A REGULARIZED TRACE FORMULA FOR
A DIFFERENTIAL OPERATOR OF SECOND ORDER WITH
UNBOUNDED OPERATOR COEFFICIENTS GIVEN IN
A FINITE INTERVAL

Erdal Gül
Department of Mathematics
Faculty of Arts and Sciences
Yıldız Technical University
Davutpaşa, 34210, Istanbul, TURKEY

Abstract: In this work, a formula for the regularized trace of second order
differential operator given in a finite interval which has unbounded operator
coefficients is found.

AMS Subject Classification: 47A70, 47A55
Key Words: Hilbert space, self-adjoint operator, kernel operator, spectrum,
resolvent

1. Introduction

Let $H$ be a separable Hilbert space and let $H_1 = L_2(H;[0,\pi])$ denotes the set
of all measurable functions $f$ with values in $H$ and such that

$$\int_0^\pi \|f(x)\|^2_H \, dx < \infty.$$

We consider the operators $L_0$ and $L$ in $H_1$ which are formed by the
differential expressions

$$l_0(y) = -y''(x) + Ay(x) \quad \text{and} \quad l(y) = -y''(x) + Ay(x) + Q(x)y(x)$$

and the same boundary conditions $y(0) = y'(\pi) = 0$ respectively. Suppose that
$A$ and $Q(x)$ in the above expressions satisfy the following conditions:
(1) $A : D(A) \to H$ is a self adjoint operator. Moreover, $A \geq I$ and $A^{-1} \in \sigma_{\infty}(H)$, where $I$ is an identity operator in $H$ and $\sigma_{\infty}(H)$ is the set of all compact operators from $H$ to $H$.

(2) For every $x \in [0, \pi]$, $Q(x) : H \to H$ is a self-adjoint compact operator. It is also a kernel operator $(Q(x) \in \sigma_1(H))$.

(3) The functions $\|Q^{(i)}(x)\|_{\sigma_1(H)}$ $(i = 0, 1, 2)$ are bounded and measurable in the interval $[0, \pi]$.

(4) For every $f \in H$, \[ \int_0^\pi (Q(x)f, f)_H \, dx = 0. \]

We denote the norms in $H$ and $H_1$ by $\|\cdot\|_H$ or $\|\cdot\|$ and $\|\cdot\|_1$ respectively and denote the sum of eigenvalues of a kernel operator $K$ by $\text{tr}K = \text{trace}K$. Moreover, $\sigma_1(H)$ denotes the space of kernel operators from $H$ to $H$ as in Cohberg and Krein [6].

The regularized trace formulas of scalar differential operators are studied by Gelfand and Levitan [9], Dikiy [7], Halberg and Kramer [12] and in many other works. In particular, the list of the works on this subject is given in Levitan Sargsyan [13] and Fulton and Prues [8].

The trace formulas for differential operators with operator coefficients are investigated by Adıgüzeloğlu [1], Chalilova [5], Maksudov et al [14], Maksudov et al [15], Adıgüzeloğlu et al [2], Albayrak et al [4], Gül [11], Adıgüzeloğlu and Bakși [3].

In this work we will firstly show how the concept of regularized trace for operator $L$ is constructed and later will obtain a formula for this regularized trace.

## 2. Definition of Regularized Trace for Operator $L$

Let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \leq \cdots$ be the eigenvalues of the operator $A$ and $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots$ be the orthonormal eigenvectors corresponding to these eigenvalues.

Moreover, $D_0$ denotes the set of the functions $y(x)$ in $H_1$ satisfying the conditions:

(1°) $y(x)$ has continuous derivative of the second order with respect to the norm in the space $H$ in the interval $[0, \pi]$.

(2°) $Ay(x)$ is continuous with respect to the norm in the space $H$.

(3°) $y(0) = y'(\pi) = 0$.

Here $D_0$ is dense in $H_1$ and the operator $L'_0 : D_0 \to H_1$ defined by $L'_0 = l_0(y)$ is symmetric. The eigenvalues of this operator are $(\frac{1}{2} + k)^2 + \gamma_j$ ($k = 0, 1, 2, \cdots$).
0, 1, 2, · · · ; j = 1, 2, · · · ) and the orthonormal eigenvectors corresponding to these eigenvalues are $M_k \sin(k + \frac{1}{2})x \cdot \varphi_j \quad (k = 0, 1, 2, · · · ; j = 1, 2, · · · )$, where $M_k = \sqrt{\frac{2}{\pi}}$ for $k = 0, 1, 2, · · ·$.

We can see that the orthonormal eigenvector system of the symmetric operator $L'_0$ is an orthonormal basis in the space $H_1$. Moreover, since this system is closed, the operator $L_0 = \overline{L'_0}$ is self-adjoint, Smirnov [17].

On the other hand, because of the fact that the operator $Q(x)$ satisfies condition (3), we can show that $Q(x)$ is a bounded, self-adjoint operator from $H_1$ to $H_1$. In this case, the operator $L = L_0 + Q$ will be a self-adjoint operator from $D(L) = D(L_0)$ to $H_1$.

