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REACTION-DIFFUSION-CONVECTION PROBLEMS WITH NON FREDHOLM OPERATORS

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Abstract: The paper is devoted to a reaction-diffusion-convection problem posed in an infinite strip. Problems of this type describe flame propagation with natural convection. If the Lewis number is different from 1, then the corresponding elliptic operator does not satisfy the Fredholm property, and the conventional methods of nonlinear analysis are not applicable. We reduce the system to an integro-differential system of equations and prove the existence of reaction-diffusion-convection waves for Lewis number close to 1.

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1. Introduction

In this work we study the propagation of convective chemical waves in a two-dimensional strip. We consider the following reaction-diffusion system coupled with the Navier-Stokes equations:

$$\frac{\partial \theta}{\partial t} - \Delta\theta + \mathbf{v}.\nabla\theta - \kappa(\theta, Y) = 0, \tag{1.1}$$

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$$\frac{\partial Y}{\partial t} - \Lambda \Delta Y + \mathbf{v}.\nabla Y + \kappa(\theta, Y) = 0, \tag{1.2}$$

$$\frac{\partial \mathbf{v}}{\partial t} - P\Delta \mathbf{v} + (\mathbf{v}.\nabla)\mathbf{v} + \nabla p - PR(\theta - \theta_0^*)\tau = 0, \tag{1.3}$$

$$\nabla.\mathbf{v} = 0. \tag{1.4}$$

Here θ is the dimensionless temperature, Y the concentration of the reactant, $\mathbf{v} = (v_1, v_2)$ the velocity of the medium, p the pressure. The equations (1.1), (1.2) describe the propagation of a premixed flame in the case of an order one reaction; the function $\kappa(\theta, Y)$ corresponds to the Arrhenius reaction term. The Navier-Stokes equations are written under the so-called Boussinesq approximation where the medium is considered as incompressible and the density is everywhere constant except for the buoyancy term, which describes the action of the gravity and appears as a result of linearization of the density. This term involves some characteristic temperature θ_0^* and $\tau = (\tau_1, \tau_2)$, a unit vector in $\mathbb{R}^2$ representing the orientation of the gravity force. Finally, the system depends on dimensionless parameters that are the inverse of the Lewis number Λ, the Prandtl number P and the Rayleigh number R.

System (1.1)-(1.4) will be considered in an infinite strip Ω with the axis forming a given angle with the vertical direction. For convenience we will suppose that the domain is fixed,

$$\Omega = \big\{(x, y) \in \mathbb{R}^2, \ \ y \in (0, 1)\big\},$$

while the vector τ can be variable. Equations (1.1)-(1.4) are supplemented with the following boundary conditions for the temperature and for the concentration

$$\frac{\partial \theta}{\partial y} = \frac{\partial Y}{\partial y} = 0 \text{ on } \partial\Omega, \tag{1.5}$$

and with the free surface boundary condition for the velocity

$$\frac{\partial v_1}{\partial y} = 0, \ \ v_2 = 0 \text{ on } \partial\Omega. \tag{1.6}$$

In order to study this problem, it is convenient to rewrite the Navier-Stokes equations in the stream function-vorticity formulation. We introduce the stream function ψ defined by:

$$\mathbf{v} = \operatorname{curl}\psi = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right), \ \ \psi = 0 \text{ on } \partial\Omega \tag{1.7}$$

and the vorticity

$$\omega = -\Delta\psi. \tag{1.8}$$

Then equations (1.3), (1.4) can be rewriten as

$$\frac{\partial \omega}{\partial t} - P\Delta\omega + \operatorname{curl}\psi.\nabla\omega - PR\operatorname{curl}\theta.\tau = 0, \tag{1.9}$$

$$\Delta\psi + \omega = 0, \tag{1.10}$$

while the boundary condition (1.6) provides

$$\omega = 0 \text{ on } \partial\Omega. \tag{1.11}$$

As already mentioned, the vector τ in (1.3) determines the orientation of the gravity force with respect to the strip. If $\tau = (1, 0)$, it is directed along the axis of the strip. If $\tau = (0, 1)$, it is oriented vertically and, consequently, perpendicular to the strip. These two cases appear to be essentially different. In the first case, the experimental results for polymerization fronts [5], [10], formal asymptotic expansions [8], and rigorous mathematical analysis for $\Lambda = 1$, [15], [12], [13] show that for small Rayleigh numbers the vertically propagating reaction-diffusion wave is stable, and convection does not appear. For sufficiently large values of the Rayleigh number it loses its stability, and a reaction-diffusion-convection (RDC) wave appears due to a bifurcation of change of stability where a real eigenvalue of the linearized problem crosses the origin. For all other directions of the gravity force, including the case where it is perpendicular to the strip, propagation of waves is accompanied by convection even for small Rayleigh numbers. Existence of RDC waves in this case is proved in [3] for $\Lambda = 1$. The corresponding experimental results are described in [1]. Existence of solutions of the evolution problem and some of their properties are studied in [9, 11], stability of RDC waves is discussed in [6].

In this work we study the existence of RDC waves in the case where $\Lambda \neq 1$. We will see below that this case is principally different in comparison with $\Lambda = 1$. The main question here is whether RDC waves are structurally stable, that is whether they persist under small change of parameters of the problem. From the mathematical point of view, structural stability can be justified by the Implicit Function Theorem. However it cannot be directly applied for the case $\Lambda \neq 1$ because the corresponding operators do not satisfy the Fredholm property, and the solvability conditions for them are not known. Structural stability of reaction-diffusion waves without convection is proved in [7] by the introduction of specially constructed integro-differential problems. Here we will develop this approach for RDC waves.

Our aim here is to study travelling wave solutions of problem (1.1)-(1.6), that is solutions of the form

$$\theta(t, x, y) = \tilde{\theta}(x + ct, y), \quad Y(t, x, y) = \tilde{Y}(x + ct, y),$$

$$\omega(t,x,y) = \tilde{\omega}(x+ct,y), \quad \psi(t,x,y) = \tilde{\psi}(x+ct,y).$$

The parameter c is the wave speed. This is an unknown real number which will be found together with the unknown functions $\tilde{\theta}$, $\tilde{Y}$, $\tilde{\omega}$ and $\tilde{\psi}$. Then system of equations (1.1)-(1.6) reduces to the following elliptic system (where the tildes are omitted):

$$-\Delta\theta + c\frac{\partial\theta}{\partial x} + \mathrm{curl}\,\psi.\nabla\theta - \kappa(\theta,Y) = 0, \tag{1.12}$$

$$-\Lambda\Delta Y + c\frac{\partial Y}{\partial x} + \mathrm{curl}\,\psi.\nabla Y + \kappa(\theta,Y) = 0, \tag{1.13}$$

$$-P\Delta\omega + c\frac{\partial\omega}{\partial x} + \mathrm{curl}\,\psi.\nabla\omega - PR\mathrm{curl}\,\theta.\tau = 0, \tag{1.14}$$

$$\Delta\psi + \omega = 0. \tag{1.15}$$

This system is completed by the boundary conditions

$$\frac{\partial\theta}{\partial y} = \frac{\partial Y}{\partial y} = \omega = \psi = 0 \text{ on } \partial\Omega, \tag{1.16}$$

together with the following limits as $x \to \pm\infty$

$$\begin{aligned} &\theta(-\infty,y)=0, \quad Y(-\infty,y)=1, \quad \omega(-\infty,y)=0, \quad \psi(-\infty,y)=0; \\ &\theta(+\infty,y)=1, \quad Y(+\infty,y)=0, \quad \omega(+\infty,y)=0, \quad \psi(+\infty,y)=0. \end{aligned} \tag{1.17}$$

