_0, \infty)$ which satisfies (1). As usual, a solution $x(t)$ of (1) is said to be oscillatory if it has arbitrary large zeros, otherwise, it is said to be nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

In [1], equation (1) is said to have "integrally small" coefficients if $\int_{t_0}^{\infty} ds/r(s) = \infty$ and $\int_t^{\infty} p(s)\, ds < \infty$ for all $t \geq t_0$, and oscillation criteria are obtained which improve earlier results for the special cases

$$x'' + p(t)f(x) = 0,$$

as well as

$$\left(r(t)x'\right)' + p(t)f(x) = 0.$$

In case equation (1) does not have integrally small coefficients, we will show that oscillation criteria can still be obtained when equation is 'convertible'. To be more precise, (1) is said to be convertible if there exists a function $\eta \in C^1([t_0, \infty), [0, \infty))$ such that:

(i) $\eta'(t) \leq 0$ for $t \geq t_0$;
(ii) $\int_t^{\infty} \eta(s)\, p(s)\, ds < \infty$ for $t \geq t_0$;
(iii) $\int_{t_0}^{\infty} \frac{r(s)(\eta'(s))^2}{\eta(s)} ds < \infty$; and
(iv) $\int_{t_0}^{\infty} \frac{ds}{\eta(s)r(s)} = \infty$.

The function $\eta(t)$ satisfying the above conditions is called a coefficient factor (relative to (1)).

We remark that convertible equations different from ours have been discussed, see e.g. [2, 3]. In particular, a result similar to the following is reported in [2].

Lemma 1. *Suppose equation (1) is convertible and $\eta(t)$ is a corresponding coefficient factor. If (1) has a nonoscillatory solution $x = x(t)$ such that $x(t) \neq 0$ for $t \geq t_1 \geq t_0$, then the function*

$$W(t) = \frac{\eta(t)\, r(t)\, \psi(x(t))\, x'(t)}{f(x(t))} \tag{2}$$

is defined for $t \geq t_1$ and satisfies

$$\int_{t_1}^{\infty} \frac{W^2(s)}{\eta(s)\, r(s)} F(x(s))\, ds < \infty, \tag{3}$$

$$\left| \int_{t_1}^{\infty} \frac{W(s)\, \eta'(s)}{\eta(s)} ds \right| < \infty, \tag{4}$$

$$\lim_{t\to\infty} W(t) = 0, \tag{5}$$

and

$$W(t) = \int_t^\infty \frac{W^2(s)}{\eta(s)r(s)} F(x(s))\,ds - \int_t^\infty \frac{W(s)\eta'(s)}{\eta(s)} ds + \int_t^\infty \eta(s)p(s)\,ds, \tag{6}$$

for $t \geq t_1$.

Proof. From (2) and (1), we obtain

$$W'(t) = -\frac{W^2(t)}{\eta(t)r(t)} F(x(t)) + \frac{W(t)\eta'(t)}{\eta(t)} - \eta(t)p(t).$$

After integrating from t_1 to t, we have

$$W(t) - W(t_1) + \int_{t_1}^t \frac{W^2(s)}{\eta(s)r(s)} F(x(s))\,ds - \int_{t_1}^t \frac{W(s)\eta'(s)}{\eta(s)} ds + \int_{t_1}^t \eta(s)p(s)\,ds = 0. \tag{7}$$

Since (1) is convertible,

$$\lim_{t\to\infty} \left[W(t) + \int_{t_1}^t \frac{W^2(s)}{\eta(s)r(s)} F(x(s))\,ds - \int_{t_1}^t \frac{W(s)\eta'(s)}{\eta(s)} ds \right] = K, \tag{8}$$

where

$$K = W(t_1) - \int_{t_1}^\infty \eta(s)p(s)ds < \infty.$$

Suppose to the contrary that (3) is not true, then in view of (8),

$$\lim_{t\to\infty} \frac{W(t) - \int_{t_1}^t \frac{W(s)\eta'(s)}{\eta(s)} ds}{\int_{t_1}^t \frac{W^2(s)}{\eta(s)r(s)} F(x(s))\,ds} ds = -1. \tag{9}$$