Let $R_0^0$ and $R_\lambda$ be resolvents of the operators $L_0$ and $L$ respectively

$$R_\lambda^0 = (L_0 - \lambda I)^{-1}, \quad R_\lambda = (L - \lambda I)^{-1}$$

and let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots$ be the eigenvalues of operator $L_0$. Here every eigenvalue is repeated according to multiplicity number. Since the eigenvalues of operator $L_0$ are $(\frac{1}{2} + k)^2 + \gamma_j \quad (k = 0, 1, 2, · · · ; j = 1, 2, · · · )$ and $\lim_{j \to \infty} \gamma_j = \infty$, we have $\lim_{n \to \infty} \mu_n = \infty$.

This means that the limit of the sequence of eigenvalues $\left\{ \frac{1}{\mu_n - \mu} \right\}_{n=1}^{\infty}$ of operator $R_\mu^0$ is zero. That is,

$$\lim_{n \to \infty} \frac{1}{\mu_n - \mu} = 0 \quad (\mu \neq \mu_n; \quad n = 1, 2, · · · ).$$

On the other hand, for every real $\mu$ which is not an eigenvalue of $L_0$, the operator $R_\mu^0$ is self-adjoint and the system of orthonormal eigenfunctions, $M_k \sin(k + \frac{1}{2})x \cdot \varphi_j \quad (k = 0, 1, 2, · · · ; j = 1, 2, · · · )$ is complete. In this case, it is well known that $R_\mu^0$ is a compact operator, Smirnov [17]. From the formula

$$R_\lambda^0 - R_\mu^0 = (\lambda - \mu)R_\lambda R_\mu^0.$$ 

It is obtained the compactness of operator $R_\lambda^0$ for every real number $\lambda \neq \mu_n \quad (n = 1, 2, · · · )$. Therefore, the operator $L_0$ has pure discrete spectrum. Since the operator $Q$ is a bounded self-adjoint operator, the spectrum of operator $L = L_0 + Q$ is also pure discrete, Smirnov [17].

Let $\lambda_1 \leq \lambda_1 \leq \cdots \leq \lambda_1 \leq \cdots$ be the eigenvalues of operator $L$. For every real $\mu$ which is not an eigenvalue of $L$, we have

$$\lim_{n \to \infty} \frac{1}{\lambda_n - \mu} = 0.$$
Thus self-adjoint operator $R_\mu = (L - \mu I)^{-1}$ is a compact operator, Naimark [16]. From the relation

$$R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu,$$

we obtain that for every $\lambda \neq \lambda_n (n = 1, 2, \cdots)$, $R_\lambda$ is a compact operator.

Let $N(\lambda)$ be the number of eigenvalues of operator $L_0$ which is not greater than a positive number $\lambda$.

If $\gamma_j \sim a_j^\alpha$ as $j \to \infty$ ($a > 0$, $\alpha > 2$) that is, if

$$\lim_{j \to \infty} \frac{\gamma_j}{a_j^\alpha} = 1$$

then it can be found that $N(\lambda) \sim d\lambda\frac{2+\alpha}{\alpha}$, where

$$d = \frac{2}{\alpha a^\alpha} \int_0^\pi \cos^2 t \sin^2 \frac{\alpha-1}{\alpha} t \, dt$$

and so

$$\mu_n \sim d_0 n^{\frac{2}{2+\alpha}} \text{ as } j \to \infty \quad (d_0 = d\frac{2}{2+\alpha}) \quad (2.1)$$

is found, Gorbacuk and Gorbacuk [10].

Now, for the eigenvalues of the operator $L = L_0 + Q$, an asymptotic formula can be found.

Since $Q$ is a self-adjoint operator from $H_1$ to $H_1$ for every $y \in H_1$ we have

$$|(Qy, y)_1| \leq \|Qy\|_1\|y\|_1 \leq \|Q\|_1\|y\|^2_1,$$

or

$$(-\|Q\|y, y)_1 \leq (Qy, y)_1 \leq (\|Q\|y, y)_1.$$

This means that

$$-\|Q\|_1 I \leq Q \leq \|Q\|_1 I.$$

And so

$$L_0 - \|Q\|_1 I \leq L = L_0 + Q \leq L_0 + \|Q\|_1 I.$$

In this case, it is well-known that

$$\mu_n - \|Q\|_1 \leq \lambda_n \leq \mu_n + \|Q\|_1,$$

Smirnov [17]. According to this, we can write

$$1 - \frac{\|Q\|_1}{\mu_n} \leq \frac{\lambda_n}{\mu_n} \leq 1 + \frac{\|Q\|_1}{\mu_n}.$$
By applying limit to each side of this inequality and by considering the equality
\[
\lim_{n \to \infty} \frac{\mu_n}{d_0 n^{2/\alpha}} = 1, \quad \text{we obtain} \quad \lim_{n \to \infty} \frac{\lambda_n}{\mu_n} = 1.
\]
Thus, we have
\[
\lim_{n \to \infty} \frac{\lambda_n}{d_0 n^{2/\alpha}} = \lim_{n \to \infty} \left( \frac{\lambda_n}{\mu_n} \cdot \frac{\mu_n}{d_0 n^{2/\alpha}} \right) = \lim_{n \to \infty} \frac{\lambda_n}{\mu_n} \lim_{n \to \infty} \frac{\mu_n}{d_0 n^{2/\alpha}} = 1,
\]
or as \(n \to \infty\), \(\lambda_n \sim d_0 n^{2/\alpha}\).