Let us specify our assumptions on the function $\kappa(\theta,Y)$. We assume that κ is of class C^2 with

$$\kappa(0,1) = \kappa(1,0) = 0, \tag{1.18}$$

and that the function

$$K(\theta) = \kappa(\theta, 1-\theta) \tag{1.19}$$

is bistable, that is there exists $\theta^* \in (0,1)$ such that

$$K(\theta) < 0 \text{ on } (0,\theta^*) \text{ and } K(\theta) > 0 \text{ on } (\theta^*,1), \tag{1.20}$$

and satisfies furthermore

$$K'(0) < 0, \quad K'(1) < 0. \tag{1.21}$$

Note that the function $\kappa(\theta,Y)$ arising in chemical kinetics is

$$\kappa(\theta,Y) = \kappa_0(\theta)Y, \text{ with } \kappa_0(\theta) \approx ke^{Z(\theta-1)}, \tag{1.22}$$

where Z is the Zeldovich number and k is a constant. Then conditions (1.18)-(1.21) consist in assuming that $\kappa_0(0) = 0$, $\kappa_0(\theta) < 0$ for $\theta \in (0,\theta^*)$, and $\kappa_0(\theta) > 0$ for $\theta > \theta^*$ instead of the standard cut-off procedure where $\kappa_0(\theta)$ is identically equal 0 on some interval $0 \le \theta \le \theta^*$, $\theta^* \in (0,1)$. Such a change is not essential from the point of view of applications to combustion waves.

In this work we will prove existence of solutions of problem (1.12)-(1.17). The proof is based on the Implicit Function Theorem. However this theorem cannot be directly applied since the Fredholm property does not hold for the operator associated to system (1.12)-(1.17), [19], [7].

The main idea of the approach developed in this work is to reformulate the system in an integro-differential form. This approach is suggested in [7] for reaction-diffusion problems without convection. The integro-differential formulation will satisfy the Fredholm property. So that solvability conditions are applicable and the Implicit Function Theorem can be used in order to prove the persistence of solutions under small perturbations.

The integro-differential system relies on the introduction of the function

$$H = \theta + Y - 1, \tag{1.23}$$

that satisfies the equation

$$-\Delta H + c\frac{\partial H}{\partial x} + \operatorname{curl}\psi.\nabla H = (1-\Lambda)\Delta\theta, \tag{1.24}$$

together with the boundary condition

$$\frac{\partial H}{\partial y} = 0 \text{ on } \partial\Omega, \tag{1.25}$$

and the limits at infinity

$$H(\pm\infty, y) = 0. \tag{1.26}$$

We will solve problem (1.24)-(1.26) and consider H as a function of θ (and possibly ψ, c and Λ).

We first investigate existence of solutions for Λ close to 1 and R close to 0. System (1.12)-(1.17) greatly simplifies when $R = 0$ since the equations (1.14), (1.15) are uncoupled from (1.12), (1.13). Then the conditions (1.16), (1.17) guarantee that $\omega = \psi = 0$. Besides, note that for $\Lambda = 1$, the solution (θ, Y) of (1.12), (1.13) satisfies $Y = 1 - \theta$, so that the two equations simplify in a single one. Therefore for $R = 0$ and $\Lambda = 1$ problem (1.12)-(1.17) reduces to the scalar equation

$$\begin{aligned} &-\Delta\theta + c\frac{\partial\theta}{\partial x} - \kappa(\theta, 1-\theta) = 0, \\ &\frac{\partial\theta}{\partial y} = 0 \text{ on } \partial\Omega, \\ &\theta(-\infty, y) = 0, \quad \theta(+\infty, y) = 1. \end{aligned} \tag{1.27}$$

Existence results for this equation are well known [4]. The operator associated to our integro-differential formulation satisfies the Fredholm property for $\Lambda = 1$ and $R = 0$. This allows us to prove the following result.

Theorem 1.1. *Assume that conditions (1.18)-(1.21) hold and that* $\int_0^1 K(s)ds \neq 0$. *Then problem (1.12)-(1.17) has a solution* $(\theta, Y, \omega, \psi, c)$ *for* Λ *sufficiently close to one and for sufficiently small* R. *This solution is continuous in* $C^1_{loc}(\bar{\Omega})^4 \times \mathbb{R}$ *with respect to* (Λ, R).

Next we study structural stability of the solutions of problem (1.12)-(1.17) for $\Lambda = 1$ and some $R_0 \neq 0$. For $\Lambda = 1$, as previously mentioned, the equations (1.12), (1.13) simplify to a single one ($Y = 1 - \theta$). Suppose now that equation (1.12)-(1.17) has a solution $(\theta_0, Y_0, \psi_0, \omega_0, c_0)$ for $\Lambda = 1$ and $R = R_0$. Then we have $Y_0 = 1 - \theta_0$ and $\omega_0 = -\Delta\psi_0$. In order to derive the structural stability of this solution, we will suppose that the system

$$-\Delta u + c_0 \frac{\partial u}{\partial x} + \operatorname{curl} \psi_0 . \nabla u + \operatorname{curl} \psi . \nabla \theta_0 - K'(\theta_0) u = \frac{\partial \theta_0}{\partial x}, \tag{1.28}$$

$$P\Delta^2 \psi - c_0 \frac{\partial \Delta\psi}{\partial x} - \operatorname{curl} \psi . \nabla \Delta \psi_0 - \operatorname{curl} \psi_0 . \nabla \Delta \psi - PR_0 \operatorname{curl} u . \tau = -\frac{\partial \Delta \psi_0}{\partial x} \tag{1.29}$$

with the corresponding boundary conditions, does not have bounded solutions.

This condition is related to the simplicity of some zero eigenvalue for the linearized operator at $(\theta_0, Y_0, \psi_0, \omega, c_0)$ corresponding to the system

$$-\Delta\theta + c\frac{\partial \theta}{\partial x} + \operatorname{curl} \psi . \nabla \theta - \kappa(\theta, 1 - \theta) = 0, \tag{1.30}$$

$$-P\Delta\omega + c\frac{\partial \omega}{\partial x} + \operatorname{curl} \psi . \nabla \omega - PR \operatorname{curl} \theta . \tau = 0, \tag{1.31}$$

together with (1.16) and under the constraint (1.15) (see Remark 5.2).

Theorem 1.2. *Under assumptions (1.18)-(1.21) suppose that problem (1.12)-(1.17) has a solution* $(\theta_0, Y_0, \omega_0, \psi_0, c_0)$ *with* $c_0 \neq 0$ *for* $\Lambda = 1$ *and* R_0. *We also suppose that system (1.28), (1.29) has no bounded solution. Then problem (1.12)-(1.17) has a solution* $(\theta, Y, \omega, \psi, c)$ *for* (Λ, R) *sufficiently close to* $(1, R_0)$.

The paper is organized as follows. We first find solvability conditions for a linear problem including (1.24)-(1.26) (Section 2). Next we rewrite system (1.12)-(1.17) in an integro-differential form (Section 3) which is used in order to obtain existence results. Next we prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5.

2. Preliminary Results

The aim of this section is to study problem (1.24)-(1.26). We first obtain solvability conditions. Then we study some regularity properties of the inverse operator.

2.1. Solvability Condition

In this section we aim to consider a class of linear equations including (1.24)-(1.26), that takes the form

$$\begin{aligned} & -\Delta H + c\frac{\partial H}{\partial x} + \operatorname{curl}\psi.\nabla H = g, \\ & \frac{\partial H}{\partial y} = 0 \text{ on } \partial\Omega, \\ & H(\pm\infty, y) = 0, \end{aligned} \tag{2.1}$$

where the real c and the function ψ are given.

For that purpose we consider the Banach spaces

$$V = \{f = (f_1, f_2) \in L^2(\Omega)^2, \nabla.f \in L^2(\Omega) \text{ and } f_2 = 0 \text{ on } \partial\Omega\}, \tag{2.2}$$

$$E = \Big\{H \in H^2(\Omega),\ \frac{\partial H}{\partial y} = 0 \text{ on } \partial\Omega\Big\}. \tag{2.3}$$

Then the following existence result holds.

