

TWO SPECIAL CLASSES OF HAMILTONIAN GRAPH

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Abstract: We study two types of Hamiltonian graphs. We give a characterization to one type, and determine the minimum number of edges in the other type.

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1. Definitions

The graphs we consider are undirected, and without multiple edges or loops. Unless otherwise stated, we follow the notation of [2]. A Hamiltonian cycle (respectively, path) in a graph is a spanning cycle (path). A graph is Hamiltonian if it contains a Hamiltonian cycle. A Hamiltonian graph is said to be *weak* if adding any new edge will not create a new Hamiltonian cycle. That is, if P is a Hamiltonian path in a weak Hamiltonian graph then the initial and final vertices of P must be adjacent. A Hamiltonian graph G is said to be *strong* if for every edge e in G there exists a Hamiltonian cycle which does not use e . That is, G is Hamiltonian, and $G - e$ is also Hamiltonian for every e in G .

We shall characterize weak Hamiltonian graphs and determine the minimum number of edges in a strong Hamiltonian graph.

Suppose G is a Hamiltonian graph on n vertices and let C be a Hamiltonian

cycle. We define i -chords of C as follows: an edge uv of G is an i -chord of C if there exists a path on C , joining u and v , and of length i . Hence, an i -chord is also an $(n-i)$ -chord. $N(v)$ denotes the neighbourhood of a vertex v .

2. Weak Hamiltonian Graphs

We characterize weak Hamiltonian graphs. An obvious example of weak Hamiltonian graphs is C_n , when the number of edges is minimum. K_n is also considered as a weak Hamiltonian graph in the defined sense; where the number of edges is maximum obviously. We look for other important weak Hamiltonian graphs in the middle. In this section, we assume that G is weak Hamiltonian. We need the following lemmas.

Lemma 1. *If an i -chord is in $E(G)$, then all of the i -chords are in $E(G)$.*

Proof. Consider the Hamiltonian cycle in Figure 1 where the vertices $v_0, v_1, v_2, \dots, v_{n-1}$ are arranged in a clockwise order. Assume without loss of generality that $v_0v_i \in E(G)$. If $v_1v_{i+1} \notin G$ then there is a Hamiltonian path joining v_1 and v_{i+1} . Namely, $v_1v_2 \dots v_i v_0 v_{n-1} v_{n-2} \dots v_{i+1}$, which is a contradiction. Hence $v_1v_{i+1} \in G$. The proof follows now inductively. \square

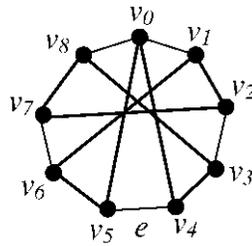


Figure 1

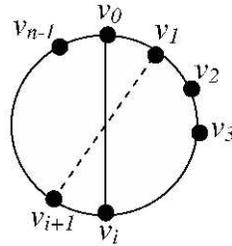


Figure 2

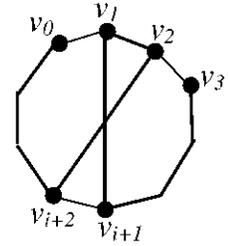


Figure 3

Lemma 2. *If there is an i -chord ($i \geq 2$), then all of the 3-chords are presented.*

Proof. We show that there is a 3-chord, and then we use Lemma 1 to deduce that all of the 3-chords are presented.