Thus there exists $t_2 \geq t_1$ such that

$$\frac{W(t) - \int_{t_1}^t \frac{W(s)\eta'(s)}{\eta(s)} ds}{\int_{t_1}^t \frac{W^2(s)}{\eta(s)r(s)} F(x(s))\,ds} ds < -\frac{1}{2} \tag{10}$$

for $t \geq t_2$. In view of the Cauchy-Schwartz inequality,

$$\left|\int_{t_1}^{t} \frac{W(s)\,\eta'(s)}{\eta(s)} ds\right| \leq \left[\int_{t_1}^{t} \frac{r(s)\,(\eta'(s))^2}{\eta(s)\,F(x(s))} ds\right]^{\frac{1}{2}} \left[\int_{t_1}^{t} \frac{W^2(s)\,F(x(s))}{\eta(s)\,r(s)} ds\right]^{\frac{1}{2}}$$
$$\leq \frac{M}{\sqrt{\varepsilon^*}} \left[\int_{t_1}^{t} \frac{W^2(s)\,F(x(s))}{\eta(s)\,r(s)} ds\right]^{\frac{1}{2}}, \quad (11)$$

where

$$M = \left\{\int_{t_1}^{\infty} \frac{r(s)(\eta'(s))^2}{\eta(s)} ds\right\}^{1/2}.$$

Furthermore, there exists $t_3 \geq t_2$ such that

$$\frac{M}{\sqrt{\varepsilon^*}\left(\int_{t_1}^{t} \frac{W^2(s)F(x(s))}{\eta(s)r(s)} ds\right)^{\frac{1}{2}}} < \frac{1}{4} \quad (12)$$

for $t \geq t_3$. From (10), (11) and (12), we see that

$$\frac{W(t)}{\int_{t_1}^{t} \frac{W^2(s)F(x(s))}{\eta(s)r(s)} ds} < -\frac{1}{4}, \quad t \geq t_3, \quad (13)$$

so that

$$\frac{\varepsilon^*}{16\eta(t)\,r(t)} \leq \frac{F(x(t))}{16\eta(t)\,r(t)}$$
$$\leq \frac{W^2(t)\,F(x(t))}{\eta(t)\,r(t)} \left(\int_{t_1}^{t} \frac{W^2(s)\,F(x(s))}{\eta(s)\,r(s)} ds\right)^{-2}, \quad t \geq t_3.$$

After integrating from t_3 to t, we see further that

$$\frac{\varepsilon^*}{16}\int_{t_3}^{t} \frac{ds}{\eta(s)\,r(s)} \leq \int_{t_3}^{t} \frac{W^2(s)\,F(x(s))}{\eta(s)\,r(s)} \left(\int_{t_1}^{s} \frac{W^2(\tau)\,F(x(\tau))}{\eta(\tau)\,r(\tau)} d\tau\right)^{-2} ds$$
$$\leq \int_{u(t_3)}^{u(t)} \frac{du}{u^2},$$

where

$$u(t) = \int_{t_1}^{t} \frac{W^2(s)F(x(s))}{\eta(s)r(s)} ds.$$

Thus,

$$\frac{\varepsilon^*}{16}\int_{t_3}^{\infty}\frac{dt}{\eta(t)\,r(t)} \le \int_{u(t_3)}^{\infty}\frac{du}{u^2} < \infty,$$

which is contrary to the definition of η.

Now that (3) holds, (4) is true in view of (11). These together with (8) lead to the fact that $\lim_{t\to\infty} W(t)$ exists. But then

$$\varepsilon^*\int_{t_1}^{\infty}\frac{W^2(s)}{\eta(s)\,r(s)}ds \le \int_{t_1}^{\infty}\frac{W^2(s)\,F(x(s))}{\eta(s)\,r(s)}ds < \infty$$

implies $\lim_{t\to\infty} W(t) = 0$. Finally, by taking limit as $t \to \infty$ on both sides of (7), we see that

$$W(t_1) = \int_{t_1}^{\infty}\frac{W^2(s)}{\eta(s)\,r(s)}F(x(s))\,ds + \int_{t_1}^{\infty}\frac{W(s)\,\eta'(s)}{\eta(s)}ds - \int_{t_1}^{\infty}\eta(s)\,p(s)\,ds$$

