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APPROXIMATING MEASURES INVARIANT UNDER PIECEWISE CONVEX TRANSFORMATIONS

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Abstract: Let $S : [0, 1] \longrightarrow [0, 1]$ be a piecewise convex transformation satisfying some conditions which guarantee the existence of an absolutely continuous invariant probability measure. In this paper, the first order Markov finite approximation method is used for computing the invariant measure, using some compactness argument for L^1-spaces.

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1. Introduction

Let $I = [0, 1]$ and $S : I \longrightarrow I$ be a nonsingular transformation. For $S : I \longrightarrow I$ which is piecewise C^2 and stretching, or piecewise C^2 and convex with a strong repellor, the existence of an invariant probability measure which is absolutely continuous with respect to the Lebesgue measure has been proved by Lasota and Yorke [6], [7].

For the computation of absolutely continuous invariant measures under one-dimensional piecewise C^2 stretching mappings on $[0, 1]$, Li [8] proved the convergence of a piecewise constant approximation method first proposed by Ulam [10]. Some high order methods have been developed in [1], [2], [4].

In this paper, we propose the first order Markov finite approximation method to compute absolutely continuous invariant measures associated with piecewise convex mappings with a strong repellor. We explore some new properties of first order Markov finite approximations and establish their convergence after we show that such approximations also preserve the monotonicity of the func-

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tion, using the fact that any subset of monotonic functions in $L^1(0,1)$ that is uniformly bounded in the L^∞-norm must be precompact in $L^1(0,1)$.

After giving some preliminaries in the next section, we prove the convergence of the first order Markov finite approximation method in Section 3.

2. Preliminaries (Piecewise Convex Transformations)

Let $S:[0,1]\longrightarrow[0,1]$ be a measurable transformation such that $m(A)=0$ implies $m(S^{-1}(A))=0$ for every Lebesgue measurable subset of $[0,1]$ with m the Lebesgue measure. The operator $P_S:L^1(0,1)\longrightarrow L^1(0,1)$ defined by

$$\int_A P_S f\,dm=\int_{S^{-1}(A)} f\,dm$$

for every measurable $A\subset[0,1]$ is called the Frobenius-Perron operator associated with S. It is well-known (see [5]) that for $f\geq 0$ and $\|f\|\equiv\int_0^1|f|dm=1$, the absolutely continuous probability measure

$$\mu(A)=\int_A f\,dm \quad \forall \text{ measurable sets } A\subset[0,1]$$

is invariant under S if and only if f is a fixed point of P_S, i.e., $P_Sf=f$. f is called the density of μ.

A basic and simple property of P_S which is useful in this paper is given below without proof. For more detailed discussion of P_S, see the monograph of Lasota and Mackey [5].

Proposition 2.1. *P_S is a positive operator that preserves the L^1-norm of nonnegative functions. Thus P_S is a Markov operator.*

Now we state the existence theorem for a class of mappings that are piecewise convex with a strong repellor, the proof of which is referred to [5] or [7].

Theorem 2.1. *Let $S:[0,1]\longrightarrow[0,1]$ satisfy that:*

(i) there is a partition $0=a_0<a_1<\cdots<a_r=1$ of $[0,1]$ such that $S|_{[a_{i-1},a_i)}$ is of C^2 for each $i=1,\ldots,r$;

(ii) $S'(x)>0$ and $S''(x)\geq 0$ for all $x\in[0,1)$, where $S'(a_i)$ and $S''(a_i)$ are right derivatives;

(iii) for each integer $i=1,\ldots,r$, $S(a_{i-1})=0$; and

(iv) $S'(0)>1$.

Then there exists a unique absolutely continuous invariant probability measure with density f^*. Furthermore, $\lim_{n\to\infty} P_S^n f = f^*$ for every density $f \in L^1(0,1)$.