Taking together with this last expression and relation (2.1) we write, if \(\gamma_j \sim a j^\alpha \quad (a, \alpha > 0)\) then as \(n \to \infty\)
\[
\mu_n, \lambda_n \sim d_0 n^{2/\alpha}.
\]

By using this formula, it can be showed that the sequence \(\{\mu_n\}_{n=1}^\infty\) has a subsequence \(\{\mu_{n_m}\}_{m=1}^\infty\) such that
\[
\mu_k - \mu_{n_m} \geq d_1 \left( k^{2/\alpha} - n_m^{2/\alpha} \right) \quad (k = n_m, n_m + 1, n_m + 2, \cdots), \quad (2.2)
\]
where \(d_1\) is a positive constant.

The limit given in the form
\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k)
\]
is called the regularized trace of operator \(L\) such that the sequence \(\{\mu_n\}_{n=1}^\infty\) has a subsequence \(\{\mu_{n_m}\}_{m=1}^\infty\) satisfying the inequality (2.2).

If \(\gamma_j \sim a j^\alpha\) as \(j \to \infty\) \((a > 0, \alpha > 2)\) then by using the inequality (2.2) we obtain that
\[
\lambda_{n_m} < \frac{1}{2} \left( \mu_{n_{m+1}} + \mu_{n_m} \right) < \lambda_{n_m} + 1
\]
for large values of \(m\).

Since \(\lambda_n, \mu_n \sim d n^{2/\alpha}\), if \(\alpha > 2\) and \(\lambda \neq \lambda_k \quad (k = 1, 2, \cdots)\) then the series \(\sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda_k - \lambda} \right)\) and \(\sum_{k=1}^{\infty} \left( \frac{\lambda}{\mu_k - \lambda} \right)\) are convergent on the circle \(|\lambda| = b_m = 2^{-1}(\mu_{n_m} + \mu_{n_{m+1}})\) for large values of \(m\). Moreover, since
\[
\lambda_k, \mu_k \sim d_0 k^{2/\alpha} \quad \text{as} \quad k \to \infty
\]
if $\alpha > 2$ and $\lambda \neq \lambda_k, \mu_k \ (k \geq 1, 2, \cdots)$ then the series $\sum_{k=1}^{\infty} |\mu_k - \lambda|^{-1}$ and $\sum_{k=1}^{\infty} |\lambda_k - \lambda|^{-1}$ are convergent. Therefore $R_\lambda^0$ and $R_\lambda$ are the kernel operators and we find that

$$\text{tr}(R_\lambda - R_\lambda^0) = \text{tr}R_\lambda - \text{tr}R_\lambda^0 = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right),$$

see Cohberg and Krein [6].

If we multiply this equation with $\frac{\lambda}{2\pi i}$ and integrate on the circle $|\lambda| = b_m = 2^{-1}(\mu_{n_m} + \mu_{n_m+1})$ then we obtain that

$$\frac{1}{2\pi i} \int_{|\lambda| = b_m} \lambda \text{tr}(R_\lambda - R_\lambda^0) d\lambda$$

$$= \frac{1}{2\pi i} \left( \int_{|\lambda| = b_m} \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda_k - \lambda} \right) d\lambda - \int_{|\lambda| = b_m} \sum_{k=1}^{\infty} \left( \frac{\lambda}{\mu_k - \lambda} \right) d\lambda \right),$$

or

$$\frac{1}{2\pi i} \int_{|\lambda| = b_m} \lambda \text{tr}(R_\lambda - R_\lambda^0) d\lambda$$

$$= - \sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi i} \int_{|\lambda| = b_m} \frac{\lambda}{\lambda - \lambda_k} d\lambda - \frac{1}{2\pi i} \int_{|\lambda| = b_m} \frac{\lambda}{\lambda - \mu_k} d\lambda \right\}. \quad (2.3)$$

Since $\mu_{n_m} < b_m < \mu_{n_m+1}$ it can be shown that for large values of $m$

$$\{\lambda_k, \mu_k\}_{1}^{n_m} \subset K(0, b_m) = \{ \lambda : |\lambda| < b_m \} \quad (2.4)$$

and

$$\lambda_k, \mu_k \in \overline{K(0, b_m)} = \{ \lambda : |\lambda| < b_m \} \quad (k \geq n_m + 1).$$

Therefore,

$$\frac{1}{2\pi i} \int_{|\lambda| = b_m} \frac{\lambda}{\lambda - \mu_k} d\lambda = \begin{cases} \mu_k & \text{if } k < n_m, \\ 0 & \text{if } k \geq n_m + 1, \end{cases}$$
and
\[
\frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{\lambda d\lambda}{\lambda - \lambda_k} = \begin{cases} 
\lambda_k & \text{if } k < n_m, \\
0 & \text{if } k \geq n_m + 1.
\end{cases}
\]

Thus, the equation (2.3) comes to the form
\[
\sum_{k=1}^{n_m} (\lambda_k - \mu_k) = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda \text{tr}(R_\lambda - R^0_\lambda) d\lambda.
\] (2.5)

