ADJACENT VERTEX-DISTINGUISHING TOTAL COLORING ON SQUARE, CUBE, Biquadratic of PATHES

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Abstract: Let $G$ be a simple graph, let $f$ be a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$, let $C_f(v) = \{f(v)\} \cup \{f(wv)|w \in V(G), vw \in E(G)\}$ for every $v \in V(G)$. If $f$ is a $k$-proper-total-coloring, and for $u, v \in V(G), uv \in E(G)$, we have that $C_f(u) \neq C_f(v)$, then $f$ is called a $k$-adjacent-vertex-distinguishing total coloring ($k$-AVDTC for short). Let $\chi_{at}(G) = \min\{k|G$ has a $k$-adjacent-vertex-distinguishing total coloring$\}$. Then $\chi_{at}(G)$ is called the adjacent-vertex-distinguishing total chromatic number. The adjacent-vertex-distinguishing total chromatic numbers on square, cube, biquadratic of paths are obtained in this paper.

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1. Introduction

In this paper, the graphs considered are connected, limited, undirected, simple graphs. A $k$-proper-total-coloring of a graph $G$ is a mapping $f$ from $V(G) \cup E(G)$ to \{1, 2, \ldots, k\} such that:

1) $\forall u, v \in V(G)$, if $uv \in E(G)$, then $f(u) \neq f(v)$;
2) $\forall e_1, e_2 \in E(G), e_1 \neq e_2$, if $e_1, e_2$ have common end vertex, then $f(e_1) \neq f(e_2)$;
3) $\forall u \in V(G), e \in E(G)$, if $u$ is the end vertex of $e$, then $f(u) \neq f(e)$.

Let $f$ be a $k$-proper-total-coloring of $G$, let $C_f(u) = \{f(u)\} \cup \{f(uw)\}_{uw \in E(G)}$ and $\overline{C}_f(u) = \{1, 2, \ldots, k\} - C_f(u)$ for every $u \in V(G)$. If $\forall u, v \in V(G), uv \in E(G)$, we have $C_f(u) \neq C_f(v)$, i.e., $\overline{C}_f(u) \neq \overline{C}_f(v)$, then $f$ is called a $k$-adjacent-vertex-distinguishing total coloring ($k$-AVDTC for short). The number $\min\{k|G\}$ has a $k$-adjacent-vertex-distinguishing total coloring is called the adjacent-vertex-distinguishing total chromatic number and is denoted by $\chi_{at}(G)$.

The theory of vertex-distinguishing proper edge-coloring has been investigated in several papers [1, 2, 3, 4]. Adjacent strong edge coloring (i.e., adjacent-vertex-distinguishing proper edge-coloring) is considered in [7] by Zhongfu Zhang et al and [8] by Xiang-en Chen et al. The concept about the adjacent-vertex-distinguishing total coloring is proposed by Zhongfu Zhang and Xiang-en Chen et al in [6]. The adjacent-vertex-distinguishing total coloring on cycle, complete graph, complete bipartite graph, fan wheel and tree are discussed in [6]. According to these results, for adjacent-vertex-distinguishing total chromatic number, a conjecture and an open problem are given in [6].

Conjecture 1.1. (see [6]) For every simple graph $G$, we have
$$\chi_{at}(G) \leq \Delta(G) + 3.$$

Open Problem 1.1. (see [6]) When do we have $\chi_{at}(H) \leq \chi_{at}(G)$, if $H$ is a subgraph of $G$?

In this paper, the adjacent-vertex-distinguishing total coloring on square, cube, biquadratic of pathes are studied and the corresponding chromatic numbers are obtained respectively. Theorem 2.1, Theorem 2.2 and Theorem 2.3 in this paper will indicate that Conjecture 1.1 is valid for square, cube, biquadratic of pathes.

Lemma 1.1. (see [6]) If $G$ does not have two distinct vertices of maximum degree which are adjacent, then $\chi_{at}(G) \geq \Delta(G) + 1$; otherwise, $\chi_{at}(G) \geq \Delta(G) + 2$. 
For the graph-theoretic terminology the reader is referred to [5].

2. Main Results

Let \( P_n = v_1v_2 \cdots v_n \) be a path with order \( n \), let \( k \) be an integer which is at least 2. We construct a new graph \( P_n^k \), where \( V(P_n^k) = V(P_n) = \{v_1, v_2, \ldots, v_n\} \), and \( E(P_n^k) = \{v_iv_{i+k} | i = 1, 2, \ldots, n - k\} \cup E(P_n) \). The new graph \( P_n^k \) is called the \( k \) cube of path \( P_n \). In this section, we give the adjacent-vertex-distinguishing total chromatic numbers of \( P_n^2, P_n^3, P_n^4 \).

