A GENERALIZED ‘USEFUL’ INACCURACY OF ORDER \( \alpha \) AND TYPE \( \beta \) AND CODING THEOREMS

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Abstract: Useful inaccuracy measures and mean codeword lengths are well known in the literature of information theory. In this communication, a new generalized ‘useful’ inaccuracy of order \( \alpha \) and type \( \beta \) has been proposed and coding theorem has been established by considering the said measure and a generalized average ‘useful’ codeword length. Our motivation for studying this is that it generalizes some results which already existing in the literature.

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1. Introduction

Consider the model given below for a finite random experiment scheme having \((x_1, x_2, ..., x_n)\) as a complete system of events, happening with respective probabilities \(P = (p_1, p_2, ..., p_n)\) and credited with utilities \(U = (u_1, u_2, ..., u_n)\), \(u_i > 0, i = 1, 2, ..., n\). Denote

\[
\chi = \left[ x_1 x_2 ..... x_n \ p_1 p_2 ..... p_n \ u_1 u_2 ..... u_n \right], \tag{1.1}
\]

we call (1.1) as utility information scheme.

Let \(Q = (q_1, q_2, ..., q_n)\) be the predicted distribution having the utility distribution \((u_1, u_2, ..., u_n)\). Taneja and Tuteja [14] have suggested and characterized the ‘useful’ inaccuracy measure.

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\[ I (P; Q; U) = -\sum_{i=1}^{n} u_i p_i \log q_i . \]  

(1.2)

By considering weighted mean codeword length, see [6]

\[ L (U) = \frac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i} . \]  

(1.3)

Taneja and Tuteja [14] derived the lower and upper bounds on \( L (U) \) in terms of \( I (P; Q; U) \).

Bhatia [3] defined the ‘useful’ average code lengths of order \( \alpha \) as

\[ L_\alpha (U) = \frac{1}{D^{\frac{\alpha-1}{\alpha}}} - 1 \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i q_i^{\frac{\alpha-1}{\alpha}}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha}} \right] , \]  

(1.4)

where \( \alpha > 0 \ (\neq 1) \) and \( \sum_{i=1}^{n} p_i \leq 1, i = 1, 2, \ldots, n \) and \( D \) is the size of the code alphabet. He also derived the bounds for the ‘useful’ average code length of order \( \alpha \) and is given by

\[ I_\alpha (P; Q; U) = \frac{1}{D^{\frac{\alpha-1}{\alpha}}} - 1 \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i q_i^{\frac{\alpha-1}{\alpha}}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{\alpha}} \right] , \]  

(1.5)

where \( \alpha > 0 \ (\neq 1) \) and \( p_i \geq 0, \sum_{i=1}^{n} p_i \leq 1, i = 1, 2, \ldots, n \) and \( D \) is the size of the code alphabet.

Under the condition

\[ \sum_{i=1}^{n} p_i q_i^{-1} D^{-l_i} \leq 1 , \]  

(1.6)

where \( D \) is the size of the code alphabet. Inequality (1.6) is a generalized Kraft’s inequality [4]. A code satisfying generalized Kraft’s inequality would be termed as personal probability code.

Longo [10], Gurdial and Pessoa [5], Autar and Khan [1], Jain and Tuteja [8], Taneja et al [15], Hooda and Bhaker [7], Bhatia [3] and Singh, Kumar and...
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Tuteja [11] considered the problem of ‘useful’ information measure and used it studying the noiseless coding theorems for sources involving utilities.

In the next section, we shall study some coding theorems for a generalized ‘useful’ inaccuracy of order $\alpha$ and type $\beta$ for incomplete probability distribution.

2. Coding Theorems

Consider the function

$$I_\alpha (P;Q;U) = \frac{1}{D^{\frac{\alpha-1}{\alpha}}} - 1 \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^\beta}{\sum_{i=1}^{n} u_i p_i^{\alpha-1}} \right)^{\frac{1}{\alpha}} \right]$$

where $\alpha > 0 (\neq 1)$, $\beta > 0$, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i \leq 1$, $i = 1, 2, ..., n$ and $D$ is the size of the code alphabet.

(a) For $\alpha \to 1$, $\beta = 1$ and $\sum_{i=1}^{n} p_i = 1$, the measure (2.1) reduces to Taneja and Tuteja [15] measure of ‘useful’ inaccuracy.

(b) For $\beta = 1$, $p_i = q_i$, $\forall i = 1, 2, ..., n$, the measure (2.1) reduces to the measure given by Autar and Khan [1] ‘useful’ information measure.

(c) For $\alpha \to 1$, $\beta = 1$ and $p_i = q_i$, $\forall i = 1, 2, ..., n$, the measure (2.1) reduces to Belis and Guiasu [2] measure of ‘useful’ information for incomplete probability distribution. Further, when utility aspect of the scheme is ignored, the measure reduces to Shannon [12] measure of entropy.

(d) When the probability distribution is complete and the utility aspect of the scheme is ignored as well as $\alpha \to 1$, $\beta = 1$. The measure (2.1) becomes Keridge’s [9] measure of inaccuracy. We call (2.1) as generalized ‘useful’ inaccuracy of order $\alpha$ and type $\beta$ for incomplete probability distribution.

