International Journal of Pure and Applied Mathematics
Volume 27 No. 2 2006, 213-218

ABOUT THE COLORING PROPERTIES OF W_m GENERAL MYCIELSKI-GRAPH

Muchun Li[1], Huiying Qiang[2], Zhongfu Zhang[3 §]
[1,2,3]Institute of Applied Mathematics
Lanzhou Jiaotong University
Lanzhou, 730070, P.R. CHINA
[2]e-mail: qhy2005ww@126. com

Abstract: $M_n(G)$ is called general Mycielski graph G, if n is natural number, and

$$V(M_n(G)) = \{v_{00}, v_{01}, ...v_{0m}; v_{10}, v_{11}, ...v_{1m}; ...; v_{n0}, v_{n1}, ...v_{nm}\},$$
$$E(M_n(G)) = E(G) \cup \{v_{ij}v_{(i+1)k} | v_{0j}v_{0k} \in E(G),\ 0 \le j,\ k \le m,\ i = 0, 1, \cdots, n-1\}.$$

The general Mycielski graph of wheel with order $(m+1)$ is noted $M_n(W_m)$. In this paper, some results of $M_n(W_m)$ graphs are obtained.

AMS Subject Classification: 05C15, 68R10, 94C15
Key Words: general Mycielski graph, total coloring, adjacent-vertex-distinguishing total coloring

1. Introduction

Definition 1. $M_n(G)$ is called the General Mycielski graph of G, if n is natural number, and

$$V(M_n(G)) = \{v_{00}, v_{01}, ...v_{0m}; v_{10}, v_{11}, ...v_{1m}; ...; v_{n0}, v_{n1}, ...v_{nm}\},$$
$$E(M_n(G)) = E(G) \cup \{v_{ij}v_{(i+1)k} | v_{0j}v_{0k} \in E(G),\ 0 \le j,\ k \le m,\ i = 0, 1, \cdots, n-1\}.$$

The general Mycielski graph of wheel with order $(m+1)$ is noted $M_n(W_m)$.

Received: November 1, 2005

§Correspondence author

Definition 2. Let $G(V,E)$ be a sample connect graph, k is natural number and f is a mapping from $V(G)\cup E(G)$ to $\{1,2,...k\}$. For all $uv, uw \in E(G)$, if $v\neq w$, then $f(uv)\neq f(uw)$. For any $uv\in E(G)$, we have $f(u)\neq f(v)$, $f(uv)\neq f(u)$, $f(v)\neq f(uv)$. Then f is called a $k-proper-total-coloring$ of G. and which is abbreviated k-PTC of G, and

$$\chi_t(G)=\min\{k|\ k\text{-PTC of } G\}$$

is called the total chromatic number of G.

Conjecture 1. (see [1]) *For a sample graph G, then $\chi_t(G)\leq\Delta(G)+2$, where $\Delta(G)$ is maximum degree of G.*

Definition 3. (see [5]) For a sample connect graph $G(V,E)$, if a k-proper-total-coloring f of G satisfied any $uv\in E(G)$, $C(u)\neq C(v)$, then f is called k-adjacent-vertex-distinguishing-total-coloring of G, which is abbreviated k-AVDTC of G, and $\chi_{at}(G)=\min\{k|\ k\text{-AVDTC of } G\}$ is called the adjacent vertex distinguishing total chromatic number of G. Here $C(u)=\{f(u)\cup f(uv)|uv\in E\}$.

Conjecture 2. (see [5]) *For a sample graph G: $\chi_{at}(G)\leq\Delta(G)+3$.*

Lemma 1. (see [5]) *If G has two adjacent maximum degree vertexes, then $\chi_{at}(G)\geq\Delta(G)+2$.*

In this paper, we will study the edge chromatic number, the toltal chromatic number and the adjacent vertex distinguishing toltal chromatic number of $M_n(W_m)$ graph. The other terminology can be found in [1].