**Theorem 2.1.** For graph \( P_n^2 \) with \( n \geq 2 \), then

\[
\chi_{at}(P_n^2) = \begin{cases} 2, & n = 2, \\ 5, & n = 3, 4, 5, \\ 6, & n \geq 6. 
\end{cases}
\]

**Proof.** Case 1. When \( n = 2, 3 \), it is obviously.

Case 3. When \( n = 4 \), there exist adjacent vertices with maximum degree in \( P_4^2 \), so \( \chi_{at}(P_4^2) \geq 5 \). To prove \( \chi_{at}(P_4^2) \geq 5 \), only give a 5-AVDTC of \( P_4^2 \).

Let \( f \) be a mapping from \( V(P_4^2) \cup E(P_4^2) \) to \( \{1, 2, 3, 4, 5\} \) such that:

1. \( f(v_1) = f(v_3v_4) = 1; \) 2. \( f(v_3) = f(v_1v_2) = 2; \)
3. \( f(v_2) = f(v_1v_3) = 3; \) 4. \( f(v_4) = f(v_2v_3) = 4; \) 5. \( f(v_2v_4) = 5. \)

Then \( f \) is a 5-AVDTC of \( P_4^2 \).

Case 4. When \( n = 5 \), there exists a vertex with degree 4 in \( P_5^2 \), so \( \chi_{at}(P_5^2) \geq 5 \). To prove \( \chi_{at}(P_5^2) \leq 5 \), only give a 5-AVDTC of \( P_5^2 \).

Let \( f \) be a mapping from \( V(P_5^2) \cup E(P_5^2) \) to \( \{1, 2, 3, 4, 5\} \) such that:

1. \( f(v_1) = f(v_3v_4) = 1; \) 2. \( f(v_4) = f(v_1v_2) = f(v_3v_5) = 2; \)
3. \( f(v_2) = f(v_1v_3) = f(v_4v_5) = 3; \) 4. \( f(v_5) = f(v_2v_3) = 4; \) 5. \( f(v_3) = f(v_2v_4) = 5. \)

Then \( f \) is a 5-AVDTC of \( P_5^2 \).

Case 5. When \( n \geq 6 \), there exist vertices with maximum degree in \( P_n^2 \), so \( \chi_{at}(P_n^2) \geq 6 \). To prove \( \chi_{at}(P_n^2) \leq 6 \), only give a 6-AVDTC of \( P_n^2 \).

Let \( f \) be a mapping from \( V(P_n^2) \cup E(P_n^2) \) to \( \{1, 2, 3, 4, 5, 6\} \) such that:

\[
f(v_i) = \begin{cases} 1, & i \equiv 1(\text{mod} \ 3), \\
3, & i \equiv 2(\text{mod} \ 3), \\
5, & i \equiv 0(\text{mod} \ 3),
\end{cases}
\]

where \( i \in \{1, 2, \cdots, n\}; \)  

\[
f(v_i v_{i+1}) = \begin{cases} 2, & i \equiv 1(\text{mod} \ 3), \\
4, & i \equiv 2(\text{mod} \ 3), \\
6, & i \equiv 0(\text{mod} \ 3),
\end{cases}
\]
where $i \in \{1, 2, \cdots, n - 1\}$;

$$f(v_i v_{i+2}) = \begin{cases} 
3, & i \equiv 1 \pmod{3}, \\
1, & i \equiv 2 \pmod{3}, \\
5, & i \equiv 0 \pmod{3}, 
\end{cases}$$

where $i \in \{1, 2, \cdots, n - 2\}$.

We will prove $f$ is a AVDTC of $P_n^2$ as follows.

It is obvious $f$ is a proper total coloring so only to prove $C(u) \neq C(v), uv \in E(P_n^2)$. Obviously $C(v_1) \neq C(v_2), C(v_1) \neq C(v_3), C(v_2) \neq C(v_3), C(v_n) \neq C(v_{n-1}), C(v_n) \neq C(v_{n-2}), C(v_{n-1}) \neq C(v_{n-2})$.

