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HYERS-ULAM-RASSIAS STABILITY OF A QUARTIC FUNCTIONAL EQUATION

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Abstract: In this paper, we obtain the general solution and the generalized Hyers-Ulam-Rassias stability of a quartic functional equation

$$f(x+2y)+f(x-2y)=2f(x)+32f(y)+48f(\sqrt{xy}\ ),$$

for all $x, y \in \mathbb{R}$ with the help of Fréchet functional equation.

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1. Introduction

S.M. Ulam [21] is the person who initiated the problem of stability of group homomorphisms. D.H. Hyers [7] answered the problem of Ulam under the assumptation that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [16]. The phenomenon that was introduced and proved by Th.M. Rassias in the year 1978, is called the Hyers-Ulam-Rassias stability. The Hyers-Ulam-Rassias stability for various functional equations have been extensively investigated by a number of mathematicians, see [3, 4, 5, 9, 10, 11, 14, 15, 18].

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Recently, we find many literatures dealing with the Hyers-Ulam-Rassias stability for Quadratic functional equation [3, 4, 10, 11, 16, 17, 19, 20] and cubic functional equation [9, 15]. The functional equation of the form

$$f(x+2y)+f(x-2y)+6f(x)=4[f(x+y)+f(x-y)+6f(y)]\,, \tag{1.1}$$

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)\,, \tag{1.2}$$

are called quartic functional equations because the function $f(x)=x^4$ becomes the solution of the equations (1.1) and (1.2).

The Hyers-Ulam stability of the quartic functional equation (1.1) was investigated by J.M. Rassias [18]. P.K. Sahoo and J.K. Chung [20] obtained the general solution of (1.1). S.H. Lee, S.M. Im and I.S. Hwang [14] discussed the Hyers-Ulam-Rassias stability and obtained its general solution of (1.2). In this paper, the authors discuss the general solution of a new type of quartic functional equation

$$f(x+2y)+f(x-2y)=2f(x)+32f(y)+48f(\sqrt{xy}\;) \tag{1.3}$$

for all $x, y \in \mathbb{R}$ with the help of Fréchet functional equation

2. Some Preliminary Results

Through out this paper, let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}$ be the set of real numbers. In this section, we list some definitions like additive, $m-$additive, $m-$additive symmetric map and Fréchet functional equation.

(i) Additive: A function $B:\mathbb{R}\to\mathbb{R}$ is called if *additive* if $B(x+y)=B(x)+B(y)$.

(ii) $m-$additive: Let a function $B_m:\mathbb{R}^m\to\mathbb{R}$ is said to be *m-additive* if it is additive in each of its variables.

(iii) Symmetric: A function B_m is said to be *symmetric* if $B_m(x_1,x_2,\ldots,x_m)=B_m(x_{\pi(1)},x_{\pi(2)},\ldots,x_{\pi(m)})$ for every permutation $\{\pi(1),\ldots,\pi(m)\}$ of $\{1,2,\ldots,m\}$.

(iv) Diagonal: If $B_m(x_1,x_2,\ldots,x_m)$ is a $m-$additive symmetric map, then $B_m(x,x,\ldots,x)$ is denoted by $B^m(x)$ is the *diagonal*.

(v) $B^{l,m-l}(x,y)$: If we substitute $x_1=x_2=\cdots=x_l=x$ and $x_{l+1}=\cdots=x_m=y$ in $B_m(x_1,x_2,\ldots,x_m)$ then the resulting expression is denoted by $B^{l,m-l}(x,y)$.