It is known that \( R_\lambda = R^0_\lambda - R_\lambda Q R^0_\lambda \) for \( \lambda \in \rho(L) \cap \rho(L_0) \). From here, for any natural number \( p \geq 2 \) we obtain the equality
\[
R_\lambda - R^0_\lambda = \sum_{j=1}^{p} (-1)^j R^0_\lambda (QR^0_\lambda)^j + (-1)^{p+1} R_\lambda (QR^0_\lambda)^{p+1}.
\]

If we substitute this expression in (2.5) we find that
\[
\sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda \text{tr} \left[ \sum_{j=1}^{p} (-1)^{j+1} R^0_\lambda (QR^0_\lambda)^j + (-1)^p R_\lambda (QR^0_\lambda)^{p+1} \right] d\lambda,
\]
or
\[
\sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \sum_{j=1}^{p} D_{mj} + D_{m}^{(p)},
\] (2.6)

where
\[
D_{mj} = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_m} \lambda \text{tr} [R^0_\lambda (QR^0_\lambda)^j] d\lambda,
\] (2.7)
\[
D_{m}^{(p)} = \frac{(-1)^p}{2\pi i} \int_{|\lambda|=b_m} \lambda \text{tr} [R_\lambda (QR^0_\lambda)^{p+1}] d\lambda.
\] (2.8)

For every natural number \( j \), it can be shown that the operator function \((QR^0_\lambda)^j\) is analytic according to the norm in \( \sigma_1(H_1) \) in the resolvent region \( \rho(L_0) \) of the operator \( L_0 \). Moreover,
\[
\text{tr}[R^0_\lambda (QR^0_\lambda)^j] = \text{tr}[(QR^0_\lambda)^{j-1}(Q(R^0_\lambda)^2)] = \text{tr}[(QR^0_\lambda)^{j-1} \frac{d}{d\lambda} (QR^0_\lambda)],
\]
\[
\text{tr}\left\{ \frac{d}{d\lambda} [(QR^0_\lambda)^j] \right\} = j\text{tr}[(QR^0_\lambda)^{j-1} \frac{d}{d\lambda} (QR^0_\lambda)].
\]
From the last two relations one obtains
\[ \operatorname{tr}[R_\lambda^0(QR_\lambda^0)^j] = \frac{1}{j} \operatorname{tr}\left\{ \frac{d}{d\lambda}[(QR_\lambda^0)^j]\right\}. \]

If this expression is substituted in (2.7), then we find that
\[ D_{mj} = \frac{(-1)^j+1}{2\pi ij} \int_{|\lambda|=b_m} \lambda \operatorname{tr}\left\{ \frac{d}{d\lambda}[(QR_\lambda^0)^j]\right\} d\lambda \]
\[ = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_m} \operatorname{tr}\left\{ \frac{d}{d\lambda}[\lambda(QR_\lambda^0)^j - (QR_\lambda^0)^j]\right\} d\lambda \]
\[ = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_m} \operatorname{tr}[(QR_\lambda^0)^j] d\lambda + \frac{(-1)^j+1}{2\pi ij} \int_{|\lambda|=b_m} \frac{d}{d\lambda}\operatorname{tr}\{\lambda(QR_\lambda^0)^j]\} d\lambda. \]

By using (2.4), we can show that
\[ \int_{|\lambda|=b_m} \frac{d}{d\lambda}\operatorname{tr}\{\lambda(QR_\lambda^0)^j]\} d\lambda = 0. \]

Because of this, we obtain that
\[ D_{mj} = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_m} \operatorname{tr}[(QR_\lambda^0)^j] d\lambda. \quad (2.9) \]

Let \( \{\psi_q(x)\}_{q=1}^\infty \) be the system of orthonormal eigenvectors corresponding to the eigenvalues \( \{\mu_q(x)\}_{q=1}^\infty \) of operator \( L_0 \) respectively. Since for \( k = 0, 1, 2, \cdots \) and \( j = 1, 2, \cdots \)
\[ M_k \sin\left(\frac{1}{2} + k\right) x \varphi_j \]
is the system of orthonormal eigenvectors corresponding to the eigenvalues
\[ \left(\frac{1}{2} + k\right)^2 + \gamma_j \] of operator \( L_0 \) respectively, we have
\[ \psi_q(x) = M_{k_q} \sin\left(\frac{1}{2} + k_q\right) x \varphi_{q_j} \quad (q = 1, 2, \cdots) \quad (2.10) \]

**Theorem 2.1.** *If the operator function \( Q(x) \) satisfies the conditions (2), (3) and (4) and if as \( j \to \infty \) \( \gamma_j \sim a_j^0 \) \( (0 < a < \infty, 2 < \alpha < \infty) \) then*
\[ \lim_{m \to \infty} D_{m1} = \frac{1}{4}[\operatorname{tr}Q(\pi) - \operatorname{tr}Q(0)]. \]
Proof. From equation (2.9) we have

\[ D_{m1} = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \text{tr}(QR^0_\lambda) \, d\lambda. \tag{2.11} \]

Since, for every \( \lambda \in \rho(L_0) \), \( QR^0_\lambda \) is a kernel operator and \( \{\psi_q(x)\}_{q=1}^{\infty} \) is an orthonormal basis of the space \( H_1 \) then the equality

\[ \text{tr}(QR^0_\lambda) = \sum_{q=1}^{\infty} (QR^0_\lambda \psi_q, \psi_q) \]

holds.