For all $i$, $2 \leq i \leq n - 2$, not loss generality, let $2 \leq i < i + 1 < i + 2 \leq n - 2$. We only to prove $C(v_i) \neq C(v_{i+1}), C(v_i) \neq C(v_{i+2}), C(v_{i+1}) \neq C(v_{i+2})$.

If $i \equiv 0 \pmod{3}$, then $C(v_i) = \{4, 5, 6, 3, 1\}$, $C(v_{i+1}) = \{6, 1, 2, 5, 3\}$, $C(v_{i+2}) = \{2, 3, 4, 1, 5\}$. So $C(v_i) \neq C(v_{i+1}), C(v_i) \neq C(v_{i+2}), C(v_{i+1}) \neq C(v_{i+2})$.

If $i \equiv 1 \pmod{3}$, then $C(v_i) = \{6, 1, 2, 5, 3\}$, $C(v_{i+1}) = \{2, 3, 4, 1, 5\}$, $C(v_{i+2}) = \{4, 5, 6, 3, 1\}$. So $C(v_i) \neq C(v_{i+1}), C(v_i) \neq C(v_{i+2}), C(v_{i+1}) \neq C(v_{i+2})$.

If $i \equiv 2 \pmod{3}$, then $C(v_i) = \{2, 3, 4, 1, 5\}$, $C(v_{i+1}) = \{4, 5, 6, 3, 1\}$, $C(v_{i+2}) = \{6, 1, 2, 5, 3\}$. So $C(v_i) \neq C(v_{i+1}), C(v_i) \neq C(v_{i+2}), C(v_{i+1}) \neq C(v_{i+2})$.

So $f$ satisfies $C(u) \neq C(v), uv \in E(P_n^2)$. After all above, we get $f$ is a AVDTC of $P_n^2$. So, $\chi_{at}(P_n^2) = 6$.

The theorem is proved.

**Theorem 2.2.** For graph $P_n^3$ with $n \geq 3$, then

$$\chi_{at}(P_n^3) = \begin{cases} 
4, & n = 3, 4, 5, \\
5, & n = 6, 7, \\
6, & n \geq 8. 
\end{cases}$$

**Proof.** Case 1. When $n = 3, 4$, it is obviously true.

Case 2. When $n = 5$ to prove $\chi_{at}(P_5^3) \leq 4$, only to give a 4-AVDTC of $P_5^3$.

Let $f$ is a mapping from $V(P_5^3) \cup E(P_5^3)$ to $\{1, 2, 3, 4\}$ such that:

1. $f(v_1) = f(v_2v_5) = f(v_4v_5) = 1$;
2. $f(v_5) = f(v_1v_2) = f(v_3v_4) = 2$;
3. $f(v_2) = f(v_4) = 3$;
4. $f(v_3) = f(v_1v_4) = f(v_2v_5) = 4$.

So $f$ is a 4-AVDTC of $P_5^3$. 

Case 3. When \( n = 6, 7 \), as there exist adjacent vertices with maximum degree in \( P^3_6 \) and \( P^3_7 \) so \( \chi_{at}(P^3_6) \geq 5 \), \( \chi_{at}(P^3_7) \geq 5 \). To prove \( \chi_{at}(P^3_6) \leq 5 \), only to give a 5-AVDT of \( P^3_6 \).

Let \( f \) be a mapping \( V(P^3_6) \cup E(P^3_6) \) to \( \{1, 2, 3, 4, 5\} \) such that:

1. \( f(v_1) = f(v_6) = f(v_2v_5) = f(v_3v_4) = 1; \)
2. \( f(v_4) = f(v_1v_2) = 2; \)
3. \( f(v_6) = f(v_4v_5) = f(v_3v_6) = 3; \)
4. \( f(v_7) = f(v_2v_3) = f(v_4v_7) = 4. \)
5. \( f(v_3) = f(v_1v_4) = f(v_5v_6) = 5. \)

So \( f \) is a 5-AVDT of \( P^3_6 \).

To prove \( \chi_{at}(P^3_7) \leq 5 \), only to give a 5-AVDT of \( P^3_7 \).

Let \( f \) be a mapping from \( V(P^3_7) \cup E(P^3_7) \) to \( \{1, 2, 3, 4, 5\} \) such that:

1. \( f(v_1) = f(v_6) = f(v_2v_5) = f(v_3v_4) = 1; \)
2. \( f(v_4) = f(v_1v_2) = f(v_6v_7) = 2; \)
3. \( f(v_7) = f(v_4v_5) = f(v_3v_6) = 3; \)
4. \( f(v_5) = f(v_2v_3) = f(v_4v_7) = 4. \)
5. \( f(v_3) = f(v_1v_4) = f(v_5v_6) = 5. \)

So \( f \) is a 5-AVDT of \( P^3_7 \).