(vi) Fréchet functional equation: Define the difference operator Δ_h as, $\Delta_h f(x)=f(x+h)-f(x)$ for $f:\mathbb{R}\to\mathbb{R}$, for $h\in\mathbb{R}$. Then it satisfies

$\Delta_h\ o\ \Delta_h^m f(x) = \Delta_h^{m+1} f(x)$ for all $m \in \mathbb{N}$ and $h \in \mathbb{R}$, $\Delta_h^0 f(x) = f(x)$. It may be observed that the functional equation $\Delta_h^{m+1} f(x)$ is equivalent to the Fréchet functional equation

$$\Delta_{h_1,h_2\ldots,h_{m+1}} f(x) = 0 \quad \text{for} \quad x, h_1, \ldots, h_{m+1} \in \mathbb{R},$$

where $\Delta_{h_1,\ldots,h_k} = \Delta_{h_k}\ o\ \ldots\ o\ \Delta_{h_1}$, $k = 2, 3, \ldots, m+1$.

3. Solution of Equation (1.3)

We state and prove some lemmas, which will be useful in proving our main Theorem to find the general solution of (1.3).

Lemma 3.1. (see [6]) *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation*

$$\Delta_{x_1,x_2,x_3,x_4} f(x_0) = 0$$

for all $x_0, x_1, x_2, x_3, x_4 \in \mathbb{R}$ if and only if f is expressed as

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x),$$

where $A^n(x)$ is the diagonal of a $n-additive$ symmetric function $A_n : \mathbb{R}^n \to \mathbb{R}$ for $n = 1, 2, 3$ and $A^0(x) = A^0$ is an arbitrary constant.

Lemma 3.2. *If the function $H : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation*

$$H(x+2y) + H(x-2y) = 2H(x) + 32H(y) + 48H(\sqrt{xy}\ ) \tag{3.1}$$

for all $x, y \in \mathbb{R}$, then H also satisfies the Fréchet functional equation

$$\Delta_{y_1,y_2,\ldots,y_5} H(y_0) = 0 \tag{3.2}$$

for all $y_0, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$.

Proof. Letting $y_0 = x + 2y$ and $z_1 = x - 2y$ in (3.1), we obtain

$$H(y_0) + H(z_1) = 2H\left(\frac{y_0+z_1}{2}\right) + 32H\left(\frac{y_0-z_1}{4}\right) + 48H\left(\sqrt{\frac{y_0^2-z_1^2}{8}}\ \right). \tag{3.3}$$

Now replacing y_0 by $y_0 + y_1$ in (3.3), we have

$$H(y_0+y_1)+H(z_1)= \ 2H\left(\frac{y_0+y_1+z_1}{2}\right)+32H\left(\frac{y_0+y_1-z_1}{4}\right)$$
$$+48H\left(\sqrt{\frac{(y_0+y_1)^2-z_1^2}{8}}\right). \quad (3.4)$$

Subtracting (3.3) from (3.4), we get

$$H(y_0+y_1)-H(y_0)=2H\left(\frac{y_0+y_1+z_1}{2}\right)-2H\left(\frac{y_0+z_1}{2}\right)$$
$$+32H\left(\frac{y_0+y_1-z_1}{4}\right)-32H\left(\frac{y_0-z_1}{4}\right)$$
$$+48H\left(\sqrt{\frac{(y_0+y_1)^2-z_1^2}{8}}\right)-48H\left(\sqrt{\frac{y_0^2-z_1^2}{8}}\right). \quad (3.5)$$

Substituting $z_1 = -4\ z_2 + y_0$ in (3.5) and therefore $z_2 = \frac{y_0-z_1}{4}$. It yields fro equation (3.5) that

$$H(y_0+y_1)-H(y_0)=2H\left(\frac{y_1}{2}-2z_2+y_0\right)-2H\left(-2z_2+y_0\right)$$
$$+32H\left(\frac{y_1}{4}+z_2\right)-32H\left(z_2\right)+48H\left(\sqrt{\frac{y_1^2}{8}+\frac{y_1y_0}{4}+z_2(-2z_2+y_0)}\right)$$
$$-48H\left(\sqrt{z_2(-2z_2+y_0)}\right). \quad (3.6)$$