If we substitute this expression in (2.11) and consider the relation

\[ R^0_\lambda \psi_q = (L_0 - \lambda I)^{-1} \psi_q = (\mu_q - \lambda)^{-1} \psi_q, \]

we obtain that

\[ D_{m1} = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \left[ \sum_{q=1}^{\infty} (QR^0_\lambda \psi_q, \psi_q) \right] \, d\lambda \]

\[ = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \left[ \sum_{q=1}^{\infty} \frac{1}{\mu_q - \lambda} (Q \psi_q, \psi_q) \right] \, d\lambda = \sum_{q=1}^{\infty} (Q \psi_q, \psi_q) \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{d\lambda}{\lambda - \mu_q}. \]

From (2.10) and by using the equality

\[ \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{d\lambda}{\lambda - \mu_q} = \begin{cases} 1, & \text{if } q \leq n_m, \\ 0, & \text{if } q > n_m, \end{cases} \]

we find that

\[ D_{m1} = \sum_{q=1}^{n_m} (Q \psi_q, \psi_q) = \sum_{q=1}^{n_m} \int_0^\pi (Q(x) \psi_q(x), \psi_q(x)) \, dx \]

\[ = \sum_{q=1}^{n_m} \int_0^\pi (Q(x) M_{k_q} \sin \left( \frac{1}{2} + k \right) x \varphi_j, M_{k_q} \sin \left( \frac{1}{2} + k \right) x \varphi_j) \, dx \]

\[ = \sum_{q=1}^{n_m} M_{k_q}^2 \int_0^\pi \sin^2 \left( \frac{1}{2} + k_q \right) x (Q(x) \varphi_j, \varphi_j) \, dx \]

\[ = \sum_{q=1}^{n_m} M_{k_q}^2 \int_0^\pi \sin^2 \left( \frac{1}{2} + k_q \right) x (Q(x) \varphi_j, \varphi_j) \, dx \]
\[ E. \text{ G"ul} \]

\[ \sum_{q=1}^{n_m} M_{k_q}^2 \int_0^\pi (1 - \cos(2k_q + 1)x)(Q(x)\varphi_{j_q}, \varphi_{j_q}) \, dx. \]

From the condition (4) for \( Q(x) \) and since

\[ M_k = \sqrt{2\pi^{-1}} \quad (k = 0, 1, 2, \cdots), \]

it follows that

\[ D_{m1} = -\frac{1}{\pi} \sum_{q=1}^{n_m} \int_0^\pi \cos(2k_q + 1)x(Q(x)\varphi_{j_q}, \varphi_{j_q}) \, dx, \quad (2.12) \]

it can be proved that the series

\[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^\pi (Q(x)\varphi_j, \varphi_j) \cos(2k + 1)x \, dx, \]

absolutely converges. It is known that

\[ \lim_{m \to \infty} \sum_{q=1}^{n_m} \int_0^\pi \left( \cos(2k_q + 1)x(Q(x)\varphi_{j_q}, \varphi_{j_q}) \right) \, dx \]

\[ = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^\pi (Q(x)\varphi_j, \varphi_j) \cos(2k + 1)x \, dx. \]

By applying limit, as \( m \to \infty \), to the equation (2.12) and by considering the last relation above

\[ \lim_{m \to \infty} D_{m1} = -\frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^\pi (Q(x)\varphi_j, \varphi_j) \cos(2k + 1)x \, dx \]

is found. By subtracting and adding the expression

\[ (Q(x)\varphi_j, \varphi_j) \cos 2kx, \]

into the integral one obtains

\[ \lim_{m \to \infty} D_{m1} = \]
\[-\frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^{\pi} [(Q(x)\varphi_j, \varphi_j)(\cos(2k+1)x + \cos 2kx) - (Q(x)\varphi_j, \varphi_j) \cos 2kx] \, dx.\]

It can be written the expression
\[-\frac{1}{\pi} \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos rx \, dx,
\]

instead of first term in the right side of this equality. Thus, we have
\[
\lim_{m \to \infty} D_{m1} = \left( -\frac{1}{\pi} \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos rx \, dx \right) + \frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos 2kx \, dx.
\]

We can write this equation in the form
\[
\lim_{m \to \infty} D_{m1} = \left( -\frac{1}{\pi} \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos rx \, dx \right) + \frac{1}{2\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos kx \, dx + (-1)^k \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos kx \, dx \right],
\]

and so we have
\[
\lim_{m \to \infty} D_{m1} = \left( -\frac{1}{2} \sum_{j=1}^{\infty} \left\{ \sum_{r=1}^{\infty} \left[ \frac{2}{\pi} \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos rx \, dx \right] \cos k0 \right\} \right) + \frac{1}{4} \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\infty} \left[ \frac{2}{\pi} \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos kx \, dx \right] \cos k0 \right\} + \sum_{k=1}^{\infty} \left[ \frac{2}{\pi} \int_0^{\pi} (Q(x)\varphi_j, \varphi_j) \cos kx \, dx \right] \cos k\pi.
\]

Since \(Q(x)\) satisfies the condition (4), the sum with respect to \(r\) in the first term on the right side of this expression is the value at 0 of Fourier series according to functions \(\{\cos rx\}_{r=0}^{\infty}\) in the interval \([0, \pi]\) of the function
\[(Q(x)\varphi_j, \varphi_j)_H\] having the derivative of second order. Similarly, the sums in the second term with respect to \(k\) are the values at the points 0 and \(\pi\) respectively of Fourier series with respect to the functions \(\{\cos mx\}_{m=0}^\infty\) in the same interval of that function.