2. Main Results

Theorem 1. *For $M_n(W_m)$ graph, $(n\geq 1, m\geq 3), \chi'(M_n(W_m))=2m$. Where $\chi'(G)$ is the edge chromatic number of G.*

Proof. *Case 1.* When $m=3$, because $\Delta(M_n(W_3))=6, \chi'(M_n(W_3)\geq 6$. Now, we give a $6-proper-edge-coloring$ of $M_n(W_3)$. Let f be:

$$f(v_{00}v_{0j})=j,\quad (j=1,2,3);\quad f(v_{0j}v_{0,j+1})=j+2,\quad (j=1,2);\quad f(v_{01}v_{03})=2;$$

$$f(v_{i0}v_{i+1,j})=f(v_{ij}v_{i+1,0})=\begin{cases} j, & \text{if } i\equiv 1 \pmod 2,\\ m+j, & \text{if } i\equiv 0 \pmod 2),\quad j=1,2,3.\end{cases}$$

$$f(v_{i1}v_{i+1,2})=f(v_{i2}v_{i+1,1})=\begin{cases} 3, & \text{if } i\equiv 1 \pmod 2,\\ 6, & \text{if } i\equiv 0 \pmod 2.\end{cases}$$

$$f(v_{i1}v_{i+1,3})=f(v_{i3}v_{i+1,1})=\begin{cases} 2, & \text{if } i\equiv 1 \pmod 2,\\ 5, & \text{if } i\equiv 0 \pmod 2.\end{cases}$$

$$f(v_{i2}v_{i+1,3}) = f(v_{i3}v_{i+1,2}) = \begin{cases} 4, & \text{if} \ \ i \equiv 1 \pmod 2, \\ 1, & \text{if} \ \ i \equiv 0 \pmod 2. \end{cases}$$

Obviously, f is $6 - PEC$ of $M_n(W_3)$.

Case 2. When $m \geq 4$, because $\Delta(M_n(W_m)) = 2m$, and the maximum degree vertexes are not adjacent, then $\chi'(M_n(W_m)) \geq 2m$. Now, we prove $\chi'(M_n(W_m)) \leq 2m$. We only prove the existence of $2m - PEC$. Let f be:

$$f(v_{00}v_{0j}) = j, \quad (j = 1, 2, \cdots m);$$

$$f(v_{0j}v_{0,j+1}) = j + 2, (j = 1, 2, \cdots m - 1); \quad f(v_{01}v_{0m}) = 2;$$

$$f(v_{ij}v_{i+1,0}) = f(v_{i0}v_{i+1,j}) = \begin{cases} j, & \text{if} \ \ i \equiv 1 \pmod 2, \\ m + j, & \text{if} \ \ i \equiv 0 \pmod 2, \quad j = 1, 2, \cdots m. \end{cases}$$

$$\begin{aligned} f(v_{ij}v_{i+1,j+1}) &= f(v_{i,j+1}v_{i+1,j}) \\ &= \begin{cases} j + 2, & \text{if} \ \ i \equiv 1 \pmod 2, \\ m + j + 2, & \text{if} \ \ i \equiv 0 \pmod 2, \quad j = 1, 2, \cdots m - 2. \end{cases} \end{aligned}$$

$$f(v_{i,m-1}v_{i+1,m}) = f(v_{im}v_{i+1,m-1}) = \begin{cases} m + 1, & \text{if} \ \ i \equiv 1 \pmod 2, \\ 1, & \text{if} \ \ i \equiv 0 \pmod 2, \end{cases}$$

$$f(v_{i1}v_{i+1,m}) = f(v_{i,m}v_{i+1,1}) = \begin{cases} 2, & \text{if} \ \ i \equiv 1 \pmod 2, \\ m + 2, & \text{if} \ \ i \equiv 0 \pmod 2. \end{cases}$$

Obviously, f is $2m - PEC$ of $M_n(W_m)$. □

Theorem 2. *For $M_n(W_m)$ graph, $(n \geq 1, \quad m \geq 3)$, then* $\chi_t(M_n(W_m)) = 2m + 1$.

Proof. Case 1. When $m = 3$, $\Delta(M_n(W_3)) = 6$, we know that $\chi_t(M_n(W_3)) \geq 7$, because the adjacent strong edge chromatic number of $M_n(W_3)$, $\chi'_{as}(M_n(W_3)) = 7$, see [3].

We fill with the color that every vertex be short of to the vertex. Then the color set of every vertex $C(v_{ij}) = \{1, 2, 3, 4, 5, 6, 7\}, \quad \chi_t(M_n(W_3) = 7$.