Now replacing y_0 by y_0+y_2 in (3.6), we obtain

$$H(y_0+y_1+y_2)-H(y_0+y_2)=2H\left(\frac{y_1}{2}-2z_2+y_0+y_2\right)$$
$$-2H\left(-2z_2+y_0+y_2\right)+32H\left(\frac{y_1}{4}+z_2\right)-32H\left(z_2\right)$$
$$+48H\left(\sqrt{\frac{y_1^2}{8}+\frac{y_1y_0}{4}+\frac{y_1y_2}{4}+z_2(-2z_2+y_0+y_2)}\right)$$
$$-48H\left(\sqrt{z_2(-2z_2+y_0+y_2)}\right). \quad (3.7)$$

Subtracting (3.6) from (3.7), we have

$$H(y_0+y_1+y_2)-H(y_0+y_1)-H(y_0+y_2)+H(y_0)$$

$$= 2H\left(\frac{y_1}{2} - 2z_2 + y_0 + y_2\right) - 2H\left(\frac{y_1}{2} - 2z_2 + y_0\right) - 2H\left(-2z_2 + y_0 + y_2\right) + 2H\left(-2z_2 + y_0\right) + 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_0}{4} + \frac{y_1 y_2}{4} + z_2(-2z_2 + y_0 + y_2)}\right) - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_0}{4} + z_2(-2z_2 + y_0)}\right) - 48H\left(\sqrt{z_2(-2z_2 + y_0 + y_2)}\right) + 48H\left(\sqrt{z_2(-2z_2 + y_0)}\right). \quad (3.8)$$

Now setting $z_3 = -2z_2 + y_0$ in (3.8) it reduces to

$$H(y_0 + y_1 + y_2) - H(y_0 + y_1) - H(y_0 + y_2) + H(y_0) = 2H\left(\frac{y_1}{2} + z_3 + y_2\right) - 2H\left(\frac{y_1}{2} + z_3\right) - 2H\left(z_3 + y_2\right) + 2H\left(z_3\right) + 48H \times \left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_0}{4} + \frac{y_1 y_2}{4} + z_2(z_3 + y_2)}\right) - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_0}{4} + z_2(z_3)}\right) - 48H\left(\sqrt{z_2(z_3 + y_2)}\right) + 48H\left(\sqrt{z_2(z_3)}\right). \quad (3.9)$$

Replacing y_0 by $y_0 + y_3$ in (3.9) and then subtracting (3.9) from the resulting expression, we have

$$H(y_0 + y_1 + y_2 + y_3) - H(y_0 + y_1 + y_2) - H(y_0 + y_1 + y_3) - H(y_0 + y_2 + y_3) + H(y_0 + y_1) + H(y_0 + y_2) + H(y_0 + y_3) - H(y_0) = 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_2}{4} + \frac{y_1 y_3}{4} + \frac{y_1 y_0}{4} + z_2(z_3 + y_2)}\right) - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_2}{4} + \frac{y_1 y_0}{4} + z_2(z_3 + y_2)}\right) - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_3}{4} + \frac{y_1 y_0}{4} + z_2(z_3)}\right) + 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1 y_0}{4} + z_2(z_3)}\right). \quad (3.10)$$

Letting $z_4 = z_2 z_3 + \frac{y_1 y_0}{4\ z_3}$ then (3.10) reduces to

$$H(y_0 + y_1 + y_2 + y_3) - H(y_0 + y_1 + y_2) - H(y_0 + y_1 + y_3) - H(y_0 + y_2 + y_3)$$

$$
\begin{aligned}
&+ H(y_0+y_1) + H(y_0+y_2) + H(y_0+y_3) - H(y_0) \\
&= 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_1y_3}{4} + z_4 + \frac{y_2z_4}{z_3} - \frac{y_0y_1y_2}{4\ z_3}}\right) \\
&\quad - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_3}{4} + z_4}\right) \\
&\quad - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + z_4 + \frac{y_2z_4}{z_3} - \frac{y_0y_1y_2}{4\ z_3}}\right) \\
&\quad + 48H\left(\sqrt{\frac{y_1^2}{8} + z_4}\right). \qquad (3.11)
\end{aligned}
$$