For this reason we obtain that
\[
\lim_{m \to \infty} D_{m1} = -\frac{1}{2} \sum_{j=1}^{\infty} [(Q(0)\varphi_j, \varphi_j)] + \frac{1}{4} \sum_{j=1}^{\infty} [(Q(0)\varphi_j, \varphi_j) + (Q(\pi)\varphi_j, \varphi_j)]
\]
\[
= \frac{1}{4} [\text{tr} Q(\pi) - \text{tr} Q(0)].
\]

This proves the theorem.

\[\square\]

3. A Formula for the Regularized Trace of the Operator \(L\)

In this section, we obtain a formula for the limit \(\lim_{m \to \infty} \sum_{k=1}^{m} (\lambda_k - \mu_k)\) that we called the regularized trace of the operator \(L\) in the previous section.

Since \(\{\psi_q(x)\}_{q=1}^\infty\) is an orthonormal basis of the space \(H_1\), for every \(y \in H_1\) we have
\[
y = \sum_{k=1}^{\infty} (y, \psi_k) \psi_k \quad \text{and} \quad R^0_L y = \sum_{k=1}^{\infty} (y, \psi_k) R^0_L \psi_k.
\]

According to this, we obtain that
\[
QR^0_L \psi_{k1} = Q\psi_{k1} / (\mu_{k1} - \lambda),
\]
\[
(QR^0_L)^2 \psi_{k1} = QR^0_L \left( \frac{Q\psi_{k1}}{\mu_{k1} - \lambda} \right) = \frac{1}{\mu_{k1} - \lambda} (Q\psi_{k1})
\]
\[
= \frac{1}{\mu_{k1} - \lambda} \left[ \sum_{k_2=1}^{\infty} (Q\psi_{k1}, \psi_{k2}) / (\mu_{k2} - \lambda) \psi_{k2} \right] = \frac{1}{\mu_{k1} - \lambda} \sum_{k_2=1}^{\infty} (Q\psi_{k1}, \psi_{k2}) / (\mu_{k2} - \lambda) Q\psi_{k2}.
\]

Similarly, it follows that
\[
(QR^0_L)^3 \psi_{k1} = \frac{1}{\mu_{k1} - \lambda} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} (Q\psi_{k1}, \psi_{k2}) / (\mu_{k2} - \lambda) (Q\psi_{k2}, \psi_{k3}) / (\mu_{k3} - \lambda) Q\psi_{k3},
\]

\[\ldots\]
\[(QR^0_\Lambda)^n \psi_k = \frac{1}{\mu_k - \lambda} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left[ \prod_{j=1}^{n-1} \frac{(Q\psi_{k_j}, \psi_{k_{j+1}})}{\mu_{k_{j+1}} - \lambda} \right] Q\psi_k. \quad (3.1)\]

This shows that

\[\text{tr} \ (QR^0_\Lambda)^n = \sum_{k_1=1}^{\infty} \left( (QR^0_\Lambda)^n \psi_{k_1}, \psi_{k_1} \right)_1\]

\[= \sum_{k_1=1}^{\infty} \left( \frac{1}{\mu_{k_1} - \lambda} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left[ \prod_{j=1}^{n-1} \frac{(Q\psi_{k_j}, \psi_{k_{j+1}})}{\mu_{k_{j+1}} - \lambda} \right] Q\psi_{k_n}, \psi_{k_n} \right) = \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left[ \prod_{j=1}^{n-1} \frac{(Q\psi_{k_j}, \psi_{k_{j+1}})}{\mu_{k_{j+1}} - \lambda} \right] \cdot (3.2)\]

where

\[\rho(j) = \begin{cases} j & \text{if } j < n, \\ 0 & \text{if } j = n. \end{cases}\]

By using this last equation (3.2), equation (2.9) comes to the form

\[D_{mj} = \frac{(-1)^j}{2\pi ij} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left[ \left( \int \left[ \prod_{q=1}^{j} (\mu_{k_q} - \lambda)^{-1} d\lambda \right] \prod_{q=1}^{j} (Q\psi_{k_q}, \psi_{k_{\rho(q)+1}}) \right) \right], \quad (3.3)\]

or

\[D_{mj} = \frac{(-1)^j}{2\pi ij} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left[ \left( \int \left[ \prod_{q=1}^{j} (\mu_{k_q} - \lambda)^{-1} d\lambda \right] \prod_{q=1}^{j} (Q\psi_{k_q}, \psi_{k_{\rho(q)+1}}) \right) \right], \quad (3.4)\]

where the symbol “*” denotes that these are numbers among \( \mu_{k_1}, \mu_{k_2}, \cdots, \mu_{k_j} \) less than or greater than \( b_{mj} \).