Case 4. When \( n \geq 8 \), as there exist adjacent vertices with maximum degree in \( P^3_n \) so \( \chi_{at}(P^3_n) \geq 6 \). We will prove \( \chi_{at}(P^3_n) \leq 6 \). Only to give a 6-AVDT of \( P^3_n \).

Let \( f \) be a mapping from \( V(P^3_n) \cup E(P^3_n) \) to \( \{1, 2, 3, 4, 5, 6\} \), such that:

1. \( f(v_i) = t_{(k+1)(l+1)}, \) where \( k \equiv i \text{ (mod 3)}, l \equiv i \text{ (mod 2)}, \) and \( t_{(k+1)(l+1)} \) is the \((k+1)\)-th line and \((l+1)\) column element of matrix \( B_1 \), where \( B_1 \) is:

\[
\begin{pmatrix}
3 & 6 \\
1 & 4 \\
5 & 2 \\
\end{pmatrix},
\]

where \( i \in \{1, 2, \cdots, n\} \).

2. \( f(v_i, v_{i+1}) = s_{(k+1)(l+1)}, \) where \( k \equiv i \text{ (mod 3)}, l \equiv i \text{ (mod 2)}, \) and \( s_{(k+1)(l+1)} \) is the \((k+1)\)-th line and \((l+1)\) column element of matrix \( B_2 \), where \( B_2 \) is:

\[
\begin{pmatrix}
1 & 4 \\
5 & 2 \\
3 & 6 \\
\end{pmatrix},
\]

where \( i \in \{1, 2, \cdots, n-1\} \).

3. \( f(v_i, v_{i+3}) = d_{(k+1)(l+1)}, \) where \( k \equiv i \text{ (mod 3)}, l \equiv i \text{ (mod 2)}, \) and \( d_{(k+1)(l+1)} \) is the \((k+1)\)-th line and \((l+1)\) column element of matrix \( B_3 \).
where $B_3$ is:

$$
\begin{pmatrix}
5 & 2 \\
3 & 6 \\
1 & 4
\end{pmatrix},
$$

where $i \in \{1, 2, \cdots, n-3\}$.

The following is similar to Theorem 2.1. The theorem is proved. □

**Theorem 2.3.** For graph $P_n^4$ with $n \geq 4$, then

$$
\chi_{at}(P_n^4) = \begin{cases}
4, & n = 4, 5, \\
5, & n = 6, 7, 8, 9, \\
6, & n \geq 10.
\end{cases}
$$

**Proof.** Case 1. When $n = 4, 5$, it is obviously true.

Case 2. When $n = 6, 7, 8, 9$, only to give a 5-AVDTC of $P_n^4$. It is similar to Theorem 2.1 and Theorem 2.2.

Case 3. When $n \geq 10$, as there exist adjacent vertices with maximum degree in $P_n^4$ so $\chi_{at}(P_n^4) \geq 6$. We will prove $\chi_{at}(P_n^4) \leq 6$. Only to give a 6-AVDTC of $P_n^4$.

Let $f$ be a mapping from $V(P_n^4) \cup E(P_n^4)$ to $\{1, 2, 3, 4, 5, 6\}$, such that:

(1) $f(v_i) = \begin{cases}
1, & i \equiv 1, (\text{ mod } 3), \\
3, & i \equiv 2, (\text{ mod } 3), \\
5, & i \equiv 0, (\text{ mod } 3),
\end{cases}$

where $i \in \{1, 2, \cdots, n\}$;

(2) $f(v_iv_{i+1}) = \begin{cases}
2, & i \equiv 1, (\text{ mod } 3), \\
4, & i \equiv 2, (\text{ mod } 3), \\
6, & i \equiv 0, (\text{ mod } 3),
\end{cases}$

where $i \in \{1, 2, \cdots, n-1\}$;

(3) $f(v_iv_{i+2}) = \begin{cases}
5, & i \equiv 1, (\text{ mod } 3), \\
1, & i \equiv 2, (\text{ mod } 3), \\
3, & i \equiv 0, (\text{ mod } 3),
\end{cases}$

where $i \in \{1, 2, \cdots, n-4\}$.

The following is similar to Theorem 2.1. So the theorem is true. □
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