Proposition 2.1. *Let $c > 0$ and let $\psi \in H^3(\Omega)$ with $\psi = 0$ on $\partial\Omega$ be given. Then system (2.1) is solvable in E if and only if $g = \nabla.f$ for some $f \in V$.*

Proof. First note that the following identity holds for any functions $H \in H^2(\Omega)$

$$\operatorname{curl}\psi.\nabla H = \nabla.(H\operatorname{curl}\psi) = -\nabla.(\psi\operatorname{curl} H). \tag{2.4}$$

This identity will be essential in the sequel.

Let us suppose that system (2.1) is solvable in E. Then, due to (2.4) we have

$$g = \nabla.f \text{ with } f = -\nabla H + c(H, 0) - \psi\operatorname{curl} H. \tag{2.5}$$

Now note that f belongs to V since $H \in E$ and $\psi = 0$ on $\partial\Omega$. This shows the necessary condition in Proposition 2.1.

Next assume that $f \in V$ is given and consider the linear equation

$$-\Delta H + c\frac{\partial H}{\partial x} + \operatorname{curl}\psi.\nabla H = \nabla.f, \tag{2.6}$$

together with the conditions

$$\frac{\partial H}{\partial y} = 0 \text{ on } \partial\Omega \text{ and } H(\pm\infty, y) = 0. \tag{2.7}$$

The existence of solution is proved in the lemma below and completes the proof of Proposition 2.1. □

Lemma 2.2. *Let $c > 0$ and let $\psi \in H^3(\Omega)$ with $\psi = 0$ on $\partial\Omega$ be given. Let $f \in V$. Then problem (2.6)-(2.7) has a unique solution H in $H^2(\Omega)$ which satisfies the estimates*

$$\|H\|_{H^1(\Omega)} \leq M(c)(1 + \|\psi\|_{H^2(\Omega)})\|f\|_{L^2(\Omega)}, \tag{2.8}$$

$$\|H\|_{H^2(\Omega)} \leq M(c)(1 + \|\psi\|_{H^3(\Omega)})\|f\|_V, \tag{2.9}$$

where $M(c)$ is a function locally bounded on $(0, +\infty)$.

Proof. The existence of a solution of problem (2.6), (2.7) is proved by introducing the following problem in a bounded domain $R_a = (-a, a) \times (0, 1)$ for $a > 0$:

$$-\Delta H + c\frac{\partial H}{\partial x} + \operatorname{curl}\psi.\nabla H = \nabla.f \text{ in } R_a, \tag{2.10}$$

together with the boundary conditions

$$\frac{\partial H}{\partial y}(x, 0) = \frac{\partial H}{\partial y}(x, 1) = 0, \tag{2.11}$$

$$-\frac{\partial H}{\partial x}(-a, y) + cH(-a, y) - \psi(-a, y)\frac{\partial H}{\partial y}(-a, y) = f_1(-a, y), \tag{2.12}$$

$$H(a, y) = 0 \text{ for } y \in (0, 1).$$

It is easy to check the existence of a solution for this problem posed in a bounded domain. Next computations similar to the ones below provide estimates on the solution of (2.10)-(2.12) that are independent of the length of the rectangle. They allow to take the limit $a \to +\infty$ and to obtain the existence of a solution of (2.6), (2.7). The details are left to the reader. We only prove here estimates (2.8), (2.9) for the problem in Ω.

For that purpose we introduce the eigenfunctions ϕ_n and eigenvalues λ_n of the operator $-u''$ in $L^2(0, 1)$ associated with the homogeneous Neumann condition, that is

$$\begin{aligned} &-\phi_n'' = \lambda_n\phi_n \text{ in } (0, 1),\ \phi_n'(0) = \phi_n'(1) = 0, \\ &0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ..., \quad \lambda_n \to +\infty \text{ for } n \to +\infty. \end{aligned} \tag{2.13}$$

For $n \geq 0$, we set

$$h_n = (H, \phi_n)_{L^2(0,1)}. \tag{2.14}$$

Multiplying equation (2.6) by H and integrating over $(0,1)$, we obtain

$$\sum_{n=1}^{+\infty} \lambda_n h_n^2 - \int_0^1 \frac{\partial^2 H}{\partial x^2} H dy + \int_0^1 (\operatorname{curl} \psi . \nabla H + c \frac{\partial H}{\partial x}) H dy \\ = \int_0^1 \nabla . f H dy. \tag{2.15}$$

Integration of equation (2.15) with respect to $x \in \mathbb{R}$ provides that

$$\begin{aligned} &\iint_\Omega (\frac{\partial H}{\partial x})^2 dxdy + \sum_{n=1}^{+\infty} \lambda_n \|h_n\|^2_{L^2(\mathbb{R})} + \iint_\Omega c \frac{\partial H}{\partial x} H dxdy \\ &+ \iint_\Omega \operatorname{curl} \psi . \nabla H H dxdy = \iint_\Omega \nabla . f H dxdy. \end{aligned} \tag{2.16}$$

Now thanks to (2.4) we have

$$\begin{aligned} \iint_\Omega \operatorname{curl} \psi . \nabla H H dxdy &= - \iint_\Omega \nabla . (\psi \operatorname{curl} H) H dxdy \\ &= \iint_\Omega \psi \operatorname{curl} H . \nabla H dxdy = 0. \end{aligned} \tag{2.17}$$

Therefore (2.16) yields

$$\lambda_1 \sum_{n=1}^{+\infty} \|h_n\|^2_{L^2(\mathbb{R})} \leq \iint_\Omega \nabla . f H dxdy = - \iint_\Omega f . \nabla H dxdy. \tag{2.18}$$

On the other hand, multiplying again equation (2.6) by H and integrating over Ω, we see that

$$\iint_\Omega |\nabla H|^2 dxdy = \iint_\Omega \nabla . f H dxdy, \tag{2.19}$$

and the Cauchy-Schwarz inequality implies

$$\iint_\Omega |\nabla H|^2 dxdy \leq \iint_\Omega |f|^2 dxdy. \tag{2.20}$$

Then, due to (2.20), we infer from (2.18) that

$$\sum_{n=1}^{+\infty} \|h_n\|^2_{L^2(\mathbb{R})} \leq \frac{1}{\lambda_1} \|f\|^2_{L^2(\Omega)}. \tag{2.21}$$

Now we need to estimate the norm of the function h_0. It satisfies the equation

$$\begin{cases} -h_0'' + ch_0' = -\int_0^1 \operatorname{curl}\psi.\nabla H dy + (\int_0^1 f_1 dy)', \\ h_0(\pm\infty) = 0. \end{cases} \tag{2.22}$$

Here for all $x \in \mathbb{R}$, we have

$$\begin{aligned} &\int_{-\infty}^{x}\int_0^1 (\psi_x H_y - \psi_y H_x) dy d\xi \\ &\qquad = \int_0^1 \int_{-\infty}^{x} \psi_x H_y d\xi dy - \int_{-\infty}^{x}\int_0^1 \psi_y H_x dy d\xi \\ &\qquad = \int_0^1 \psi(x,y) H_y(x,y) dy - \int_0^1 \int_{-\infty}^{x} \psi(\xi,y) H_{x,y}(\xi,y) d\xi dy \\ &+ \int_{-\infty}^{x}\int_0^1 \psi(\xi,y) H_{x,y}(\xi,y) d\xi dy = \int_0^1 \psi(x,y) H_y(x,y) dy. \end{aligned} \tag{2.23}$$