Again replacing y_0 by $y_0 + y_4$ in (3.11) and then subtracting (3.11) from the resulting expression, we obtain

$$
\begin{aligned}
&H(y_0+y_1+y_2+y_3+y_4) - H(y_0+y_1+y_2+y_3) - H(y_0+y_1+y_2+y_4) \\
&- H(y_0+y_1+y_3+y_4) - H(y_0+y_2+y_3+y_4) + H(y_0+y_1+y_2) + H(y_0+y_1+y_3) \\
&+ H(y_0+y_1+y_4) + H(y_0+y_2+y_3) + H(y_0+y_2+y_4) + H(y_0+y_3+y_4) \\
&- H(y_0+y_1) - H(y_0+y_2) - H(y_0+y_3) - H(y_0+y_4) + H(y_0) \\
&= 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_1y_3}{4} + \frac{y_2z_4}{z_3} + z_4 - \frac{y_0y_1y_2}{4\ z_3} - \frac{y_1y_2y_4}{4\ z_3}}\right) \\
&\quad - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_2z_4}{z_3} - \frac{y_0y_1y_2}{4\ z_3} - \frac{y_1y_2y_4}{4\ z_3} + z_4}\right) \\
&\quad - 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_1y_3}{4} + z_4 + \frac{y_2z_4}{z_3} - \frac{y_0y_1y_2}{4\ z_3}}\right) \\
&\quad + 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_2z_4}{z_3} + z_4 - \frac{y_0y_1y_2}{4\ z_3}}\right). \qquad (3.12)
\end{aligned}
$$

Letting $z_5 = z_4 - \frac{y_1y_2y_0}{4\ z_3}$ (3.12), the above equation becomes

$$
\begin{aligned}
&H(y_0+y_1+y_2+y_3+y_4) - H(y_0+y_1+y_2+y_3) - H(y_0+y_1+y_2+y_4) \\
&- H(y_0+y_1+y_3+y_4) - H(y_0+y_2+y_3+y_4) + H(y_0+y_1+y_2) + H(y_0+y_1+y_3)
\end{aligned}
$$

$$+ H(y_0 + y_1 + y_4) + H(y_0 + y_2 + y_3) + H(y_0 + y_2 + y_4) + H(y_0 + y_3 + y_4)$$
$$- H(y_0 + y_1) - H(y_0 + y_2) - H(y_0 + y_3) - H(y_0 + y_4) + H(y_0)$$
$$= 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_1y_3}{4} + \frac{y_2z_4}{z_3} + z_5 - \frac{y_1y_2y_4}{4\ z_3}}\right)$$
$$- 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_2z_4}{z_3} + z_5 - \frac{y_1y_2y_4}{4\ z_3}}\right)$$
$$- 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_1y_3}{4} + \frac{y_2z_4}{z_3} + z_5}\right)$$
$$+ 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_2z_4}{z_3} + z_5}\right). \quad (3.13)$$

Now replace y_0 by $y_0 + y_5$ in (3.13), we obtain

$$H(y_0+y_1+y_2+y_3+y_4+y_5) - H(y_0+y_1+y_2+y_3+y_5) - H(y_0+y_1+y_2+y_4+y_5)$$
$$- H(y_0 + y_1 + y_3 + y_4 + y_5) - H(y_0 + y_2 + y_3 + y_4 + y_5) + H(y_0 + y_1 + y_2 + y_5)$$
$$+ H(y_0 + y_1 + y_3 + y_5)H(y_0 + y_1 + y_4 + y_5) + H(y_0 + y_2 + y_3 + y_5)$$
$$+ H(y_0 + y_2 + y_4 + y_5) + H(y_0 + y_3 + y_4 + y_5) - H(y_0 + y_1 + y_5) - H(y_0 + y_2 + y_5)$$
$$- H(y_0 + y_3 + y_5) - H(y_0 + y_4 + y_5) + H(y_0 + y_5)$$
$$= 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_1y_3}{4} + \frac{y_2z_4}{z_3} + z_5 - \frac{y_1y_2y_4}{4\ z_3}}\right)$$
$$- 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_2z_4}{z_3} + z_5 - \frac{y_1y_2y_4}{4\ z_3}}\right)$$
$$+ 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_2z_4}{z_3} + z_5}\right)$$
$$- 48H\left(\sqrt{\frac{y_1^2}{8} + \frac{y_1y_2}{4} + \frac{y_1y_3}{4} + \frac{y_2z_4}{z_3} + z_5}\right). \quad (3.14)$$

