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ON THE VERTEX-DISTINGUISHING EDGE COLORING OF JOIN GRAPH WITH STAR AND COMPLETE BALANCED BIPARTITE GRAPH

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Abstract: This paper deals with the vertex-distinguishing edge coloring of join graph with star and complete balanced bipartite graph.

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1. Introduction

Graph theory is a sort of models which can be applied in various science fields such as computer science, physics, biology, chemistry, strategy, etc. Graph coloring is one of the chief topics in graph research. The four-color conjecture is firstly brought up in vertex coloring, which develops the research work in graph theory. Later on, based on many theoretical and practical problems, numbers of mathematical experts began to study vertex coloring, edge coloring, total coloring, list coloring, etc.

This paper deals with the vertex-distinguishing edge coloring of join graph with star and complete balanced bipartite graph.

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Definition 1.1. (see [1]-[4], [6]) G is a sample graph, f is a proper edge coloring of G, for any vertices $u, v \in G(V)$, suppose $C(u) \neq C(v)$, where $C(u) = \{f(uw)|uw \in E(G)\}$, then f is called a k-vertex-distinguishing edge-coloring of G, and which is abbreviated k-$VDEC$ of G, and $\chi'_{vd}(G) = \min\{k \mid \text{G has a } k\text{-}VDEC\}$ is called the k-vertex-distinguishing edge-chromatic number of G.

C(u) in Definition 1.1 is called the color set of vertex u and $\{1, 2, \cdots, k\}\backslash C(u)$ is denoted by $\overline{C}(u)$.

Definition 1.2. (see [1]-[4]) G is a sample graph, n_i denotes the number of vertex which has i degrees, δ, Δ denote minimum, maximum degree of G respectively, then

$$\mu(G) = \max\{\min\{\lambda| \binom{\lambda}{i} \geq n_i\},\ \delta \leq i \leq \Delta\}$$

is called combinatorial degree of G.

Conjecture 1.1. (see [1]-[4]) *Let G be a connected graph with $|V(G)| \geq 3$, then:*

$$\mu(G) \leq \chi'_{vd}(G) \leq \mu(G) + 1.$$

Definition 1.3. (see [4]-[7]) Suppose G and H are two simple graphs of which vertices and edges are disconnected,

$$V(G \vee H) = V(G) \cup V(H), E(G \vee H) = E(G) \cup E(H) \cup \{uv|u \in V(G)v \in V(H)\},$$

then $G \vee H$ is called join-graph of G and H.

In this paper, we studied the conjecture of $S_m \bigvee K_{n,n}$. For others terminologies we refer to reference [5].

2. Main Results

From Definition 1.2, it is easy to obtain the following lemmas.

Lemma 2.1. *G is a graph without isolated edge and one isolated vertex at most, $\chi'_{vd}(G) \geq \mu(G)$.*

Lemma 2.2. *If $m \geq 1, n \geq 2$, we have*

$$\mu(S_m \bigvee K_{n,n}) = \begin{cases} 4, & m = n = 1; \\ m + 3, & m > n = 1; \\ 2n + 2, & n > m = 1. \end{cases}$$

Lemma 2.3. *For complete graph K_n with order $n \geq 3$, we have*

$$\chi'_{vd}(K_n) = \begin{cases} n, & n \equiv 1 (\text{mod } 2); \\ n+1, & n \equiv 0 (\text{mod } 2). \end{cases}$$

Theorem 2.1. *For star S_m with order $m+1$ and complete balanced bipartite graph $K_{n,n}$, when $m = 0$, then:*

$$\chi'_{vd}(S_0 \bigvee K_{n,n}) = \begin{cases} 3, & n = 1; \\ 2n, & n \geq 2. \end{cases}$$

Proof. Suppose $V(S_0) = \{w\}, V(K_{n,n}) = \{v_1, v_2, \cdots, v_n\} \cup \{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}$

Case 1. When $n = 1$, we have $S_0 \bigvee K_{1,1} = K_3$, from Lemma 2.3, obviously, $\chi'_{vd}(S_0 \bigvee K_{1,1}) = 3$.

Case 2. When $n \geq 2$, we can work out:

$$\mu(S_0 \bigvee K_{n,n}) = \max\{\min\{\lambda| \begin{pmatrix} \lambda \\ 2n \end{pmatrix} \geq 1\}, \min\{\lambda| \begin{pmatrix} \lambda \\ n+1 \end{pmatrix} \geq 2n\}\} = 2n.$$

So, we need only to prove that $S_0 \bigvee K_{n,n}$ has a $2n - VDEC$.