Remark 2.1. It was shown in [5] that for the piecewise convex transformation S with a strong repellor, the Frobenius-Perron operator P_S leaves the set of nonnegative decreasing functions invariant, and for any decreasing density $f \in L^1(0,1)$,

$$P_S f(x) \le \frac{1}{\lambda} f(0) + K, \tag{1}$$

where $\lambda = S'(0) > 1$ and $K = \sum_{i=2}^{r} 1/(a_{i-1}S'(a_{i-1}))$. Thus, we have

$$P_S^n f(x) \le f(0) + \frac{\lambda K}{\lambda - 1}, \quad \forall n$$

which guarantees the existence of f^*.

In the following section we analyze the convergence of the piecewise linear Markov approximation scheme for computing the fixed point of P_S when S is piecewise convex with a strong repellor.

3. The First Order Markov Finite Approximation

Divide the interval $[0,1]$ into n equal parts. Let $I_i = [x_{i-1}, x_i]$ with $x_i = ih$, $i = 1, \ldots, n$, $x_0 = 0$ and $x_n = 1$. Each subinterval I_i has length $h = \frac{1}{n}$. Denote by Δ_n the space of continuous piecewise linear functions corresponding to the above partition. Then Δ_n is a linear subspace of $L^1(0,1)$ with dimension $n+1$. First of all, we choose a basis for Δ_n. Let

$$e_i(x) = e(\frac{x - x_i}{h}), \quad i = 0, 1, \ldots, n,$$

where $e(x) = (1 - |x|)\chi_{[-1,1]}(x)$, $-\infty < x < \infty$, with χ_A the characteristic function of A. This basis has the property that

$$\sum_{i=0}^{n} e_i(x) \equiv 1, \qquad x \in [0,1].$$

Note that the support of e_i is $I_i \cup I_{i+1}$ for $i = 1, 2, \ldots, n-1$, and that of e_0 and e_n is I_1 and I_n, respectively. In the following, denote

$$f_i = \frac{1}{h} \int_{I_i} f dm$$

which is the average value of f over the i-th subinterval I_i. Now define $Q_n : L^1(0,1) \longrightarrow \Delta_n$ as follows

$$Q_n f = f_1 e_0 + \sum_{i=1}^{n-1} \frac{f_i + f_{i+1}}{2} e_i + f_n e_n.$$

It was proved in [1] that Q_n is a Markov operator of finite rank, $\lim_{n\to\infty} Q_n f = f$ strongly for all $f \in L^1(0,1)$, and $V_0^1 Q_n f \le V_0^1 f$. It is also easy to see that $\|Q_n f\|_\infty \le \|f\|_\infty$ for all $f \in L^\infty(0,1)$. From Remark 2.1, the Frobenius-Perron operator P_S corresponding to the piecewise convex transformation S maps the set of nonnegative decreasing functions into itself. The following result indicates that Q_n has the same property.

Lemma 3.1. *If $f \in L^1(0,1)$ is decreasing, then $Q_n f$ is also decreasing.*

Proof. Let $0 \le x \le y \le 1$ and $x \in I_i$ for some $i = 1, 2, \dots, n$. First suppose $1 < i < n$ and $y \in I_i$. Then

$$Q_n f(x) = \frac{f_{i-1} + f_i}{2} e_{i-1}(x) + \frac{f_i + f_{i+1}}{2} e_i(x),$$
$$Q_n f(y) = \frac{f_{i-1} + f_i}{2} e_{i-1}(y) + \frac{f_i + f_{i+1}}{2} e_i(y).$$

Since f is decreasing, $f_{i-1} \ge f_{i+1}$. Noting that $e_{i-1}(x) = 1 - e_i(x) \ge e_{i-1}(y) = 1 - e_i(y)$, we have

$$\begin{aligned} Q_n f(x) - Q_n f(y) &= \frac{f_{i-1} + f_i}{2}[e_{i-1}(x) - e_{i-1}(y)] + \frac{f_i + f_{i+1}}{2}[e_i(x) - e_i(y)] \\ &= \frac{f_{i-1} + f_i}{2}[e_{i-1}(x) - e_{i-1}(y)] - \frac{f_i + f_{i+1}}{2}[e_{i-1}(x) - e_{i-1}(y)] \\ &= \frac{f_{i-1} - f_{i+1}}{2}[e_{i-1}(x) - e_{i-1}(y)] \ge 0. \end{aligned}$$