for any $t_1 \ge t_0$. The proof is complete. □

2. Main Results

Now we introduce, for each $a \ne 0$, a function

$$\Gamma_a(x) = \int_a^x \frac{\psi(u)}{f(u)}du, \quad x \ne 0.$$

When $a > 0$, $\Gamma_a(x)$ is strictly increasing for $x > 0$, and when $a < 0$, $\Gamma_a(x)$ is strictly decreasing for $x < 0$. Therefore, when $a > 0$, $\Gamma_a^{-1}(y)$ is bounded below by a and is strictly increasing on $[0,\infty)$; and when $a < 0$, $\Gamma_a^{-1}(y)$ is bounded above by a and is strictly decreasing for $y \ge 0$. Thus when $a \ne 0$, $F\left(\Gamma_a^{-1}(y)\right)$ is nondecreasing on $[0,\infty)$.

Lemma 2. *Suppose equation (1) is convertible and $\eta(t)$ is a corresponding coefficient factor. Let $x = x(t)$ be a nonoscillatory solution of (1) such that $x(t) \ne 0$ for $t \ge t_1 \ge t_0$. If*

$$\int_t^{\infty}\left(\eta(s)\,p(s) - \frac{r(s)\,(\eta'(s))^2}{4\varepsilon^*\eta(s)}\right)ds \ge 0 \tag{14}$$

for $t \ge t_1$, then the function $W(t)$ defined by (2) satisfies

$$W(t) \;=\; \int_t^{\infty}\frac{W^2(s)}{\eta(s)\,r(s)}F\left(\Gamma_{x(t_1)}^{-1}\left(\int_{t_1}^{s}\frac{W(\tau)}{\eta(\tau)\,r(\tau)}d\tau\right)\right)ds$$

$$-\int_t^\infty \frac{W(s)\,\eta'(s)}{\eta(s)}ds + \int_t^\infty \eta(s)\,p(s)\,ds \tag{15}$$

for $t \geq t_1 \geq t_0$.

Proof. In view of (6), we have

$$\begin{aligned} W(t) = & \int_t^\infty \frac{F(x(s))}{\eta(s)\,r(s)}\left(W(s) - \frac{\eta'(s)\,r(s)}{2F(x(s))}\right)^2 ds \\ & + \int_t^\infty \left(\eta(s)\,p(s) - \frac{r(s)\,(\eta'(s))^2}{4\eta(s)\,F(x(s))}\right) ds \\ & \geq \int_t^\infty \left(\eta(s)\,p(s) - \frac{r(s)\,(\eta'(s))^2}{4\varepsilon^*\eta(s)}\right) ds \geq 0 \end{aligned} \tag{16}$$

for $t \geq t_1 \geq t_0$. Now that $W(t) \geq 0$ for $t \geq t_1$, we may infer from the definition of $W(t)$ that $x'(t) \geq 0$ for $t \geq t_1$ when $x(t) > 0$ for $t \geq t_1$ and $x'(t) \leq 0$ for $t \geq t_1$ when $x(t) < 0$ for $t \geq t_1$. Therefore, $x(t) \geq x(t_1) > 0$ or $x(t) \leq x(t_1) < 0$ for $t \geq t_1 \geq t_0$. But then

$$\int_{t_1}^t \frac{W(s)}{\eta(s)\,r(s)}ds = \int_{x(t_1)}^{x(t)} \frac{\psi(u)}{f(u)}du = \Gamma_{x(t_1)}(x(t)), \tag{17}$$

so that

$$x(t) = \Gamma_{x(t_1)}^{-1}\left(\int_{t_1}^t \frac{W(s)}{\eta(s)\,r(s)}ds\right), \quad t \geq t_1 \geq t_0. \tag{18}$$

If we now substitute $x(t)$ in (18) into (6), (15) is obtained. The proof is complete. □

Lemma 3. *Suppose that equation (1) is convertible,* $\eta(t)$ *is a coefficient factor and (14) holds for* $t \geq t_1 \geq t_0$. *Then (1) has a nonoscillatory solution* $x = x(t)$ *which satisfies* $x(t) \neq 0$ *for* $t \geq t_1 \geq t_0$ *if, and only if, there are function* $\varphi \in C^1([t_0,\infty),[0,\infty))$ *and some constant* $a \neq 0$ *such that*

$$\varphi'(t) + \frac{\varphi^2(t)}{\eta(t)\,r(t)}F\left(\Gamma_a^{-1}\left(\int_{t_1}^t \frac{\varphi(s)}{\eta(s)\,r(s)}ds\right)\right) - \frac{\varphi(t)\,\eta'(t)}{\eta(t)} \leq -\eta(t)\,p(t), \quad t \geq t_1. \tag{19}$$