Therefore setting $g_0(x) = \int_0^1 \psi(x,y)H_y(x,y)dy$ and $f_0(x) = \int_0^1 f_1 dy$, equation (2.22) rewrites as

$$\begin{cases} -h_0'' + ch_0' = g_0' + f_0', \\ h_0(\pm\infty) = 0. \end{cases} \tag{2.24}$$

Next integrating equation (2.24), we obtain the following expression for the function h_0

$$h_0 = (f_0 + g_0) \star (e^{cx}\mathbb{1}_{x\le 0}). \tag{2.25}$$

The Young formula for the convolution product provides the estimate

$$\begin{aligned} \|h_0\|_{L^2(\mathbb{R})} &\le \|f_0 + g_0\|_{L^2} \|e^{cx}\mathbb{1}_{x\le 0}\|_{L^1} \\ &\le \frac{1}{c}(\|f_1\|_{L^2(\Omega)} + \|\psi\|_\infty \|H_y\|_{L^2(\Omega)}). \end{aligned} \tag{2.26}$$

Since the sequence (ϕ_n) is an Hilbertian basis in $L^2(0,1)$, we have

$$\|H\|^2_{L^2(\Omega)} = \sum_{n=0}^{+\infty} \|h_n\|^2_{L^2(\mathbb{R})}. \tag{2.27}$$

Finally we infer from (2.20), (2.21) and (2.26) that

$$\|H\|_{H^1(\Omega)} \le M(c)(1 + \|\psi\|_\infty)\|f\|_{L^2(\Omega)}, \tag{2.28}$$

where M is a function locally bounded for $c > 0$.

The estimate in $H^2(\Omega)$ follows from equation (2.6). More precisely we have

$$\begin{aligned} \|\Delta H\|_{L^2(\Omega)} &\leq \|c\frac{\partial H}{\partial x} + \operatorname{curl}\psi.\nabla H - \nabla.f\|_{L^2(\Omega)} \\ &\leq (c + \|\psi\|_{W^{1,\infty}})\|H\|_{H^1} + \|f\|_V. \end{aligned} \tag{2.29}$$

Thanks to the two-dimensional embedding $H^3(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we obtain

$$\|H\|_{H^2} \leq M(c)(1 + \|\psi\|_{H^3})\|f\|_V. \tag{2.30}$$

Uniqueness of the solution easily follows from this inequality. Lemma 2.2 is proved. □

2.2. Regularity of the Inverse Operator

In this section we will consider the operator corresponding to the resolution of problem (2.6), (2.7) and study its regularity. For that purpose we introduce the Banach space

$$F = \left\{\psi \in H^3(\Omega),\ \psi = 0 \text{ on } \partial\Omega\right\}. \tag{2.31}$$

Recall that Proposition 2.1 allows us to solve system (2.1) for $c > 0$, $\psi \in F$ and $f \in V$. Therefore we can define the operator

$$\begin{aligned} \Phi :& V \times F \times (0, +\infty) \to E \\ &(f, \psi, c) \to H, \end{aligned} \tag{2.32}$$

where H is the solution of problem (2.6), (2.7).

Let us first prove the continuity of the operator Φ.

Lemma 2.3. *The operator Φ is continuous from $V \times F \times (0, +\infty)$ into $H^2(\Omega)$.*

Proof. The proof of this lemma is based on estimate (2.9) and identity (2.4). Let

$$(f_1, \psi_1, c_1) \in V \times F \times (0, +\infty), \quad (f_2, \psi_2, c_2) \in V \times F \times (0, +\infty)$$

be given. We set $H_1 = \Phi(f_1, \psi_1, c_1)$, $H_2 = \Phi(f_2, \psi_2, c_2)$ and $H = H_1 - H_2$. Then H satisfies the equation

$$-\Delta H + c_1\frac{\partial H}{\partial x} + \operatorname{curl}\psi_1.\nabla H = \nabla.(f_1 - f_2) + (c_2 - c_1)\frac{\partial H_2}{\partial x}$$

$$+ \nabla H_2.\mathrm{curl}\,(\psi_2 - \psi_1). \quad (2.33)$$

By virtue of (2.4), we have the identity

$$\nabla H_2.\mathrm{curl}\,(\psi_2 - \psi_1) = -\nabla.((\psi_2 - \psi_1)\mathrm{curl}\,H_2). \quad (2.34)$$

Note that $(\psi_2 - \psi_1)\mathrm{curl}\,H_2 \in V$ since the functions ψ_1 and ψ_2 vanish on $\partial\Omega$.

Using (2.34), we can write equation (2.33) in the form

$$\begin{aligned} &-\Delta H + c_1 \frac{\partial H}{\partial x} + \mathrm{curl}\,\psi_1.\nabla H \\ &\qquad = \nabla.\big(f_1 - f_2 + ((c_2 - c_1)H_2, 0) - (\psi_2 - \psi_1)\mathrm{curl}\,H_2\big), \quad (2.35)\end{aligned}$$

so that estimate (2.9) implies that

$$\begin{aligned} &\|H\|_{H^2} \le M(c_1)(1 + \|\psi_1\|_{H^3}) \\ &\qquad \times \big(\|f_1 - f_2\|_V + |c_1 - c_2|\|H_2\|_{H^1} + \|H_2\|_{H^2}\|\psi_1 - \psi_2\|_{H^3}\big). \quad (2.36)\end{aligned}$$

This estimate proves the continuity of the operator Φ. □

Next we will study some differential properties of the operator Φ.

Lemma 2.4. *The operator Φ is of class $\mathcal{C}^1$ from $V \times F \times (0, +\infty)$ into $H^2(\Omega)$. Its derivatives are given by the equalities*

$$\Phi'_f(f_0, \psi_0, c_0)f = \Phi(f, \psi_0, c_0)\,, \quad (2.37)$$

$$\Phi'_c(f_0, \psi_0, c_0) = -\Phi\Big(\big(\Phi(f_0, \psi_0, c_0), 0\big), \psi_0, c_0\Big), \quad (2.38)$$

$$\Phi'_\psi(f_0, \psi_0, c_0)\psi = \Phi\Big(\psi\mathrm{curl}\,\Phi(f_0, \psi_0, c_0), \psi_0, c_0\Big)\,. \quad (2.39)$$

Proof. We first note that $\Phi(f, \psi, c)$ depends linearly on f, so that its derivative with respect to f is given by (2.37).

Let $(f_0, \psi_0, c_0) \in V \times F \times (0, +\infty)$ and c be sufficiently small so that $c + c_0 > 0$. We set

$$H_0 = \Phi(f_0, \psi_0, c_0), \quad H = \Phi(f_0, \psi_0, c_0 + c), \quad \tilde{H} = H - H_0. \quad (2.40)$$

Clearly $\tilde{H}$ satisfies the equation

$$-\Delta\tilde{H} + c_0 \frac{\partial \tilde{H}}{\partial x} + \mathrm{curl}\,\psi_0.\nabla\tilde{H} = -c\frac{\partial H}{\partial x}, \quad (2.41)$$

and (2.8) yields that

$$\|\tilde{H}\|_{H^1} \le M(c_0)(1 + \|\psi_0\|_{H^3})c\|H\|_{L^2}. \quad (2.42)$$

Next we introduce the solution $\bar{H}$ of the equation

$$-\Delta \bar{H} + c_0 \frac{\partial \bar{H}}{\partial x} + \operatorname{curl} \psi_0 . \nabla \bar{H} = -c \frac{\partial H_0}{\partial x}. \tag{2.43}$$

Note that H is well defined by virtue of Lemma 2.2 since $(H_0, 0) \in V$. Then we have

$$-\Delta(\tilde{H} - \bar{H}) + c_0 \frac{\partial(\tilde{H} - \bar{H})}{\partial x} + \operatorname{curl} \psi_0 . \nabla(\tilde{H} - \bar{H}) = -c \frac{\partial \tilde{H}}{\partial x}, \tag{2.44}$$

and in view of (2.9) we obtain the estimate

$$\begin{aligned} \|\tilde{H} - \bar{H}\|_{H^2} &\leq M(c_0)(1 + \|\psi_0\|_{H^3}) c \|\tilde{H}\|_{H^1} \\ &\leq M(c_0)^2 (1 + \|\psi_0\|_{H^3})^2 c^2 \|H\|_{L^2}. \end{aligned} \tag{2.45}$$

This shows that Φ is differentiable with respect to c and $\Phi'_c(f_0, \psi_0, c_0).c = \bar{H}$.