If there is an i -chord ($i \geq 2$), v_1v_{i+1} say. Then, by Lemma 1 all of the i -chords are presented. In particular v_2v_{i+2} . Therefore there is a Hamiltonian path joining v_0 and v_3 , namely $v_0v_{n-1}v_{n-2} \dots v_{i+2}v_2v_1v_{i+1}v_iv_{i-1} \dots v_3$ (see Figure 2). Hence the 3-chord v_0v_3 must be there. \square

Lemma 3. *If there is an i -chord ($i \geq 2$), then all of the $(2k + 1)$ -chords are presented where $k \geq 1$.*

Proof. By induction on k . When $k = 1$, by Lemma 2 all of the 3-chords are there. Suppose G contains all $(2l - 1)$ -chords, in particular v_0v_{2l-1} . Then we may use v_1v_{2l} and $v_{n-1}v_{2l-2}$ to deduce that there is a Hamiltonian path joining v_0 and v_{2l+1} (see Figure 3). Hence G contains v_0v_{2l+1} , and by Lemma 1 all of the $(2l + 1)$ -chords are presented. Hence all of the odd chords are in $E(G)$. \square

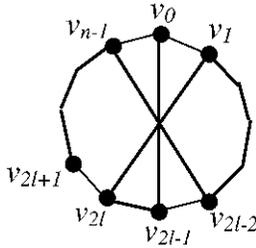


Figure 4

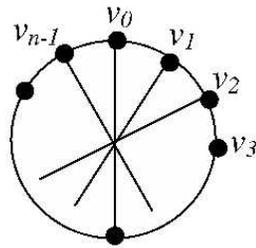


Figure 5

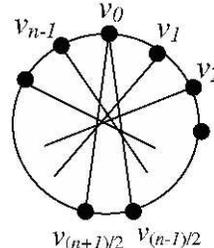


Figure 6

Proposition 4. *If G is a weak Hamiltonian graph on n vertices, where n is even, then G is isomorphic to C_n or contains $K_{\frac{n}{2}, \frac{n}{2}}$ as a subgraph.*

Proof. If $G \not\cong C_n$, then considering a Hamiltonian cycle C , there is an i -chord ($i \geq 2$). Hence by Lemma 3 all odd chords are presented. This proves that $K_{\frac{n}{2}, \frac{n}{2}}$ is contained in G . \square

We may use the following result which is due to Bondy, see [1].

Theorem 5. *If G is Hamiltonian and $|E(G)| \geq (\frac{n}{2})^2$, then either $G \cong K_{\frac{n}{2}, \frac{n}{2}}$ or G is pancyclic.*

Corollary 6. *If G is a weak Hamiltonian graph on n vertices, where n is even, then G is isomorphic to either C_n , K_n , or $K_{\frac{n}{2}, \frac{n}{2}}$.*

Proof. Suppose $G \not\cong C_n$. Then by Proposition 4, G contains $K_{\frac{n}{2}, \frac{n}{2}}$ as a subgraph, and so $|E(G)| \geq (\frac{n}{2})^2$. Now by Theorem 5 we have either $G \cong K_{\frac{n}{2}, \frac{n}{2}}$ or G is pancyclic. If $G \not\cong K_{\frac{n}{2}, \frac{n}{2}}$, then G is pancyclic. Therefore, if we consider a Hamiltonian cycle C , then all odd chords are presented (Lemma 3), and all even chords are presented as well (as G is pancyclic and using Lemma 1). Hence $G \cong K_n$. \square

Corollary 7. *If G is a weak Hamiltonian graph on n vertices, where n is odd, then G is isomorphic to either C_n or K_n .*

Proof. Suppose $G \not\cong C_n$. Then considering a Hamiltonian cycle C , there is an i -chord ($i \geq 2$). Hence by Lemma 3 all odd chords are presented. But

since an i -chord can be regarded as an $(n - i)$ -chord, then even chords can be regarded as odd chords as well. This means that all even chords must be presented too, and $G \cong K_n$. \square

Combining Corollary 6 and Corollary 7, we state our final result.

Proposition 8. *If G is a weak Hamiltonian graph on n vertices, then G is isomorphic to either C_n , K_n , or $K_{\frac{n}{2}, \frac{n}{2}}$.*

3. Strong Hamiltonian Graphs

The obvious example here is K_n . Thus, strong Hamiltonian graphs of minimal size are to be sought. Suppose that $\varsigma(n)$ is the minimum number of edges in a strong Hamiltonian graph with n vertices. Clearly, the minimum degree δ is at least 3. Hence, if we could construct a class of cubic graphs or *nearly cubic* (a graph is nearly cubic if each vertex is of degree 3 except one vertex that is of degree 4), then we may conclude that $\varsigma(n)$ is $\frac{3n}{2}$ when n is even, and $\frac{3n+1}{2}$ when n is odd.