Suppose $C = \{1, 2, ..., 2n-1, 0\}, \overline{C}(u) = C \setminus C(u)$, let f be: $f(wv_i) = i$ $(i = 1, 2, \cdots, 2n-1)$; $f(wv_{2n}) = 0$; $f(v_iv_j) = i + j - n$ $(i = 1, 2, \cdots, n-1;$ $j = n+1, n+2, \cdots, 2n)$; $f(v_nv_j) = j - n - 1$ $(j = n+1, n+2, \cdots, 2n)$.

Such that, $\overline{C}(w) = \phi$; $C(v_i) = \{i, i+1, \cdots, i+n\}$ $(i = 1, 2, \cdots, n-1)$; $C(v_n) = \{0, 1, 2, \cdots, n\}$; $C(v_i) = \{i-n-1, i-n+1, i-n+2, \cdots, i\}$ $(i = n+1, n+2, \cdots, 2n-1)$;$C(v_{2n}) = \{n-1, n+1, n+2, \cdots, 2n-1, 0\}$.

Obviously, $\forall u, v \in V(S_0 \bigvee K_{n,n})$, if $u \neq v$, we have $C(u) \neq C(v)$. So, f is a $2n - VDEC$ of $S_0 \bigvee K_{n,n}$.

From all of above, Theorem 2.1 is true. □

Theorem 2.2. *For star S_m with order $m+1$ and complete balanced bipartite graph $K_{n,n}$, when $m \geq 1, n = 1$, then*

$$\chi'_{vd}(S_m \bigvee K_{1,1}) = \begin{cases} 5, & m = 1; \\ m+3, & m \geq 2. \end{cases}$$

Proof. Suppose $V(S_m) = \{w\} \cup \{u_1, u_2, \cdots, u_m\}, V(K_{1,1}) = \{v_1\} \cup \{v_2\}$.

Case 1. When $m = 1$, $S_1 \bigvee K_{1,1} = K_4$, from Lemma 2.3, $\chi'_{vd}(S_1 \bigvee K_{1,1}) = 5$.

Case 2. When $m \geq 2$, we can work out:

$$\mu(S_m \bigvee K_{1,1}) = \max\{\min\{\lambda| \begin{pmatrix} \lambda \\ m+2 \end{pmatrix} \geq 3\}, \min\{\lambda| \begin{pmatrix} \lambda \\ 3 \end{pmatrix} \geq m\}\} = m+3.$$

So, we need only to prove that $S_m \bigvee K_{1,1}$ has a $(m+3)-VDEC$.

Suppose $C = \{1, 2, ..., m+2, 0\}, \overline{C}(u) = C \setminus C(u)$, Let f be:

$f(wu_i) = i-1 \ \ (i = 1, 2, \cdots, m); f(wv_1) = m; f(wv_2) = m+1;\ f(u_iv_j) = i+j-1 \ \ (i = 1, 2, \cdots, m-1;\ \ j = 1, 2);\ f(u_mv_1) = m+1; f(u_mv_2) = 0; f(v_1v_2) = m+2.$

Such that, $\overline{C}(w) = \{m+2\}$;$\overline{C}(v_1) = \{0\}$; $\overline{C}(v_2) = \{1\}$;$C(u_i) = \{i-1, i, i+1\} \ \ (i = 1, 2, \cdots, m-1)$; $C(u_m) = \{0, m-1, m+1\}$.

Obviously, $\forall u, v \in V(S_m \bigvee K_{1,1})$, if $u \neq v$, we have $C(u) \neq C(v)$. So, f is a $(m+3)-VDEC$ of $S_m \bigvee K_{1,1}$.

From all of above, Theorem 2.2 is true. □

Theorem 2.3. *For star S_m with order $m+1$ and complete balanced bipartite graph $K_{n,n}$, when $m = 1, n \geq 2$, then $\chi'_{vd}(S_1 \bigvee K_{n,n}) = 2n+2$.*

Proof. Suppose $V(S_1) = \{w\} \cup \{u_1\}, V(K_{n,n}) = \{v_1, v_2, \cdots, v_n\} \cup \{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}$. We can work out:

$$\mu(S_1 \bigvee K_{n,n}) = \max\{\min\{\lambda| \begin{pmatrix} \lambda \\ 2n+1 \end{pmatrix} \geq 2\},$$

$$\min\{\lambda| \begin{pmatrix} \lambda \\ n+2 \end{pmatrix} \geq 2n\}\} = 2n+2 .$$

So, we need only to prove that $S_1 \bigvee K_{n,n}$ has a $(2n+2)-VDEC$.