Finally, let $(f_0, \psi_0, c_0) \in V \times E_2 \times (0, +\infty)$ and $\psi \in E_2$. We set

$$H_0 = \Phi(f_0, \psi_0, c_0), \quad H = \Phi(f_0, \psi_0 + \psi, c_0), \quad \tilde{H} = H - H_0. \tag{2.46}$$

Clearly $\tilde{H}$ satisfies the equation

$$-\Delta \tilde{H} + c_0 \frac{\partial \tilde{H}}{\partial x} + \operatorname{curl} \psi_0 . \nabla \tilde{H} = -\operatorname{curl} \psi . \nabla H = -\nabla . (\psi \operatorname{curl} H), \tag{2.47}$$

and (2.9) yields the estimate

$$\|\tilde{H}\|_{H^2} \leq M(c_0)(1 + \|\psi_0\|_{H^3}) \|\psi\|_{H^3} \|H\|_{H^1}. \tag{2.48}$$

We introduce the solution $\bar{H}$ of the equation

$$\Delta \bar{H} + c_0 \frac{\partial \bar{H}}{\partial x} + \operatorname{curl} \psi_0 . \nabla \bar{H} = \nabla . (\psi \operatorname{curl} H_0). \tag{2.49}$$

Note that Lemma 2.2 can be applied since $\psi \operatorname{curl} H_0 \in V$. We obtain

$$-\Delta(\tilde{H} - \bar{H}) + c_0 \frac{\partial(\tilde{H} - \bar{H})}{\partial x} + \operatorname{curl} \psi_0 . \nabla(\tilde{H} - \bar{H}) = \nabla . (\psi \operatorname{curl} \tilde{H}), \tag{2.50}$$

and in view of (2.9) we have:

$$\begin{aligned} \|\tilde{H} - \bar{H}\|_{H^2} &\leq M(c_0)(1 + \|\psi_0\|_{H^3}) \|\psi\|_{H^3} \|\tilde{H}\|_{H^2} \\ &\leq M(c_0)^2 (1 + \|\psi_0\|_{H^3})^2 \|\psi\|_{H^3}^2 \|H\|_{H^1}. \end{aligned} \tag{2.51}$$

This shows that Φ is differentiable with respect to ψ and $\Phi'_\psi(f_0, \psi_0, c_0).\psi = \bar{H}$.

It remains to show the continuity of the derivatives (2.37), (2.38) and (2.39). Let us start with operator (2.37). Let (f, ψ_1, c_1) and (f, ψ_2, c_2) be given. We set $W = \Phi(f, \psi_1, c_1) - \Phi(f, \psi_2, c_2)$. Then the function W satisfies the equation

$$\begin{aligned} &- \Delta W + c_1 \frac{\partial W}{\partial x} + \operatorname{curl} \psi_1 . \nabla W - (c_1 - c_2) \frac{\partial \Phi(f, \psi_2, c_2)}{\partial x} \\ &\qquad\qquad - \operatorname{curl}(\psi_1 - \psi_2) . \nabla \Phi(f, \psi_2, c_2) \\ &\quad = -\nabla . \Big((c_1 - c_2)(\Phi(f, \psi_2, c_2), 0) - (\psi_1 - \psi_2) \operatorname{curl} \Phi(f, \psi_2, c_2) \Big). \end{aligned} \tag{2.52}$$

From (2.9) we obtain

$$\begin{aligned} &\|W\|_{H^2} \le M(c_1)(1 + \|\psi_1\|_{H^3}) \\ &\qquad \times \|(c_1 - c_2)(\Phi(f, \psi_2, c_2), 0) - (\psi_1 - \psi_2) \operatorname{curl} \Phi(f, \psi_2, c_2)\|_V . \end{aligned} \tag{2.53}$$

The continuity of the operator $\Phi(., \psi, c)$ follows from this estimate. The continuity of the operators (2.38) and (2.39) can be proved in the same way. □

Remark 2.5. In this section we assumed that $c > 0$. If we change x by $-x$, we can obtain analogous results for $c < 0$. However the operator Φ is not defined for $c = 0$.

3. Reformulation of the Problem

In this section, we introduce an integro-differential operator in order to study problem (1.12)-(1.17).

We first rewrite system (1.12)-(1.17) using the variable $(\theta, H, \psi, \omega, c)$ where $H = \theta + Y - 1$. By adding (1.12) and (1.13) we easily obtain the system

$$-\Delta \theta + c \frac{\partial \theta}{\partial x} + \operatorname{curl} \psi . \nabla \theta - \kappa(\theta, 1 - \theta + H) = 0, \tag{3.1}$$

$$-\Delta H + c \frac{\partial H}{\partial x} + \operatorname{curl} \psi . \nabla H = (1 - \Lambda) \Delta \theta, \tag{3.2}$$

$$-P \Delta \omega + c \frac{\partial \omega}{\partial x} + \operatorname{curl} \psi . \nabla \omega - PR \operatorname{curl} \theta . \tau = 0, \tag{3.3}$$

$$\Delta \psi + \omega = 0, \tag{3.4}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial H}{\partial y} = \omega = \psi = 0 \text{ on } \partial \Omega, \tag{3.5}$$

$$\begin{aligned} &\theta(-\infty, y) = 0, \quad H(-\infty, y) = 0, \quad \omega(-\infty, y) = 0, \quad \psi(-\infty, y) = 0; \\ &\theta(+\infty, y) = 1, \quad H(+\infty, y) = 0, \quad \omega(+\infty, y) = 0, \quad \psi(+\infty, y) = 0. \end{aligned} \tag{3.6}$$

For the functional setting of this problem it is convenient to consider the spaces E and F defined by (2.3), (2.31) and to introduce the space

$$G = \Big\{ \omega \in H^2(\Omega),\ \omega = 0 \text{ on } \partial\Omega \Big\}. \tag{3.7}$$

In addition, to take care of the nonzero limits at infinity for the temperature θ we introduce a function $\phi : \mathbb{R} \to \mathbb{R}$ of class C^∞ such that

$$\phi(x) = 0 \ \text{ for } x < -1 \text{ and } \phi(x) = 1 \ \text{ for } x > 1, \tag{3.8}$$

and consider the new unknown $u = \theta - \phi$.

Now if $u \in E$, then H satisfies (2.6) where $f = (1-\Lambda)\nabla(u+\phi)$ belongs to V. Therefore using the operator introduced in Section 2.2, we have

$$H = \Phi\Big((1-\Lambda)\nabla(u+\phi), \psi, c\Big). \tag{3.9}$$

This leads us to consider H as a function of u, ψ, c and Λ and to set

$$H = \mathcal{H}(\Lambda, u, \psi, c) = \Phi\Big((1-\Lambda)\nabla(u+\phi), \psi, c\Big). \tag{3.10}$$

The operator $\mathcal{H}$ is acting from $(0,+\infty) \times E \times F \times (0,+\infty)$ into E.