Proof. Let $x = x(t)$ be a nonoscillatory solution of (1) which satisfies $x(t) \neq 0$ for $t \geq t_1 \geq t_0$ and let $W(t)$ be defined by (2). Then in view of

Lemma 2, $W(t)$ satisfies (15). If we now differentiate (15), we see that $W(t)$ satisfies (19) with $a = x(t_1)$.

Conversely, suppose there are function $\varphi \in C^1([t_0,\infty),[0,\infty))$ and some nonzero constant a such that (19) holds. Integrating (19) from t to T ($T \geq t \geq t_1$), we have

$$\varphi(T) + \int_t^T \frac{\varphi^2(s)}{\eta(s)\,r(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^s \frac{\varphi(\tau)}{\eta(\tau)\,r(\tau)} d\tau\right)\right) ds - \int_t^T \frac{\varphi(s)\,\eta'(s)}{\eta(s)} ds + \int_t^T \eta(s)\,p(s)\,ds \leq \varphi(t). \quad (20)$$

By taking limits on both sides of (20) as $T \to \infty$,

$$\int_t^\infty \frac{\varphi^2(s)}{\eta(s)\,r(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^s \frac{\varphi(\tau)}{\eta(\tau)\,r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{\varphi(s)\,\eta'(s)}{\eta(s)} ds + \int_t^\infty \eta(s)\,p(s)\,ds \leq \varphi(t), \quad (21)$$

for $t \geq t_1$. Now we define a sequence of functions on the interval $[t_1,\infty)$ as follows: $y_0(t) \equiv 0$ and

$$y_{n+1}(t) = \int_t^\infty \frac{y_n^2(s)}{\eta(s)\,r(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^s \frac{y_n(\tau)}{\eta(\tau)\,r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{y_n(s)\,\eta'(s)}{\eta(s)} ds + \int_t^\infty \eta(s)\,p(s)\,ds, \quad (22)$$

for $n = 0,1,2,\ldots$. It is clear that (14) implies $y_1(t) \geq 0$. By induction, we may then show

$$y_n(t) \leq y_{n+1}(t) \leq \varphi(t), \quad n = 0,1,2,3,\ldots \quad . \quad (23)$$

Therefore,

$$\lim_{n\to\infty} y_n(t) = y(t) \leq \varphi(t).$$

By Lebesgue's Dominated Convergence Theorem, we may then infer from (22) that

$$y(t) = \int_t^\infty \frac{y^2(s)}{\eta(s)\,r(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^s \frac{y(\tau)}{\eta(\tau)\,r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{y(s)\,\eta'(s)}{\eta(s)} ds + \int_t^\infty \eta(s)\,p(s)\,ds. \quad (24)$$

Let

$$x(t) = \Gamma_a^{-1}\left(\int_{t_1}^{t} \frac{y(s)}{\eta(s)\, r(s)} ds\right), \tag{25}$$

then

$$\Gamma_a(x(t)) = \int_{t_1}^{t} \frac{y(s)}{\eta(s)\, r(s)} ds, \quad t \geq t_1. \tag{26}$$

After differentiation, we obtain

$$\frac{\psi(x(t))\, x'(t)}{f(x(t))} = \frac{y(t)}{\eta(t)\, r(t)}, \quad t \geq t_1. \tag{27}$$

Thus,

$$\begin{aligned}\big(r(t)\,\psi(x(t))\,x'(t)\big)' &= \left(\frac{y(t)\, f(x(t))}{\eta(t)}\right)' \\ &= \frac{y'(t)\, f(x(t))\,\eta(t) + y(t)\, f'(x(t))\, x'(t)\,\eta(t) - \eta'(t)\, y(t)\, f(x(t))}{\eta^2(t)}, \\ & \qquad t \geq t_1. \end{aligned} \tag{28}$$