Subracting (3.13) from (3.14) yields

$$
\begin{aligned}
H(y_0 + y_1 + y_2 + y_3 + y_4 + y_5) &- H(y_0 + y_1 + y_2 + y_3 + y_4) \\
&- H(y_0 + y_1 + y_2 + y_3 + y_5) - H(y_0 + y_1 + y_2 + y_4 + y_5) \\
&- H(y_0 + y_1 + y_3 + y_4 + y_5) - H(y_0 + y_2 + y_3 + y_4 + y_5) \\
&+ H(y_0 + y_1 + y_2 + y_3) + H(y_0 + y_1 + y_2 + y_4) \\
&+ H(y_0 + y_1 + y_2 + y_5) + H(y_0 + y_1 + y_3 + y_4) \\
&+ H(y_0 + y_1 + y_3 + y_5) + H(y_0 + y_1 + y_4 + y_5) \\
&+ H(y_0 + y_2 + y_3 + y_4) + H(y_0 + y_2 + y_3 + y_5) \\
&+ H(y_0 + y_2 + y_4 + y_5) + H(y_0 + y_3 + y_4 + y_5) \\
&- H(y_0 + y_1 + y_2) - H(y_0 + y_1 + y_3) \\
&- H(y_0 + y_1 + y_4) - H(y_0 + y_1 + y_5) \\
&- H(y_0 + y_2 + y_3) - H(y_0 + y_2 + y_4) - H(y_0 + y_2 + y_5) \\
&- H(y_0 + y_3 + y_4) - H(y_0 + y_3 + y_5) - H(y_0 + y_4 + y_5) \\
&+ H(y_0 + y_1) + H(y_0 + y_2) + H(y_0 + y_3) \\
&+ H(y_0 + y_4) + H(y_0 + y_5) - H(y_0) = 0\,, \qquad (3.15)
\end{aligned}
$$

which gives

$$\Delta_{y_1,\ldots,y_5} H(y_0) = 0\,.$$

Hence the lemma is proved. □

The following lemma is a special case of a more general result due to Lemma 3.1.

Lemma 3.3. *The function $H : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation (3.2) for all $y_0, y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$ if and only if H is expressed as*

$$H(y) = B^4(y) + B^3(y) + B^2(y) + B^1(y) + B^0(y),$$

where $B^m(x)$ is the diagonal of a m-additive symmetric function $B_m : \mathbb{R}^m \to \mathbb{R}$ for $m = 1, 2, 3, 4$ and $B^0(y) = B^0$ is an arbitrary constant.

Theorem 3.4. *If $B^4(x)$ is the diagonal of the 4-additive symmetric function $B_4 : \mathbb{R}^4 \to \mathbb{R}$ then the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the quartic functional equation (1.3) for all $x, y \in \mathbb{R}$ if and only if f is of the form*

$$f(x) = B^4(x).$$

Proof. By Lemma 3.3, we see that f satisfies the Fréchet functional equation (3.2) is given by

$$f(x) = B^4(x) + B^3(x) + B^2(x) + B^1(x) + B^0(x), \qquad (3.16)$$

where $B^m(x)$ is the diagonal of a $m-$additive symmetric function $B_m : \mathbb{R}^m \to \mathbb{R}$ for $m = 1, 2, 3, 4$ and $B^0(x) = B^0$ is an arbitrary constant. But we have f is an even function. Hence $B^3(x) \equiv 0$ and $B^1(x) \equiv 0$. Hence the equation (3.16) reduces to