Then, system (3.1)-(3.6) is equivalent to the following integro-differential formulation involving the variables $(u, \psi, \omega, c) \in E \times F \times G \times \mathbb{R}$

$$-\Delta\theta + c\frac{\partial\theta}{\partial x} + \operatorname{curl}\psi.\nabla\theta - \kappa\Big(\theta, 1-\theta+\mathcal{H}(\Lambda,u,\psi,c)\Big) = 0, \qquad \theta = u + \phi, \tag{3.11}$$

$$-P\Delta\omega + c\frac{\partial\omega}{\partial x} + \operatorname{curl}\psi.\nabla\omega - PR\operatorname{curl}\theta.\tau = 0, \tag{3.12}$$

$$\Delta\psi + \omega = 0. \tag{3.13}$$

Solutions of system (3.11)-(3.13) provide solutions of problem (1.12)-(1.17) by setting $Y = 1 - \theta + \mathcal{H}(\Lambda, u, \psi, c)$.

In what follows we will use this integro-differential system in order to prove the existence of solutions of problem (1.12)-(1.17).

4. Implicit Function Theorem for $R = 0$

The aim of this section is to show that problem (1.12)-(1.17) possesses a solution for Λ close to one and for R close to zero. More precisely we will prove the following existence result.

Theorem 4.1. *Suppose that assumptions (1.18)-(1.21) hold and that* $\int_0^1 K(s)ds \neq 0$. *Then there exists* $\epsilon > 0$ *such that problem (1.12)-(1.17) has a solution* $(\theta, Y, \psi, \omega, c)$ *for all values of* Λ *and* R *such that*

$$|\Lambda - 1| + |R| < \epsilon. \tag{4.1}$$

This solution is continuous with respect to (Λ, R) *in the topology of* $C^1_{loc}(\bar{\Omega})^4 \times \mathbb{R}$.

Proof. The proof of this theorem relies on the Implicit Function Theorem applied to the integro-differential system (3.11)-(3.13). Recall that for this system the unknowns are

$$(u, \psi, \omega, c) \in X = E \times F \times G \times \mathbb{R}. \tag{4.2}$$

The system also involves the fixed parameter $P \in (0, +\infty)$ and the two parameters

$$\Lambda \in (0, +\infty), \quad R \in \mathbb{R} \tag{4.3}$$

that will be allowed to vary.

For $\Lambda = 1$, the solution of (3.2) is $H \equiv 0$ so that we have

$$\mathcal{H}(1, u, \psi, c) \equiv 0. \tag{4.4}$$

Also for $R = 0$, the equations (3.12), (3.13) are uncoupled from (3.11) and in view of (3.6), we have $\psi = \omega = 0$.

In particular for $\Lambda = 1$ and $R = 0$, system (3.11)-(3.13) reduces to

$$-\Delta\theta + c\frac{\partial\theta}{\partial x} - \kappa(\theta, 1-\theta) = 0, \quad \theta = u + \phi, \tag{4.5}$$

$$\frac{\partial\theta}{\partial y} = 0 \ \text{ on } \partial\Omega, \tag{4.6}$$

$$\theta(-\infty, y) = 0, \quad \theta(+\infty, y) = 1. \tag{4.7}$$

It is known that under conditions (1.18)-(1.21), problem (4.5)-(4.7) has a unique solution $(\theta_0 = u_0 + \phi, c_0)$, see [4]. The function θ_0 is independent of y and monotone increasing with respect to x. Also the sign of c_0 is the one of $\int_0^1 K(s)ds$. For simplicity we will assume in the sequel that $c_0 > 0$.

Let us denote by

$$\mathcal{A} = \mathcal{A}(u, \omega, \psi, c, \Lambda, R) \tag{4.8}$$

the operator associated to the integro-differential system (3.11)-(3.13). This operator is acting from

$$X \times (0, +\infty) \times \mathbb{R}, \ X \text{ given by (4.2)}, \tag{4.9}$$

into

$$Y = L^2(\Omega) \times L^2(\Omega) \times H^1(\Omega). \tag{4.10}$$

Then the solutions of (3.11)-(3.13) are the zeros of $\mathcal{A}$. Also we have

$$\mathcal{A}\big(u_0, 0, 0, c_0, 1, 0\big) = 0. \tag{4.11}$$

We first derive some regularity result for this operator.

Lemma 4.2. *The operator $\mathcal{A} : X \times (0, +\infty) \times \mathbb{R} \to Y$ is of class C^1.*

Proof. The continuity of the operator $\mathcal{A}$ is obvious. The existence and the continuity of its Fréchet derivative follow from the embeddings $H^2 \to L^\infty$ and $H^3 \to W^{1,\infty}$ in the two dimensional case, and from the regularity of the nonlinear term κ. □

We aim to apply the Implicit Function Theorem to the operator $\mathcal{A}$ given by (4.8) in the neighborhood of the zero $(u_0, 0, 0, c_0, 1, 0)$. For that purpose it is necessary to investigate the linearized operator with respect to (u, ω, ψ, c) at the point $(u_0, 0, 0, c_0, 1, 0)$. Let us denote by A this operator.

Let us study the problem

$$A(u, \psi, \omega, c) = (f, g, h)\,,$$

where $(f, g, h) \in Y$ is given. In view of system (3.11)-(3.13) and property (4.4), this problem reads:

$$-\Delta u + c_0 \frac{\partial u}{\partial x} + c(u_0 + \phi)' + \psi_y (u_0 + \phi)' - K'(\theta_0) u = f, \tag{4.12}$$

$$-P\Delta\omega + c_0 \frac{\partial \omega}{\partial x} = g, \tag{4.13}$$

$$\Delta\psi + \omega = h, \tag{4.14}$$

together with the conditions

$$\begin{aligned} &\frac{\partial u}{\partial y} = \psi = \omega = 0 \text{ on } \partial\Omega, \\ &u(\pm\infty, y) = \psi(\pm\infty, y) = \omega(\pm\infty, y) = 0. \end{aligned} \tag{4.15}$$

Here we recall that the function K is defined in (1.19).

We first note that problem

$$-P\Delta\omega + c_0\frac{\partial\omega}{\partial x} = g, \quad \omega = 0 \text{ on } \partial\Omega, \tag{4.16}$$

possesses a unique solution ω in $H^2(\Omega)$ for any $g \in L^2(\Omega)$. Next the problem

$$\Delta\psi + \omega = h, \quad \psi = 0 \text{ on } \partial\Omega, \tag{4.17}$$

is also uniquely solvable in $H^3(\Omega)$ for any $h \in H^1(\Omega)$.

Now let us denote by A_0 the operator defined by

$$A_0 u = -\Delta u + c_0\frac{\partial u}{\partial x} - K'(\theta_0)u. \tag{4.18}$$

Then equation (4.12) can be written in the form

$$A_0 u = f - c(u_0 + \phi)' - \psi_y(u_0 + \phi)'. \tag{4.19}$$

Since assumption (1.21) holds, the operator A_0 from E into $L^2(\Omega)$ satisfies the Fredholm property and it has the zero index [19]. Therefore the equation $A_0 u = g$ is solvable in E if and only if

$$\iint_\Omega g v^* dxdy = 0, \tag{4.20}$$

where v^* is an eigenfunction corresponding to the zero eigenvalue of the formally adjoint operator,

$$-\Delta v^* - c_0\frac{\partial v^*}{\partial x} - K'(\theta_0)v^* = 0. \tag{4.21}$$

Thus the solvability condition applied to equation (4.19) reads

$$\iint_\Omega (f - c(u_0 + \phi)' - \psi_y(u_0 + \phi)')v^* dxdy = 0.$$

Therefore for any $f \in L^2(\Omega)$ we can choose c such that it is satisfied provided that

$$\iint_\Omega (u_0 + \phi)' v^* dxdy \neq 0. \tag{4.22}$$

Condition (4.22) holds true due to the monotonicity of $\theta_0 = u_0 + \phi$ and the positiveness of v^* so that problem (4.19) possesses a solution in E.