Case 2. When $m = 4, \Delta(M_n(W_4)) = 8$, then $\chi_t(M_n(W_4)) \geq 9$. Now, we prove $\chi_t(M_n(W_4)) \leq 9$. We only prove the existence of 9-PTC of $M_n(W_4)$. Let f be:

$$f(v_{i0}) = 9; \quad f(v_{ij}) = j + 1; \quad i = 0, 1, 2, \cdots n, \quad j = 1, 2, 3, 4;$$

$$f(v_{00}v_{0j}) = j, \quad j = 1, 2, 3, 4; \quad f(v_{0j}v_{0,j+1}) = j + 3, \quad j = 1, 2, 3; \quad f(v_{01}v_{04}) = 9;$$

$$f(v_{ij}v_{i+1,0}) = f(v_{i0}v_{i+1,j}) = \begin{cases} m + j, & \text{if} \ \ i \equiv 0 \pmod 2, \\ j, & \text{if} \ \ i \equiv 1 \pmod 2 \quad j = 1, 2, 3, 4; \end{cases}$$

$$f(v_{ij}v_{i+1,j+1}) = f(v_{i+1,j}v_{i,j+1}) = \begin{cases} j+3, & \text{if } i \equiv 1 \pmod 2, j=1,2,3; \ , \\ 6+j, & \text{if } i \equiv 0 \pmod 2, \ j=1,2. \end{cases}$$

$$f(v_{i3}v_{i+1,4}) = f(v_{i+1,3}v_{i4}) = 1; \quad i \equiv 1 (\bmod 2);$$

$$f(v_{i1}v_{i+1,4}) = f(v_{i+1,1}v_{i4}) = \begin{cases} 9, & \text{if } i \equiv 1 \pmod 2, \\ 3, & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

Obviously, f is $9-PTC$ of $M_n(W_4)$.

Case 3. When $m \geq 5$, because $\Delta(M_n(W_m)) = 2m$. Let f be:

$$f(v_{i0}) = 2m+1, \quad i = 0,1,2,\cdots n; \quad f(v_{ij}) = j+1, \quad j = 1,2,\cdots m; \quad i = 0,1,2,\cdots n.$$

$$f(v_{00}v_{0j}) = j, \quad j = 1,2,\cdots m; \quad f(v_{0j}v_{0,j+1}) = j+3; \quad j = 1,2,\cdots m-1. \quad f(v_{01}v_{0m}) = 2m-1;$$

$$f(v_{ij}v_{i+1,0}) = f(v_{i0}v_{i+1,j}) = \begin{cases} m+j, & \text{if } i \equiv 0 \pmod 2, \\ j, & \text{if } i \equiv 1 \pmod 2, \quad j = 1,2,\cdots m. \end{cases}$$

$$f(v_{ij}v_{i+1,j+1}) = f(v_{i,j+1}v_{i+1,j}) = \begin{cases} m+j+2, & \text{if } i \equiv 0 \pmod 2, \\ m+j-1, & \text{if } i \equiv 1 \pmod 2, \quad j = 1,2,\cdots m-1. \end{cases}$$

$$f(v_{i1}v_{i+1,m}) = f(v_{i+1,1}v_{i,m}) = \begin{cases} 3, & \text{if } i \equiv 0 \pmod 2, \\ 4, & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

Obviously, f is $(2m+1)-PTC$ of $M_n(Wm)$. □

Theorem 3. *For $M_n(W_m)$ graph $(n \geq 1, m \geq 3)$, we have*

$$\chi_{at}(M_n(W_m)) = \begin{cases} 8, & \textit{if } m = 3 \\ 2m+1, & \textit{if } m \geq 4. \end{cases}$$

Proof. Case 1. When $m = 3$, $\Delta(M_n(W_3)) = 6$, and the maximum vertex are adjacent, from Lemma 1 we know: $\chi_{at}(M_n(W_3) \geq 8$. Now, we give a $8-AVDTC$of $M_n(W_3)$. Let f be :

$$f(v_{00}v_{0j}) = j, \quad j = 1,2,3; \quad f(v_{01}v_{03}) = 2; \quad f(v_{0j}v_{0,j+1}) = j+2, \quad j = 1,2;$$

$$f(v_{i0}) = 8; \quad f(v_{i1}) = 7; \quad f(v_{i2}) = 6; \quad f(v_{i3}) = 1; \quad i = 0,1,2,\cdots n.$$