The case $i = 1$ or $i = n$ is similar to prove. Now suppose $y \in I_j$ with $1 < i < j < n$. Then

$$Q_n f(y) = \frac{f_{j-1} + f_j}{2} e_{j-1}(y) + \frac{f_j + f_{j+1}}{2} e_j(y).$$

Since f is decreasing, we have

$$\frac{f_{i-1} + f_i}{2} \ge \frac{f_i + f_{i+1}}{2} \ge \frac{f_{j-1} + f_j}{2} \ge \frac{f_j + f_{j+1}}{2}.$$

From $e_{i-1}(x) + e_i(x) = 1$ and $e_{j-1}(y) + e_j(y) = 1$, we obtain

$$Q_n f(x) \geq \frac{f_i + f_{i+1}}{2} \geq \frac{f_{j-1} + f_j}{2} \geq Q_n f(y).$$

Similarly we can prove the result in the case $i = 1$ or $j = n$. □

Let $P_n = Q_n P_S|_{\Delta_n}$. Then $P_n : \Delta_n \longrightarrow \Delta_n$ is a Markov operator of finite rank, $\lim_{n\to\infty} P_n f = P_S f$ strongly for any $f \in L^1(0,1)$, and there is a fixed density f_n of P_n in Δ_n [1]. The following lemma shows that f_n can actually be taken to be decreasing.

Lemma 3.2. *$P_n : \Delta_n \longrightarrow \Delta_n$ has a continuous piecewise linear fixed density function $f_n \in D_n$ where*

$$D_n = \{f \in \Delta_n \mid \ f \geq 0, \quad \|f\| = 1, \quad f \ \text{ is decreasing}\}.$$

Proof. See [9]. □

Now we can prove the convergence of the piecewise linear Markov approximation method.

Theorem 3.1. *Suppose $S : [0,1] \longrightarrow [0,1]$ satisfies the conditions of Theorem 2.1. Let $f_n \in D_n$ be a sequence of continuous decreasing piecewise linear fixed density functions of P_n. Then f_n converge to the unique fixed density f^* of P_S in $L^1(0,1)$.*

Proof. First we show that the sequence of nonnegative numbers $f_n(0)$ is bounded above. In fact, from (1), we have for $x \in [0,1]$,

$$0 \leq f_n(x) = P_n f_n(x) = Q_n P_S f_n(x) \leq \max_{x\in[0,1]} P_S f_n(x) \leq \frac{1}{\lambda} f_n(0) + K.$$

In particular,

$$f_n(0) \leq \frac{1}{\lambda} f_n(0) + K.$$

Since $\lambda > 1$,

$$f_n(0) \leq \frac{\lambda K}{\lambda - 1}, \quad \forall n.$$

Since f_n is nonnegative and decreasing, it follows that

$$V_0^1 f_n = f_n(0) - f_n(1) \leq f_n(0) \leq \frac{\lambda K}{\lambda - 1}, \quad \forall n.$$

By Helly's Theorem, $\{f_n\}$ is precompact in $L^1(0,1)$. Suppose $\lim_{k\to\infty} f_{n_k} = g$ for some subsequence $\{n_k\}$ of positive integers. Then from

$$\|P_S g - g\| \quad \leq \quad \|g - f_{n_k}\| + \|f_{n_k} - P_{n_k} f_{n_k}\|$$

$$\begin{aligned} &+ \quad \|P_{n_k} f_{n_k} - P_{n_k} g\| + \|P_{n_k} g - P_S g\| \\ &\leq \quad \|g - f_{n_k}\| + \|f_{n_k} - g\| + \|P_{n_k} g - P_S g\| \to 0, \end{aligned}$$

we have $P_S g = g = f^*$. Hence $\lim_{n\to\infty} f_n = f^*$ since all convergent subsequences of f_n converge to f^*.

Remark 3.1. It can be shown that for n sufficiently large, P_n has a unique fixed density in Δ_n, see [3]. Thus any sequence of fixed densities of P_n in Δ_n converges to f^*.

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