Proposition 9. *If G is a strong Hamiltonian graph on n vertices, then:*

$$\varsigma(n) = \begin{cases} \frac{3n}{2} & \text{when } n \text{ is even,} \\ \frac{3n+1}{2} & \text{when } n \text{ is odd.} \end{cases}$$

Proof. As we have mentioned, it is clear that the minimum degree δ is at least 3, and enough to find a cubic graph when n is even and nearly cubic when n is odd. Assume n is even, and consider the Mobius ladder in Figure 4 where the vertices are arranged in a clockwise direction around the Hamiltonian cycle C and $N(v_i) = \{v_{i-1}, v_{i+1}, v_{i+\frac{n}{2}}\}$. The indices are to be taken modulo $n - 1$.

Using symmetry we may check upon deleting only two chords, a 1-chord (an edge of C) and an $\frac{n}{2}$ -chord. The latter is trivial as C does use only 1-chords. So assume e is a 1-chord, v_0v_1 without loss of generality. Then, $v_0v_{n-1}v_{n-2}\dots v_{\frac{n}{2}+1}v_1v_2v_3\dots v_{\frac{n}{2}}v_0$ is a Hamiltonian cycle that does not use e .

Suppose now that n is odd. Let G be the nearly cubic graph modified from Mobius ladder (see Figure 5), where the obvious Hamiltonian cycle is shown, $N(v_0) = \{v_1, v_{n-1}, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}\}$, and $N(v_i) = \{v_{i-1}, v_{i+1}, v_{i+\frac{n+1}{2}}\}$ ($i \neq 0$). Indices are to be taken modulo $n - 1$.

Once again, it is trivial to see that $G - e$ is still Hamiltonian when e is an i -chord ($i \neq 1$). Therefore, we may inspect only 1-chords. Note that there are $\frac{n-1}{2}$ of what we call the *cross configuration*, that is two crossed $\frac{n-1}{2}$ -chords. Namely $\{v_{i-1}v_{i-1+\frac{n+1}{2}}, v_i v_{i+\frac{n-1}{2}}\}$, where $i \neq \frac{n+1}{2}$ (see Figure 6).

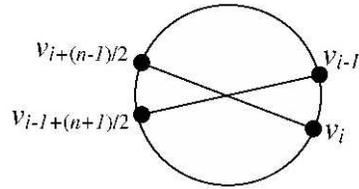


Figure 7:

If $e = v_{i-1}v_i$ ($i \neq \frac{n+1}{2}$), then using a cross configuration, it is easy to spot a Hamiltonian cycle for $G - e$, namely $v_i v_{i+1} \dots v_{i-1+\frac{n+1}{2}} v_{i-1} v_{i-2} \dots v_{i+\frac{n+1}{2}} v_i$.

If $e = v_{\frac{n-1}{2}} v_{\frac{n+1}{2}}$, then taking all the i -chords ($i \neq 1$) together with these $\frac{n-1}{2}$ 1-chords: $v_1 v_2, v_3 v_4, \dots, v_{n-2} v_{n-1}$ gives us a Hamiltonian cycle for $G - e$. The case when $n = 9$ is shown in Figure 7 where bold edges depict the Hamiltonian cycle.

This proves the result. □

4. Conclusion

We have characterized weak Hamiltonian graphs, and evaluated the minimum number of edges in a strong Hamiltonian graph.

References

- [1] B. Bollobas, *Extremal Graph Theory*, Academic Press Inc., London (1978).
- [2] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, Elsevier Science Publishing Co., Inc. (1976).