We have shown that the problem

$$A(u, \psi, \omega, c) = (f, g, h) \tag{4.23}$$

is solvable in X for any $(f,g,h) \in Y$. However this solution is not unique. Indeed, A_0 has the zero eigenvalue θ_0', and any function $(u+\tau\theta_0', \psi, \omega, c)$ with $\tau \in \mathbb{R}$ is also a solution of problem (4.23).

In order to construct an invertible operator, we need to introduce an appropriate subspace of X. Let

$$(\ker A_0)^{\perp} = \{u \in E, \iint_{\Omega} uv\,dxdy = 0 \;\; \forall v \in \ker A_0\}.$$

Then the operator A acting from

$$(\ker A_0)^{\perp} \times F \times G \times \mathbb{R}$$

onto Y is invertible.

This leads us to consider the space

$$X_1 \times (0,+\infty) \times \mathbb{R}, \quad \text{where} \quad X_1 = (u_0 + \ker A_0^{\perp}) \times F \times G \times \mathbb{R}. \tag{4.24}$$

Then $(u_0, 0, 0, c_0, 1, 0)$ belongs to that space and we can apply the Implicit Function Theorem to $\mathcal{A}$ in that space. This readily allows to obtain the conclusions of Theorem 4.1.

$\square$

5. Implicit Function Theorem for $R \neq 0$

The aim of this section is to consider problem (1.12)-(1.17) for Λ close to 1 and for R not necessarily small. More precisely we suppose that for $\Lambda = 1$ and for some $R = R_0$ problem (1.12)-(1.17) possesses a solution and we aim to show the existence of solutions for Λ close to 1 and R close R_0

We first introduce an integro-differential formulation for (1.12)-(1.17). As in Section 3, equations (1.12), (1.13) can be replaced by (3.11). Also it is convenient to rewrite the equations (1.14), (1.15) as a fourth order elliptic equation for the stream function ψ. Then problem (1.12)-(1.17) amounts in finding (θ, ψ, c) satisfying

$$-\Delta\theta + c\frac{\partial\theta}{\partial x} + \operatorname{curl}\psi.\nabla\theta - \kappa\big(\theta, 1-\theta+\mathcal{H}(\Lambda,u,\psi,c)\big) = 0, \;\; \theta = u + \phi, \tag{5.1}$$

$$P\Delta^2\psi - c\frac{\partial\Delta\psi}{\partial x} - \operatorname{curl}\psi.\nabla\Delta\psi - PR\operatorname{curl}\theta.\tau = 0, \tag{5.2}$$

$$\frac{\partial\theta}{\partial y} = 0, \;\; \psi = \Delta\psi = 0 \text{ on } \partial\Omega, \tag{5.3}$$

$$u(\pm\infty, y) = \psi(\pm\infty, y) = 0. \tag{5.4}$$

Here the unknown (u, ψ, c) lies in

$$\hat{X} = E \times \hat{F} \times (0, +\infty), \tag{5.5}$$

where

$$\hat{F} = \big\{\psi \in H^4(\Omega), \ \ \psi = \Delta\psi = 0 \text{ on } \partial\Omega\big\}. \tag{5.6}$$

Let us denote by

$$\mathcal{B} = \mathcal{B}(u, \psi, c, \Lambda, R) \tag{5.7}$$

the operator associated to the integro-differential system (5.1), (5.2). This operator is acting from

$$\hat{X} \times (0, +\infty) \times \mathbb{R} \tag{5.8}$$

into

$$\hat{Y} = L^2(\Omega) \times L^2(\Omega). \tag{5.9}$$

For $\Lambda = 1$, since $\mathcal{H}(1, u, \psi, c) \equiv 0$, the operator $\mathcal{B}$ takes a simpler form. We assume that for $\Lambda = 1$ and for some $R = R_0$ problem (5.1)-(5.4) has a solution (u_0, ψ_0, c_0) with $c_0 \neq 0$.

Then we can prove the following stability result.

Theorem 5.1. *Suppose that assumptions (1.18)-(1.21) hold. Moreover suppose that for $\Lambda = 1$ and for some $R_0 \neq 0$ problem (5.1)-(5.4) has a solution (u_0, ψ_0, c_0) with $c_0 \neq 0$ and that system (1.28), (1.29) has no bounded solution. Then there exists $\epsilon > 0$ such that problem (5.1)-(5.4) has a solution (θ, ψ, c) for all Λ and R such that*

$$|\Lambda - 1| + |R - R_0| < \epsilon. \tag{5.10}$$

This solution is continuous for the topology of $C^1_{loc}(\bar{\Omega})^2 \times \mathbb{R}$ with respect to the parameters (Λ, R).

Remark 5.2. The linearized operator $\mathcal{B}$ has a zero eigenvalue because of the invariance of the solution with respect to translation in space. The assumption that system (1.28), (1.29) does not have bounded solution corresponds to the simplicity of the zero eigenvalue for the following linear operator B corresponding to the left hand-side of the following system:

$$-\Delta u + c_0 \frac{\partial u}{\partial x} + \operatorname{curl} \psi_0 . \nabla u + \operatorname{curl} \psi . \nabla \theta_0 - K'(\theta_0) u = \frac{\partial \theta_0}{\partial x}, \tag{5.11}$$

$$\begin{aligned} \Delta^{-1}(P\Delta^2\psi - c_0 \frac{\partial \Delta\psi}{\partial x} - \operatorname{curl} \psi . \nabla\Delta\psi_0 - \operatorname{curl} \psi_0 . \nabla\Delta\psi - PR_0 \operatorname{curl} u . \tau) \\ = -\frac{\partial \psi_0}{\partial x}. \end{aligned} \tag{5.12}$$

Here Δ^{-1} is the linear integral operator acting from $L^2(\Omega)$ into $\{v \in H^2(\Omega), v|_{\partial\Omega} = 0\}$ which associates to a given function $f \in L^2(\Omega)$ the unique function $v \in H^2(\Omega)$ such that $v|_{\partial\Omega} = 0$ and $\Delta v = f$. The operator B is considered as acting from $E \times \hat{F}$ into $H^2(\Omega)$. The assumption that the zero eigenvalue is simple implies in particular that it is not a bifurcation point.

The operator B can be introduced by considering the following eigenvalue problem with constraint: find three functions (u, ψ, ω) and a real number λ such that

$$-\Delta u + c_0 \frac{\partial u}{\partial x} + \operatorname{curl} \psi_0 . \nabla u + \operatorname{curl} \psi . \nabla \theta_0 - K'(\theta_0) u = \lambda u, \tag{5.13}$$

$$-P\Delta\omega + c_0 \frac{\partial \omega}{\partial x} - \operatorname{curl} \psi . \nabla \Delta \psi_0 + \operatorname{curl} \psi_0 . \nabla \omega - PR_0 \operatorname{curl} u . \tau = \lambda \omega, \tag{5.14}$$

$$\Delta\omega + \psi = 0, \tag{5.15}$$

together with the corresponding boundary conditions. Then we easily see that this eigenvalue problem corresponds to a classical eigenvalue problem for the operator B.

Proof. The proof of Theorem 5.1 is based on the Implicit Function Theorem applied to the operator $\mathcal{B}$. We first note that this operator is of the class C^1 from $\hat{X} \times (0, +\infty) \times \mathbb{R}$ into $\hat{Y}$.