Further, consider a generalized ‘useful’ mean length credited with utilities and probabilities as

$$L_\alpha^\beta (U) = \frac{1}{D^{\frac{\alpha-1}{\alpha}}} - 1 \left[ 1 - \frac{1}{\sum_{i=1}^{n} p_i^\beta} \left( \frac{\sum_{i=1}^{n} u_i}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right)^{\frac{1}{\alpha}} \right] D^{-\frac{\alpha-1}{\alpha}}$$

where $\alpha > 0 (\neq 1)$, $\beta > 0$, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i \leq 1$, $i = 1, 2, ..., n$ and $D$ is the size of the code alphabet.
For $\alpha \to 1, \beta = 1$, the measure (2.2) reduces to ‘useful’ mean length $L(U)$ of the code, given by Guiasu and Picard [6].

(b) when the utility aspect of the scheme is ignored by taking $u_i = 1$, $\forall i = 1, 2, .., n$, $\sum_{i=1}^{n} p_i = 1$ and $\alpha \to 1, \beta = 1$, the mean length (2.2) becomes optimal code length identical to Shannon, see [12].

Now we find the bounds for $L^\beta_\alpha (U)$ in terms of $I^\beta_\alpha (P;Q;U)$ under the condition

$$\sum_{i=1}^{n} p_i^\beta q_i^{\alpha-1} D^{-l_i} \leq 1,$$

where $D$ is the size of the code alphabet.

**Theorem 2.1.** For all integers $D$ ($D > 1$), let $l_i$ satisfy (2.3), then the generalized average ‘useful’ codeword length satisfies

$$L^\beta_\alpha (U) \geq I^\beta_\alpha (P;Q;U),$$

and the equality holds iff

$$l_i = - \log \left( \frac{u_i q_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\alpha-1}} \right).$$

Proof. By Hölder’s inequality, see [13]

$$\sum_{i=1}^{n} x_i y_i \geq \left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1, p < 1 (\neq 0)$ and $x_i, y_i > 0, i = 1, 2, .., n$, we see the equality holds if and only if there exists a positive constant $c$ such that

$$x_i^p = c y_i^q.$$

Making the substitution

$$p = \frac{\alpha - 1}{\alpha}, \quad q = 1 - \alpha, \quad x_i = p_i^\beta \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{\beta}} D^{-l_i}.$$
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\[ y_i = p_i^{\frac{\beta}{1-\alpha}} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{1-\alpha}} q_i^{-1} \]

in (2.6), using (2.3) and after making suitable operations we get (2.4) for \((D^{\frac{\alpha-1}{\alpha}} - 1) \neq 0\) according as \(\alpha \neq 1\).

**Theorem 2.2.** For every code with lengths \(\{l_i\}, i = 1, 2, ..., n\) of Theorem 2.1, \(L_\alpha^\beta (U)\) can satisfy the inequality

\[ L_\alpha^\beta (U) < I_\alpha^\beta (P;Q;U) D^{\frac{1-\alpha}{\alpha}} + \frac{1 - D^{\frac{1-\alpha}{\alpha}}}{D^{\frac{\alpha-1}{\alpha}} - 1}. \]  

(2.8)

**Proof.** Let \(l_i\) be the positive integer satisfying the inequality

\[ -\log \left( \frac{n}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) \leq l_i < -\log \left( \frac{n}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) + 1. \]  

(2.9)

Consider the intervals

\[ \delta_i = \left[ -\log \left( \frac{n}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right), -\log \left( \frac{n}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) + 1 \right] \]  

(2.10)

of length 1. In every \(\delta_i\), there lies exactly one positive integer \(l_i\) such that

\[ 0 < -\log \left( \frac{n}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) \leq l_i < -\log \left( \frac{n}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) + 1. \]  

(2.11)

We will first show that the sequences \(l_1, l_2, ..., l_n\), thus defined satisfy (2.3). From (2.11), we have

\[ -\log \left( \frac{n}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{-1}} \right) \leq l_i, \]
or
\[
\left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\alpha-1}} \right) \geq D^{-l_i}.
\]

Multiply both sides by \( p_i^{\beta-1} \) and summing over \( i = 1, 2, \ldots, n \), we get (2.3). The last inequality of (2.11) gives
\[
l_i < -\log \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\alpha-1}} \right) + 1,
\]
or
\[
D^{-l_i} < \left( \frac{u_i q_i^\alpha}{\sum_{i=1}^{n} u_i p_i^\beta q_i^{\alpha-1}} \right)^{-1} D.
\]

Multiplying both sides by \( p_i^{\beta} \left( \frac{u_i}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{\alpha}} \) and summing over \( i, i = 1, 2, \ldots, n \) and after suitable operations, we get
\[
L_\alpha^\beta(U) < I_\alpha^\beta(P; Q; U) D^{\frac{1-\alpha}{\alpha}} + \frac{1 - D^{\frac{1-\alpha}{\alpha}}}{D^{\frac{\alpha-1}{\alpha}} - 1}. \quad \square \quad (2.12)
\]

**Remark.** For \( 0 < \alpha < 1 \) and since \( D \geq 2 \), from (2.12), we have
\[
\frac{1 - D^{\frac{1-\alpha}{\alpha}}}{D^{\frac{\alpha-1}{\alpha}} - 1} > 1
\]
from which it follows that the upper bound of \( L_\alpha^\beta(U) \) in (2.8) is greater than unity.

**References**