We may calculate $y'(t)$ from (24) by differentiation, and substitute it into (28) to yield

$$\big(r(t)\,\psi(x(t))\,x'(t)\big)' + p(t)\, f(x(t)) = 0, \quad t \geq t_1.$$

Thus the function $x(t)$ defined by (25) is a nonoscillatory solution of (1). The proof is complete. □

Let us now formally construct a sequence of functions on the interval $[t_1, \infty)$ as follows:

$$Q_0(t) = \int_t^\infty \eta(s)\, p(s)\, ds,$$

$$Q_1(t) = \int_t^\infty \frac{Q_0^2(s)}{\eta(s)\, r(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^{s} \frac{Q_0(\tau)}{\eta(\tau)\, r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{Q_0(s)\,\eta'(s)}{\eta(s)} ds,$$

$$Q_{n+1}(t) = \int_t^\infty \frac{(Q_0(s) + Q_n(s))^2}{\eta(s)\, r(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^{s} \frac{(Q_0(\tau) + Q_n(\tau))}{\eta(\tau)\, r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{(Q_0(s) + Q_n(s))\,\eta'(s)}{\eta(s)} ds \tag{29}$$

for $n = 0, 1, 2, ...$. In view of (14), we have $Q_0(t) \geq 0$. Furthermore, by mathematical induction, it is easily seen that for $n = 1, 2, 3, ...,$

$$Q_{n+1}(t) \geq Q_n(t), \quad t \geq t_1. \tag{30}$$

Theorem 1. *Suppose equation (1) is convertible, $\eta(t)$ is a coefficient factor and (14) holds. Then (1) has a nonoscillatory solution $x = x(t)$ which satisfies $x(t) \neq 0$ for $t \geq t_1 \geq t_0$ if, and only if, there is some constant $a \neq 0$ such that the sequence of functions $\{Q_n\}$ (formally) defined above is well defined and*

$$\lim_{t\to\infty} Q_n(t) = Q(t) < \infty, \quad t \geq t_1 \geq t_0, \tag{31}$$

Proof. Let $x = x(t)$ be a nonoscillatory solution of (1) which satisfies $x(t) \neq 0$ for $t \geq t_1 \geq t_0$. Then (15) and (16) hold by Lemma 2. Therefore,

$$W(t) \geq Q_0(t) \geq 0, \quad t \geq t_1. \tag{32}$$

Let

$$U(t) = \int_t^\infty \frac{W^2(s)}{\eta(s)\, r(s)} F\left(\Gamma_{x(t_1)}^{-1}\left(\int_{t_1}^s \frac{W(\tau)}{\eta(\tau)\, r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{W(s)\, \eta'(s)}{\eta(s)} ds. \tag{33}$$

Then, in view of (32), we have

$$U(t) \geq \int_t^\infty \frac{Q_0^2(s)}{\eta(s)\, r(s)} F\left(\Gamma_{x(t_1)}^{-1}\left(\int_{t_1}^s \frac{Q_0(\tau)}{\eta(\tau)\, r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{Q_0(s)\, \eta'(s)}{\eta(s)} ds = Q_1(t). \tag{34}$$

By (15) and (34), we have

$$W(t) \geq Q_0(t) + U(t) \geq Q_0(t) + Q_1(t) \geq 0, \quad t \geq t_1. \tag{35}$$

By (33) and (34), $U(t) \geq Q_2(t)$, thus

$$W(t) \geq Q_0(t) + Q_2(t) \geq 0, \quad t \geq t_1. \tag{36}$$

By mathematical induction, we then see that

$$W(t) \geq Q_0(t) + Q_n(t), \quad t \geq t_1, \quad n = 1, 2, 3, \cdots . \tag{37}$$

Finally, from (30) and (37), we may assert $\{Q_n(t)\}$ is well defined and converges to some $Q(t)$.