Consider the operator $\mathcal{B}$ linearized with respect to (u, ψ, c) about $(u_0, \psi_0, c_0, 1, R_0)$. Let $(f, g) \in L^2(\Omega) \times L^2(\Omega)$ be given and consider the equation

$$\mathcal{B}'_{(u,\psi,c)}(u_0, \psi_0, c_0, 1, R_0).(u, \psi, c) = (f, g). \tag{5.16}$$

This problem takes the form: find $(u, \psi, c) \in \hat{X}$ such that

$$-\Delta u + c_0 \frac{\partial u}{\partial x} + \operatorname{curl} \psi_0 . \nabla u + \operatorname{curl} \psi . \nabla \theta_0 - K'(\theta_0) u = f - c \frac{\partial \theta_0}{\partial x}, \tag{5.17}$$

$$P\Delta^2 \psi - c_0 \frac{\partial \Delta \psi}{\partial x} - \operatorname{curl} \psi . \nabla \Delta \psi_0 - \operatorname{curl} \psi_0 . \nabla \Delta \psi - PR_0 \operatorname{curl} u . \tau = g + c \frac{\partial \Delta \psi_0}{\partial x}. \tag{5.18}$$

We now introduce the operator B_0 which is associated to the left-hand side of this system of equations. This operator acts on the variables (u, ψ). System (5.17), (5.18) can be written in the form

$$B_0(u, \psi) = (f, g) - c\left(\frac{\partial \theta_0}{\partial x}, -\frac{\partial \Delta \psi_0}{\partial x}\right). \tag{5.19}$$

The following lemma provides solvability conditions for the operator B_0.

Lemma 5.3. *Under assumptions (1.18)-(1.21), the operators B_0 and and its formally adjoint B_0^* are Fredholm with the zero index. Equation (5.19) is solvable if and only if its right-hand side is orthogonal in $L^2(\Omega)$ to the kernel of the operator B_0^*.*

Proof. The proof of the Fredholm property together with the computations of the index can be found in [13]. It is shown that the operator $B_0 - \lambda$ is normally solvable with a finite dimensional kernel for all nonnegative λ. Since it is invertible for large λ, it satisfies the Fredholm property with the zero index. The same is true for the formally adjoint operator.

In the case of Fredholm operators with the zero index, solvability conditions can be formulated in terms of orthogonality to the kernel of the formally adjoint operator [2]. □

We now apply these solvability conditions to equation (5.19). It is solvable for any (f, g) for a suitable value of c if the following condition is satisfied:

$$\iint_\Omega (u^* \frac{\partial \theta_0}{\partial x} - \psi^* \frac{\partial \Delta\psi_0}{\partial x}) dxdy \neq 0, \tag{5.20}$$

where (u^*, ψ^*) is a nonzero solution of the equation

$$B_0^*(u^*, \psi^*) = 0. \tag{5.21}$$

Suppose now that condition (5.20) does not hold true that is

$$\iint_\Omega (u^* \frac{\partial \theta_0}{\partial x} - \psi^* \frac{\partial \Delta\psi_0}{\partial x}) dxdy = 0. \tag{5.22}$$

Due to Lemma 5.3 the equation

$$B_0(u, \psi) = (\frac{\partial \theta_0}{\partial x}, -\frac{\partial \Delta\psi_0}{\partial x}), \tag{5.23}$$

has a solution $(u, \psi) \in E \times \hat{F}$. This provides a nonzero and bounded solution of the system of equations

$$-\Delta u + c_0 \frac{\partial u}{\partial x} + \operatorname{curl}\psi_0.\nabla u + \operatorname{curl}\psi.\nabla\theta_0 - K'(\theta_0)u = \frac{\partial \theta_0}{\partial x}, \tag{5.24}$$

$$P\Delta^2\psi - c_0 \frac{\partial \Delta\psi}{\partial x} - \operatorname{curl}\psi.\nabla\Delta\psi_0 - \operatorname{curl}\psi_0.\nabla\Delta\psi - PR_0 \operatorname{curl} u.\tau = -\frac{\partial \Delta\psi_0}{\partial x}. \tag{5.25}$$

This is in contradiction with our assumption on system (1.28), (1.29). Therefore condition (5.20) holds true and problem (5.19) is solvable for any $(f, g) \in L^2(\Omega)^2$ and a suitable value of c.

Next we can conclude the proof as in Theorem 4.1 by choosing a convenient submanifold where the solution of equation (5.19) is unique, and the operator B_0 is invertible. □

References

[1] M. Bazile, H.A. Nichols, J.A. Pojman, V. Volpert, Effect of orientation on thermoset frontal polymerization, *Journal of Polymer Science, Part A: Polymer Chemistry*, **40** (2002), 3504-3508.

[2] M. Belk, *Stabilité Structurelle de Solutions Invariantes par Translation. Applications à des Problèmes de Réactions-Diffusion avec Convection*, Thèse, Université Lyon 1 (2003).

[3] M. Belk, B. Kazmierczak, V. Volpert, Existence of reaction-diffusion-convection waves in unbounded cylinders, *Preprint*, University Lyon 1 (2003).

[4] H. Berestycki, L. Nirenberg, Travelling fronts in cylinders, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **9** (1992), 497-572.

[5] G. Bowden, M. Garbey, V. Ilyashenko, J. Pojman, S. Solovyov, V. Volpert, The effect of convection on a propagating front with a solid product: comparison of theory and experiments, *J. Chem. Physics B*, **101** (1997), 678-686.

[6] P. Constantin, A. Kiselev, L. Ryzhik, Fronts in reactive convection: bounds, stability, and instability, *Comm. Pure Appl. Math.*, **56** (2003), 1781-1804.

[7] A. Ducrot, M. Marion, V. Volpert, Reaction-diffusion problems with non Fredholm operators, *Preprint*, University Lyon 1 (2004).

[8] M. Garbey, A. Taik, V. Volpert, Influence of natural convection on stability of reaction fronts in liquids, *Quart. Appl. Math.*, **53** (1998), 1-35.

[9] S. Malham, J.X. Xin, Global solutions to a reactive Boussinesq system with front data on an infinite domain, *Comm. Math. Phys.*, **193** (1998), 287-316.

[10] B. McCaughey, J.A. Pojman, C. Simmons, V.A. Volpert, The effect of convection on a propagating front with a liquid product: comparison of theory and experiments, *Chaos*, **8** (1998), 520-529.

[11] O. Manley, M. Marion, R. Temam, Equations of combustion in the presence of complex chemistry, *Indiana Univ. Math. J.*, **42** (1993), 941-967.

[12] R. Texier, V. Volpert, Problèmes de réaction-diffusion-convection dans des cylindres non bornès, *CRAS*, **333**, Serie I (2001), 1077-1082.

[13] R. Texier-Picard, V.A. Volpert, Reaction-diffusion-convection problems in unbounded cylinders, *Revista Matematica Complutense*, **16** (2003), 233-276.

[14] V. Volpert, A. Volpert, Properness and topological degree for general elliptic operators, *Abstract and Applied Analysis*, **3** (2003), 129-181.

[15] V. Volpert, A. Volpert, Convective instability of reaction fronts. Linear stability analysis, *Eur. J. Appl. Math.*, **9** (1998), 507-525.

[16] A. Volpert, V. Volpert, Existence of multidimensional travelling waves and systems of waves, *Comm. PDE*, **26** (2001), 421-459.

[17] A.I. Volpert, V.A. Volpert, Spectrum of elliptic operators and stability of traveling waves, *Asymptotic Analysis*, **23** (2000), 111-134.

[18] A.I. Volpert, V.A. Volpert, Applications of the rotation theory of vector fields to the study of wave solutions of parabolic equations, *Trans. Moscow Math. Soc.*, **52** (1990), 59-108.

[19] V. Volpert, A. Volpert, J.F. Collet, Topological degree for elliptic operators in unbounded cylinders, *Adv. Diff. Eq.*, **4** (1999), 777-812.

[20] A. Volpert, Vit. Volpert, Vl. Volpert, Traveling wave solutions of parabolic systems, *Translation of Mathematical Monographs*, **140**, Amer. Math. Society, Providence (1994).

[21] F.A. Williams, *Combustion Theory*, Benjamin-Cummings, Menlo Park CA, Second Edition (1985).