$$f(x) = B^4(x) + B^2(x) + B^0(x). \tag{3.17}$$

Substituting (3.17) into (1.3), we get

$$\begin{aligned} &2B^4(x) + 2B^4(2y) + 12B^{2,2}(x, 2y) + 2B^2(x) + 2B^4(2y) + 2B^0 \\ &\quad = 2B^4(x) + 2B^2(x) + +2B^0 + 32B^4(y) + 32B^2(y) + 32B^0 \\ &\qquad +48B^4(\sqrt{xy}\,) + 48B^2(\sqrt{xy}\,) + 48B^0\,. \end{aligned} \tag{3.18}$$

In the above equation (3.18) comparing the corresponding terms on both sides and observing that

$$B^4(2y) = 16B^4(y), \quad B^2(2y) = 4B^2(y), \quad B^{2,2}(x, 2y) = 0,$$

we obtain,

$$24B^2(y) + 48B^4(\sqrt{xy}\,) + 48B^2(\sqrt{xy}\,) + 80B^0 = 0\,, \tag{3.19}$$

which gives $B^2(y) \equiv 0, B^4(\sqrt{xy}\,) \equiv 0, 48B^2(\sqrt{xy}\,) \equiv 0, B^0 \equiv 0$. Thus (3.17) reduces to

$$f(x) = B^4(x),$$

which completes the proof of the theorem. □

4. Hyers-Ulam-Rassias Stability of (1.3)

In this section, we investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.3).

Let X be a real vector space and let Y be a Banach space. Define a function $\varphi : X^2 \to \mathbb{R}^+$ such that

$$\sum_{i=0}^{\infty} \frac{\varphi(0, 2^i y)}{16^i} \text{ converges and } \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{16^n} = 0 \tag{4.1}$$

for all $x, y \in X$. Now suppose,

$$Df(x, y) = f(x + 2y) + f(x - 2y) - 2f(x) - 32f(y) - 48f(\sqrt{x}y\,).$$

We state the following theorem.

Theorem 4.1. *Suppose $f : X \to Y$ satisfies*

$$\| Df(x,y) \| \leq \varphi(x,y) \tag{4.2}$$

for all $x, y \in X$, then there exits a unique quartic function $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\| f(y) - Q(y) \| \leq \frac{1}{32} \sum_{i=0}^{\infty} \frac{\varphi(0, 2^i y)}{16^i} \tag{4.3}$$

for all $y \in X$. The function Q is given by

$$Q(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n} \tag{4.4}$$

for all $y \in X$.

Proof. Setting $x = 0$ in (4.2), we obtain

$$\left\| f(2y) + f(-2y) - 2f(0) - 32f(y) - 48f(\sqrt{0}\ ) \right\| \leq \varphi(0,y)\,,$$

which gives,

$$\left\| f(y) - \frac{f(2y)}{16} \right\| \leq \frac{1}{32}\varphi(0,y) \tag{4.5}$$

for all $y \in X$. Replacing y by $2y$ in (4.5) and dividing the resulting expression by 16, we obtain

$$\left\| \frac{f(2y)}{16} - \frac{f(2^2 y)}{16^2} \right\| \leq \frac{1}{32 \cdot 16}\varphi(0,2y). \tag{4.6}$$