From equation (3.4) it can be seen that
\[ D_{m2} = \frac{1}{4\pi i} \sum_{k_1=1}^{n_m} \sum_{k_2=n_m+1}^{\infty} \left[ \int_{|\lambda|=b_m} \frac{d\lambda}{(\lambda - \mu_{k_1})(\lambda - \mu_{k_2})} \right] (Q\psi_{k_1}, \psi_{k_2})_1 (Q\psi_{k_2}, \psi_{k_1})_1 + \frac{1}{4\pi i} \sum_{k_1=n_m+1}^{\infty} \sum_{k_2=1}^{n_m} \left[ \int_{|\lambda|=b_m} \frac{d\lambda}{(\lambda - \mu_{k_1})(\lambda - \mu_{k_2})} \right] (Q\psi_{k_1}, \psi_{k_2})_1 (Q\psi_{k_2}, \psi_{k_1})_1 \]

\[ = \frac{1}{2\pi i} \sum_{k=1}^{n_m} \sum_{j=n_m+1}^{\infty} \left[ \int_{|\lambda|=b_m} \frac{d\lambda}{(\lambda - \mu_k)(\lambda - \mu_j)} \right] (Q\psi_k, \psi_j)_1 (Q\psi_j, \psi_k)_1 + 1 \]

Since \( k \leq n_m \) and \( j \geq n_m + 1 \) we have

\[ \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{d\lambda}{(\lambda - \mu_k)(\lambda - \mu_j)} = \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{1}{(\mu_k - \mu_j)} \left[ \frac{1}{(\lambda - \mu_k)} - \frac{1}{(\lambda - \mu_j)} \right] d\lambda = \frac{1}{\mu_k - \mu_j} \left[ \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{d\lambda}{(\lambda - \mu_k)} - \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{d\lambda}{(\lambda - \mu_j)} \right] = \frac{1}{\mu_k - \mu_j}. \]

This implies that

\[ |D_{m2}| \leq \|Q\|_1^2 \Omega_m. \]  

(3.6)
where
\[ \Omega_m = \sum_{j=n_m+1}^{\infty} (\mu_j - \mu_{n_m})^{-1} \quad (m = 1, 2, \cdots). \]

In a similar form, by using (3.4) and considering the inequalities (2.2) and
\[ x^{1+\delta} - (x-1)^{1+\delta} > x^\delta \quad (x > 1, \; \delta > 0), \]
it can be shown that
\[ |D_{m3}| \leq \|Q\|^2 \Omega_m (\Omega_m + 4d_1^{-1}n_m^{-\delta}), \quad d_1 = \frac{d_0}{4}, \; \delta = \frac{\alpha - 2}{\alpha + 2}. \tag{3.7} \]

Moreover, if \( \gamma_j \sim a_j^\alpha \) as \( j \to \infty \) \((0 < a < \infty, 2 < \alpha < \infty)\) then it is satisfied that
\[ \|R_\lambda\|_{\sigma_1(H_1)} < \text{const.} n_m^{-\delta} \quad (\delta = \frac{\alpha - 2}{\alpha + 2}), \tag{3.8} \]
on the circle \(|\lambda| = b_m\).

On the other hand, since \( Q(x) \) is a bounded self-adjoint operator from \( H_1 \) to \( H_1 \) and
\[ \mu_n - \|Q\|_1 \leq \lambda_n \leq \mu_n + \|Q\|_1, \]
if \( \gamma_j \sim a_j^\alpha \) as \( j \to \infty \) \((a > 0, \; \alpha > 2)\) then for the large values of \( m \), the inequality
\[ \|R_\lambda\|_1 < \frac{d_1}{4} n_m^\delta \quad (\delta = \frac{\alpha - 2}{\alpha + 2}), \tag{3.9} \]
is also satisfied on the circle \(|\lambda| = b_m\).

Now we are ready to prove the following theorem.

**Theorem 3.1.** Suppose that \( \gamma_j \sim a_j^\alpha \) as \( j \to \infty \) \((0 < a < \infty, 2 < \alpha < \infty)\). If the operator function \( Q(x) \) satisfies the conditions (2) and (3) then for \( j \geq 2 \)
\[ \lim_{m \to \infty} D_{mj} = 0. \]

**Proof.** We can give a restriction to for the magnitude of the expression \( D_{m1} \): From (2.9) we have
\[
|D_{mj}| \leq \frac{1}{2\pi j} \int_{|\lambda| = b_m} |\text{tr} (QR^0_\lambda)^j||d\lambda| \leq \frac{1}{2\pi j} \int_{|\lambda| = b_m} \|QR^0_\lambda\|_{\sigma_1(H_1)}^j |d\lambda|
\]
\[
\leq \frac{1}{2\pi j} \int_{|\lambda| = b_m} \|QR^0_\lambda\|_{\sigma_1(H_1)} \|QR^0_\lambda\|_{\sigma_1(H_1)}^{j-1} |d\lambda|
\]
\[
\leq \frac{1}{2\pi j} \int_{|\lambda|= b_m} \|Q\|_1 \|R_0^\lambda\|_{\sigma_1(H_1)} \|QR_0^\lambda\|^{j-1}_{\sigma_1(H_1)} |d\lambda|
\]

\[
\leq \frac{1}{2\pi j} \int_{|\lambda|= b_m} \|Q\|^2_1 \|R_0^\lambda\|_{\sigma_1(H_1)} \|R_0^\lambda\|^{j-1}_{\sigma_1(H_1)} |d\lambda|. \quad (3.10)
\]

If we choose \(Q(x) \equiv 0\) identically then we have \(R_\lambda = R_0^\lambda\).