Case 1. When $n = 2$, we need only to prove that $S_1 \bigvee K_{2,2}$ has a $6-VDEC$. Let f be:

$f(wv_i) = i \ (i = 1, 2, 3, 4); f(u_1v_i) = i+1 \ (i = 1, 2, 3, 4);$

$f(wu_1) = 0; f(v_1v_3) = 5; f(v_1v_4) = 0; f(v_2v_3) = 0; f(v_2v_4) = 1.$

Such that, $\overline{C}(w) = \{5\}$; $\overline{C}(u_1) = \{1\}$; $\overline{C}(v_1) = \{3, 4\}$; $\overline{C}(v_2) = \{4, 5\}$; $\overline{C}(v_3) = \{1, 2\}$; $\overline{C}(v_4) = \{2, 3\}$.

Obviously, f is a $6-VDEC$ of $S_1 \bigvee K_{2,2}$, the conclusion is true.

Case 2. When $n \geq 3$, let f be:

$$f(wv_i) = i \ \ (i = 1, 2, \cdots, 2n);\ \ f(u_1v_i) = i+1 \ \ (i = 1, 2, \cdots, 2n);\ \ f(wu_1) = 0;$$

$f(v_iv_j) = i+j-n+1 \ (i = 1, 2, \cdots, n-2; \ \ j = n+1, n+2, \cdots, 2n)$; $f(v_iv_j) = j+2 \ (i = n-1; \ \ j = n+1, n+2, \cdots, 2n)$; $f(v_iv_j) = j+4 \ (i = n; \ \ j = n+1, n+2, \cdots, 2n)$.

Such that, $\overline{C}(w) = \{2n+1\}$; $\overline{C}(u_1) = \{1\}$; $\overline{C}(v_i) = \{i+n+2, i+n+3, \cdots, i+2n+1\}$ in take $\bmod(2n+2) \ \ (i = 1, 2, \cdots, n-2)$; $\overline{C}(v_{n-1}) = \{n+$

$1, n+2, 2n+3, 2n+4, \cdots, 3n\}$ in take mod$(2n+2)$; $\overline{C}(v_n) = \{n+2, n+3, n+4, 2n+5 \cdots, 3n+1\}$ in take mod$(2n+2)$; $\overline{C}(v_i) = \{i+3, i+5, i+6, \cdots, i+n+3\}$ in take mod$(2n+2)$ $(i = n+1, n+2, \cdots, 2n)$.

Obviously, $\forall u, v \in V(S_1 \bigvee K_{n,n})$, if $u \neq v$, we have $C(u) \neq C(v)$

So f is a $(2n+2) - VDEC$ of $S_1 \bigvee K_{n,n}$.

From all of above, Theorem 2.3 is true.

Theorem 2.4. For star S_m with order $m+1$ and complete balanced bipartite graph $K_{n,n}$, when $m \geq 3, n \geq 2$, then $\chi'_{vd}(S_m \bigvee K_{n,n}) = m + 2n$.

Proof. Suppose

$$V(S_m) = \{w\} \cup \{u_1, u_2, \cdots, u_m\},$$

$$V(K_{n,n}) = \{v_1, v_2, \cdots, v_n\} \cup \{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}.$$

We can work out:

$$\mu(S_m \bigvee K_{n,n}) = \max\{\min\{\lambda | \binom{\lambda}{m+2n} \geq 1\}, \min\{\lambda | \binom{\lambda}{2n+1} \geq m\},$$

$$\min\{\lambda | \binom{\lambda}{m+n+1} \geq 2n\}\} = m + 2n.$$

So we need only to prove $S_m \bigvee K_{n,n}$ has a $(m+2n) - VDEC$.

Case 1. When $m > n \geq 2$,

Case 1.1. When $m = 3, n = 2$, we need only to prove $S_3 \bigvee K_{2,2}$ has a $7 - VDEC$

Suppose $C = \{1, 2, \cdots, 7\}$, $\overline{C}(u) = C \setminus C(u)$, let f be:

$f(wu_i) = i$ $(i = 1, 2, 3)$; $f(wv_i) = i+3$ $(i = 1, 2, 3, 4)$; $f(u_1v_1) = 5$; $f(u_1v_2) = 6$; $f(u_1v_3) = 7$; $f(u_1v_4) = 3$; $f(u_2v_1) = 1$; $f(u_2v_2) = 3$; $f(u_2v_3) = 4$; $f(u_2v_4) = 5$; $f(u_3v_1) = 2$; $f(u_3v_2) = 7$; $f(u_3v_3) = 5$; $f(u_3v_4) = 4$; $f(v_1v_3) = 3$; $f(v_1v_4) = 6$; $f(v_2v_3) = 1$; $f(v_2v_4) = 2$.