$$f(v_{ij}v_{i+1,0}) = f(v_{i0}v_{i+1,j}) = \begin{cases} m+j, & \text{if } i \equiv 0 \pmod 2, \\ j, & \text{if } i \equiv 1 \pmod 2 \quad j = 1, 2, \cdots m; \end{cases}$$

$$f(v_{i1}v_{i+1,2}) = f(v_{i2}v_{i+1,1}) = \begin{cases} 8, & \text{if } i \equiv 0 \pmod 2, \\ 3, & \text{if } i \equiv 1 \pmod 2 \end{cases}$$

$$f(v_{i2}v_{i+1,3}) = f(v_{i3}v_{i+1,2}) = \begin{cases} 7, & \text{if } i \equiv 0 \pmod 2, \\ 4, & \text{if } i \equiv 1 \pmod 2 \end{cases}$$

$$f(v_{i1}v_{i+1,3}) = f(v_{i3}v_{i+1,1}) = \begin{cases} 5, & \text{if } i \equiv 0 \pmod 2, \\ 2, & \text{if } i \equiv 1 \pmod 2 \end{cases}$$

Suppose $\overline{C(u)} = C - C(u)$. Then $\overline{C(v_{i0})} = \{7\}$, $\overline{C(v_{i1})} = \{6\}$, $\overline{C(v_{i2})} = \{1\}$, $\overline{C(v_{i3})} = \{8\}$, $(i = 0, 1, 2, \cdots n)$. Obviously, for $\forall uv \in M_n(W_3)$, $C(u) \neq C(v)$. So f is $8 - AVDTC$ of $M_n(W_3)$.

Case 2. When $m = 4$, from Case 2 of Theorem 2, we know: $\overline{C(v_{i0})} = \{1, 2, \cdots 9\}$, $\overline{C(v_{i1})} = \{6, 8\}$, $\overline{C(v_{i2})} = \{1, 9\}$, $\overline{C(v_{i3})} = \{2, 9\}$, $\overline{C(v_{i4})} = \{2, 7\}$, $i = 0, 1, 2, \cdots n$.

Obviously, f is $9 - AVDTC$ of $M_n(W_4)$.

Case 3. When $m \geq 5$, from above Case 3 of Theorem 2, we know:

$$C(v_{i0}) = \{1, 2, \cdots 2m+1\}, \quad i = 0, 1, 2, \cdots n;$$

$$C(v_{01}) = \{1, 2, 3, 4, m+1, m+3, 2m-1\};$$

$$C(v_{i1}) = \{1, 2, 3, 4, m, m+1, m+3, \}; \quad i = 1, 2, \cdots n;$$

$$C(v_{0j}) = \{j, j+1, j+2, j+3, m+j, m+j+1, m+j+2\}, \quad j = 2, 3, \cdots m-1;$$

$$C(v_{0m}) = \{3, m, m+1, m+2, 2m-1, 2m, 2m+1\};$$

$$C(v_{ij}) = \{j, j+1, m+j-2, m+j-1, m+j, m+j+1, m+j+2\}, \quad j = 2, 3, \cdots m-1; \quad i = 1, 2, \cdots n;$$

$$C(v_{im}) = \{3, 4, m, m+1, 2m-2, 2m, 2m+1\}, \quad i = 1, 2, \cdots n.$$

Obviously, for $\forall uv \in M_n(W_m)$, $C(u) \neq C(v)$. So f is $(2m+1) - AVDTC$ of $M_n(W_m)$. The theorem is proved. □

Acknowledgments

This research is supported by NSFC of P.R. China (No. 19871036).

References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applicatives*, New York, The Macmillan Press (1976).

[2] Li Jinwen, The total chromatic number of Mycielski graph of some graphs, *Journal of Lanzhou Railway University*, **22** (2003), 7-10.

[3] Chen Yi, About the adjacent strong edge chromatic of general Mycielski graph of wheel graph, *Mathematics in Economics*.

[4] Zhang Zhongfu, Chen Xiangen, Li Jinwen, About the adjacent vertex distinguishing total chromatic number of graphs, *Science of China Ser. A.*, **34**, No. 5 (2004), 574-583.

[5] Zhang Zhongfu, et al, Adjacent strong edge chromatic of graphs, *Applied Mathematics Letters*, **15** (2002), 623-626.