Conversely, suppose (31) holds. Then, by (30), $Q(t) \geq Q_n(t)$, for $n = 1, 2, 3, \ldots$. By means of Lebesgue's Dominated Convergence Theorem, we may infer from (29) that

$$Q(t) = \int_t^\infty \frac{(Q_0(s) + Q(s))^2}{\eta(s) r(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^s \frac{Q_0(\tau) + Q(\tau)}{\eta(\tau) r(\tau)} d\tau\right)\right) ds - \int_t^\infty \frac{(Q_0(s) + Q(s)) \eta'(s)}{\eta(s)} ds, \quad (38)$$

for $t \geq t_1$. After differentiation, we have

$$(Q_0(t) + Q(t))' + \frac{(Q_0(t) + Q(t))^2}{\eta(t) r(t)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^t \frac{Q_0(s) + Q(s)}{\eta(s) r(s)} ds\right)\right) - \frac{(Q_0(t) + Q(t)) \eta'(t)}{\eta(t)} = -\eta(t) p(t) \quad (39)$$

for $t \geq t_1$. In view of Lemma 3, equation (1) has a nonoscillatory solution. The proof is complete. □

The above derivations will enable us to obtain a Hille-Wintner Type Comparison Theorem between (1) and a similar equation of the form

$$\left(\overline{r}(t) \psi(x) x'\right)' + \overline{p}(t) f(x) = 0, \quad t \geq t_0, \quad (40)$$

where f and ψ are the same functions in (1) and $\overline{r}(t)$, $\overline{p}(t)$ satisfy the same conditions satisfied by $r(t)$ and $p(t)$ respectively.

Theorem 2. *Suppose equations (1) and (40) are convertible, and $\eta(t)$, $\overline{\eta}(t)$ are corresponding coefficient factors respectively. Suppose further that*

$$\int_t^\infty \left(\eta(s) p(s) - \frac{r(s) (\eta'(s))^2}{4\varepsilon^* \eta(s)}\right) ds \geq 0, \quad t \geq t_0,$$

$$\int_t^\infty \left(\bar{\eta}(s) \bar{p}(s) - \frac{\bar{r}(s) (\bar{\eta}'(s))^2}{4\varepsilon^* \bar{\eta}(s)}\right) ds \geq 0, \quad t \geq t_0,$$

$$0 \leq \overline{\eta}(t) \overline{r}(t) \leq \eta(t) r(t), \quad t \geq t_0,$$

$$0 \leq \int_t^\infty \eta(s) p(s) ds \leq \int_t^\infty \bar{\eta}(s) \bar{p}(s) ds, \quad t \geq t_0,$$

and

$$\frac{\overline{\eta'}(t)}{\overline{\eta}(t)} \le \frac{\eta'(t)}{\eta(t)}, \quad t \ge t_0.$$

If equation (1) is oscillatory, so is equation (40). If equation (40) is nonoscillatory, so is equation (1).

Proof. Let

$$\bar{Q}_0(t) = \int_t^\infty \bar{\eta}(s)\,\bar{p}(s)\,ds,$$

$$\bar{Q}_1(t) = \int_t^\infty \frac{\bar{Q}_0^2(s)}{\bar{\eta}(s)\,\bar{r}(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^s \frac{\bar{Q}_0(\tau)}{\bar{\eta}(\tau)\,\bar{r}(\tau)}d\tau\right)\right)ds - \int_t^\infty \frac{\bar{Q}_0(s)\,\bar{\eta}'(s)}{\bar{\eta}(s)}ds,$$

$$\bar{Q}_{n+1}(t) = \int_t^\infty \frac{\left(\bar{Q}_0(s)+\bar{Q}_n(s)\right)^2}{\bar{\eta}(s)\,\bar{r}(s)} F\left(\Gamma_a^{-1}\left(\int_{t_1}^s \frac{\left(\bar{Q}_0(\tau)+\bar{Q}_n(\tau)\right)}{\bar{\eta}(\tau)\,\bar{r}(\tau)}d\tau\right)\right)ds - \int_t^\infty \frac{\left(\bar{Q}_0(s)+\bar{Q}_n(s)\right)\bar{\eta}'(s)}{\bar{\eta}(s)}ds$$

for $n = 1, 2, \ldots$. Then our assumptions imply

$$Q_n(t) \le \overline{Q}_n(t), \quad n = 0, 1, 2, \ldots . \tag{41}$$

If equation (40) has a nonoscillatory solution, then the sequence $\left\{\overline{Q}_n(t)\right\}$ converges to some function $\overline{Q}(t)$. In view of (30) and (41), $\{Q_n(t)\}$ is nondecreasing and bounded, thus $\lim_{t\to\infty} Q_n(t) = Q(t)$ exists. By Theorem 1, equation (1) has a nonoscillatory solution. The proof is complete. □

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