Such that, $\overline{C}(w) = \phi$; $\overline{C}(u_1) = \{2, 4\}$; $\overline{C}(u_2) = \{6, 7\}$; $\overline{C}(u_3) = \{1, 6\}$, $\overline{C}(v_1) = \{7\}$; $\overline{C}(v_2) = \{4\}$; $\overline{C}(v_3) = \{2\}$; $\overline{C}(v_4) = \{1\}$.

Obviously, f is a $7 - VDEC$ of $S_3 \bigvee K_{2,2}$, the conclusion is true.

Case 1.2. When $m - n = 1, m > 3$, we need only to prove $S_{n+1} \bigvee K_{n,n}$ has a $(3n+1) - VDEC$, suppose $C = \{1, 2, ..., 3n, 0\}$, $\overline{C}(u) = C \setminus C(u)$, let f be:

$f(wv_i) = i - 1$ $(i = 1, 2, \cdots, 2n)$; $f(wu_i) = 2n + i - 1$ $(i = 1, 2, \cdots, n+1)$; $f(u_iv_j) = 2n + i + j - 1(\text{mod } 3n+1)(i = 1, 2, \cdots, n; j = 1, 2, \cdots, 2n)$; $f(v_iv_j) = i + j - 1$ $(i = 1, 2, \cdots, n;$ $j = n+1, n+2, \cdots, 2n)$; $f(u_{n+1}v_j) = n + j - 1$ $(j = 1, 2, \cdots, n)$; $f(u_{n+1}v_j) = n + j$ $(j = n+1, n+2, \cdots, 2n-1)$; $f(u_{n+1}v_{2n}) = 0$.

Such that, $\overline{C}(w) = \phi$;$\overline{C}(u_i) = \{i+n-1, i+n, \cdots, i+2n-2\}$ $(i = 1, 2, \cdots, n)$;$\overline{C}(u_{n+1}) = \{1, 2, \cdots, n-1, 2n\}$ $\overline{C}(v_i) = \{i, i+1, \cdots, i+n-2\}$ $(i = 1, 2, \cdots, n)$; $\overline{C}(v_i) = \{n+i+1, n+i+2, \cdots, 2n+i-1\}$*in take* mod$(3n+1)$ $(i = n+1, n+2, \cdots, 2n-1)$;$\overline{C}(v_{2n}) = \{1, 2, \cdots, n-2, 3n\}$.

Obviously, $\forall u, v \in V(S_{n+1} \bigvee K_{n,n})$, if $u \neq v$, we have $C(u) \neq C(v)$

So f is a $(3n+1)-VDEC$ of $S_{n+1} \bigvee K_{n,n}$, the conclusion is true.

Case 1.3. When $m - n \geq 2$, suppose $C = \{1, 2, ..., m+2n-1, 0\}$, $\overline{C}(u) = C \setminus C(u)$, let f be:
$f(wv_i) = i-1 (i = 1, 2, \cdots, 2n)$; $f(wu_i) = 2n+i-1 (i = 1, 2, \cdots, m)$; $f(u_iv_j) = 2n+i+j-1(\text{mod } m+2n)(i = 1, 2, \cdots, m-1; j = 1, 2, \cdots, 2n)$; $f(u_mv_j) = n+j-1(j = 1, 2, \cdots, n)$; $f(u_mv_j) = n+j(j = n+1, n+2, \cdots, 2n)$; $f(v_iv_j) = i+j-1$ $(i = 1, 2, \cdots, n;\ j = n+1, n+2, \cdots, 2n)$.

Such that: $\overline{C}(w) = \phi$;$\overline{C}(u_i) = \{i+2n-m, i+2n-m+1, \cdots, i+2n-2\}(i = 1, 2, \cdots, m-1)$; $C(u_m) = \{n, n+1, \cdots, 2n-1, 2n+1, 2n+2, \cdots, 3n, m+2n-1\}$ $\overline{C}(v_i) = \{i, i+1, \cdots, i+n-2\}(i = 1, 2, \cdots, n)$; $\overline{C}(v_i) = \{n+i+1, n+i+2, \cdots, 2n+i-1\}$*in take* mod$(m+2n)(i = n+1, n+2, \cdots, 2n)$.