It means that
\[
\|R_\lambda\|_1 < \frac{d_1}{4} n_m^{-\delta} \quad (\delta = \frac{\alpha - 2}{\alpha + 2}). \quad (3.11)
\]

From (3.8), (3.10) and (3.11), it follows that
\[
|D_{mj}| \leq \text{const.} \int_{|\lambda|= b_m} n_m^{1-\delta} n_m^{-\delta(j-1)} |d\lambda| \leq \text{const.} \mu_{n_m} n_m^{1-\delta j}.
\]

Since \(\mu_{n_m} \leq \text{const.} n_m^{1+\delta}\) we have
\[
|D_{mj}| \leq \text{const.} n_m^{2-\delta(j-1)}.
\]

Clearly, if \(j > 1 + 2\delta^{-1}\) then
\[
\lim_{m \to \infty} D_{mj} = 0.
\]

For \(j = 2\) since \(\lim_{m \to \infty} \Omega_m = 0\), from (3.6) we obtain that
\[
\lim_{m \to \infty} D_{m2} = 0.
\]

Similarly, from (3.7) we see that
\[
\lim_{m \to \infty} D_{m3} = 0.
\]

It follows that for \(j = 2, 3, \ldots, |2\delta^{-1}| + 1\)
\[
\lim_{m \to \infty} D_{mj} = 0 \quad \square
\]

Our main result in this paper is given by the following theorem.
Theorem 3.2. Suppose that $\gamma_j \sim aj^\alpha$ as $j \to \infty (0 < a < \infty, 2 < \alpha < \infty)$. If the operator function $Q(x)$ satisfies the conditions (2), (3) and (4) then the formula
\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4} \left[ \text{tr} Q(\pi) - \text{tr} Q(0) \right]
\]
is satisfied.

Proof. By using Theorem 2.1 and Theorem 3.1, from equation (2.6) we write that
\[
\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4} \left[ \text{tr} Q(\pi) - \text{tr} Q(0) \right] + \lim_{m \to \infty} D_m^{(p)}.
\] (3.12)

Here, by (2.8) we have
\[
D_m^{(p)} = \frac{(-1)^p}{2\pi i} \int_{|\lambda|=b_m} \lambda \text{tr} \left[ R_\lambda (QR_\lambda^0)^{p+1} \right] d\lambda.
\]

We can give a restriction to the magnitude of this expression as in the following
\[
|D_m^{(p)}| \leq \frac{1}{2\pi i} \int_{|\lambda|=b_m} |\lambda| |\text{tr} \left[ R_\lambda (QR_\lambda^0)^{p+1} \right]| d\lambda
\]
\[
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda \|_1 \| (QR_\lambda^0)^{p+1} \|_{\sigma_1(H_1)} |d\lambda|
\]
\[
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda \|_1 \| (QR_\lambda^0)^p \|_{\sigma_1(H_1)} \| (QR_\lambda^0)^{p+1} \|_{\sigma_1(H_1)} |d\lambda|
\]
\[
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda \|_1 \| Q \|_p \| R_\lambda^0 \|_1 \| Q \|_p \| R_\lambda^0 \|_{\sigma_1(H_1)} |d\lambda|
\]
\[
\leq b_m \int_{|\lambda|=b_m} \| R_\lambda \|_1 \| Q \|_p \| R_\lambda^0 \|_1 \| Q \|_p \| R_\lambda^0 \|_{\sigma_1(H_1)} |d\lambda|.
\]

From (3.8) and (3.9) we obtain that
\[
|D_m^{(p)}| \leq \text{const.} b_m^2 n_m^{-(p+1)\delta} n_m^{1-\delta}.
\]
Since $b_m \leq \text{const.} \, n_m^{1+\delta}$ then we have

$$|D_m^{(p)}| \leq \text{const.} \, n_m^{-(p+2)\delta+1} n_m^{2(1+\delta)} = \text{const.} \, n_m^{3-p\delta}.$$  

It follows that for $p > 3\delta^{-1}$

$$\lim_{m \to \infty} D_m^{(p)} = 0.$$  

If we substitute this result in equation (3.12) we obtain the regularized trace formula of operator $L$ as

$$\lim_{m \to \infty} \sum_{k=1}^{n_m} (\lambda_k - \mu_k) = \frac{1}{4} [\text{tr} Q(\pi) - \text{tr} Q(0)].$$  

The proof is done.

**Acknowledgments**

We offer our deepest gratitude to Professor Dr. Ehliman Adigüzelov for his sincerity and the thoughtfulness he showed to us during the exploration of this work.

**References**