Obviously, $\forall u, v \in V(S_m \bigvee K_{n,n})$, if $u \neq v$, we have $C(u) \neq C(v)$

So f is a $(m+2n)-VDEC$ of $S_m \bigvee K_{n,n}$, the conclusion is true.

Case 2. When $m = n \geq 3$, we can work out:

$$\mu(S_m \bigvee K_{m,m}) = \max\{\min\{\lambda| \begin{pmatrix} \lambda \\ 3m \end{pmatrix} \geq 1\}, \min\{\lambda| \begin{pmatrix} \lambda \\ 2m+1 \end{pmatrix} \geq 3m\}\} = 3m.$$

So we need only to prove $S_m \bigvee K_{m,m}$ has a $3m-VDEC$.

Suppose $V(S_m) = \{w\} \cup \{u_1, u_2, \cdots, u_m\}$, $V(K_{m,m}) = \{v_1, v_2, \cdots, v_m\} \cup \{v_{m+1}, v_{m+2}, \cdots, v_{2m}\}$, $C = \{1, 2, ..., 3m-1, 0\}$, $\overline{C}(u) = C \setminus C(u)$, let f be:

$f(wv_i) = i-1$ $(i = 1, 2, \cdots, 2m)$; $f(wu_i) = 2m+i-1$ $(i = 1, 2, \cdots, m)$; $f(v_iv_j) = i+j-1$ $(i = 1, 2, \cdots, m;$ $j = m+1, m+2, \cdots, 2m)$; $f(u_iv_j) = 2m+i+j-1(\text{mod } 3m)(i = 1, 2, \cdots, m-1; j = 1, 2, \cdots, 2m)$; $f(u_mv_j) = m+j-1$ $(j = 1, 2, \cdots, m)$; $f(u_mv_j) = j-m-1$ $(j = m+1, m+2, \cdots, 2m)$.

Such that: $\overline{C}(w) = \phi$;$C(u_i) = \{i+m, i+m+1, \cdots, i+2m-2\}$ $(i = 1, 2, \cdots, m)$; $\overline{C}(v_i) = \{i, i+1, \cdots, m+i-2\}$ $(i = 1, 2, \cdots, m)$.$\overline{C}(v_i) = \{m+i, m+i+1, \cdots, 2m+i-2\}$ *in take* mod$(3m)(i = m+1, m+2, \cdots, 2m)$.

Obviously, $\forall u, v \in V(S_m \bigvee K_{m,m})$, if $u \neq v$, we have $C(u) \neq C(v)$

So f is a $3m-VDEC$ of $S_m \bigvee K_{m,m}$, the conclusion is true.

Case 3. When $n > m \geq 3$,

suppose $C = \{1, 2, ..., m+2n-1, 0\}$, $\overline{C}(u) = C \setminus C(u)$, let f be:

$f(wv_i) = i-1 \quad (i = 1,2,\cdots,2n)$; $f(wu_i) = 2n+i-1 \quad (i = 1,2,\cdots,m)$; $f(u_iv_j) = 2n+i+j-1(\text{mod } m+2n)(i = 1,2,\cdots,m-1;\ j = 1,2,\cdots,2n)$; $f(u_mv_j) = n+j-1\ (j = 1,2,\cdots,n)$; $f(u_mv_j) = j-n-1\ (j = n+1,n+2,\cdots,2n)$; $f(v_iv_j) = i+j-1\ (i = 1,2,\cdots,n;\ \ j = n+1,n+2,\cdots,2n)$.

Such that: $\overline{C}(w) = \phi$;$C(u_i) = \{2n-m+i, 2n-m+i+1,\cdots,i+2n-2\}$ $(i = 1,2,\cdots,m)$; $\overline{C}(v_i) = \{i, i+1,\cdots,n+i-2\}$ $(i = 1,2,\cdots,n)$.$\overline{C}(v_i) = \{i-n, i-n+1, n+i, n+i+1,\cdots,2n+i-4\}$ *in take* $\text{mod}(m+2n)(i = n+1, n+2,\cdots,2n)$.

Obviously, $\forall u,v \in V(S_m \bigvee K_{m,m})$, if $u \neq v$, we have $C(u) \neq C(v)$

So f is a $(m+2n)-VDEC$ of $S_m \bigvee K_{n,n}$, the conclusion is true.

From all of above, Theorem 2.4 is true.

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