Now,

$$\begin{aligned} \left\| f(y) - \frac{f(2^2 y)}{16^2} \right\| &\leq \left\| f(y) - \frac{f(2y)}{16} \right\| + + \left\| \frac{f(2y)}{16} - \frac{f(2^2 y)}{16^2} \right\| \\ &\leq \frac{1}{32}\left[\varphi(0,y) + \frac{\varphi(0,2y)}{16} \right] \end{aligned} \tag{4.7}$$

for all $y \in X$. From (4.5) and (4.7) and using induction on a positive integer n we obtain that

$$\left\| f(y) - \frac{f(2^n y)}{16^n} \right\| \leq \frac{1}{32} \sum_{i=0}^{n-1} \frac{\varphi(0, 2^i y)}{16^i} \leq \frac{1}{32} \sum_{i=0}^{\infty} \frac{\varphi(0, 2^i y)}{16^i} \tag{4.8}$$

for all $y \in X$. We shall prove that $\{f(2^n y)/16^n\}$ is Cauchy sequence. For any $m, n > 0$, consider

$$\left\| \frac{f(2^m y)}{16^m} - \frac{f(2^n y)}{16^n} \right\| \leq \left\| \frac{f(2^{m+n-n} y)}{16^{m+n-n}} - \frac{f(2^n y)}{16^n} \right\|$$
$$\leq \frac{1}{32} \sum_{i=0}^{n-1} \frac{\varphi(0, 2^{i+n} y)}{16^{i+n}} \leq \frac{1}{32} \sum_{i=0}^{\infty} \frac{\varphi(0, 2^i x)}{16^i}. \tag{4.9}$$

Since the right-hand side of the inequality (4.9) tends to 0 as n tends to infinity. Hence the sequence $\{f(2^n y)/16^n\}$ is a Cauchy sequence and define $Q(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n}$ for all $y \in X$. By letting $n \to \infty$ in (4.8), we obtain our result (4.3). Now we shall show that Q satisfies (1.3). Replacing $x,\ y$ by $2^n x,\ 2^n y$ in (4.2) and divide by 16^n we get,

$$\frac{1}{16^n} \| f(2^n(x+2y)) + f(2^n(x-2y)) - 2f(2^n(x)) - 32 f(2^n(y))$$
$$-48 f(\sqrt{2^n x 2^n y}\,) \Big\| \leq \frac{1}{16^n} \|\varphi(2^n x, 2^n y)\| \,.$$

Taking the limit as $n \to \infty$, using (4.1) and (4.4), we see that

$$\left\| Q(x+2y) + Q(x-2y) - 2Qf(x) - 32Q(y) - 48Q(\sqrt{x}y\ ) \right\| \leq 0\,,$$

which shows that Q satisfies (1.3).

Also letting $x = 0$ and using evenness of Q, we obtain $2Q(2y) = 32Q(y)$ and $Q(2y) = 16Q(y)$. In general $Q(2^n y) = 16^n Q(y)$, which proves Q is quartic.

To prove the uniqueness of Q: Let $Q' : X \to Y$ be another quartic function of (1.3), then we have, $Q(2^n y) = 16^n Q(y)$ and $Q'(2^n y) = 16^n Q'(y)$ for all $y \in X$ and $n \in \mathbb{N}$. But

$$\begin{aligned}\left\| Q(y) - Q'(y) \right\| &\leq \frac{1}{16^n} \left\| Q(2^n y) - Q'(2^n y) \right\| \\ &\leq \frac{1}{16^n} \left(\|Q(2^n y) - f(2^n y)\| + \left\| f(2^n y) - Q'(2^n y) \right\| \right) \\ &\leq \frac{1}{16^n} \left[\frac{1}{32} \sum_{i=0}^{\infty} \frac{\varphi(0, 2^i y)}{16^i} + \frac{1}{32} \sum_{i=0}^{\infty} \frac{\varphi(0, 2^i y)}{16^i} \right] \\ &\leq \frac{1}{16} \sum_{i=0}^{\infty} \frac{\varphi(0, 2^i y)}{16^{i+n}} \quad \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

for all $y \in X$. Therefore Q is unique. This completes the proof of the theorem. □

We obtain the following corollaries concerning the stability of the equation (1.3).

Corollary 4.2. *Let X be a real normed space and Y be a Banach space. Let ϵ, p, q be real numbers such that $\epsilon \geq 0, q \geq 0$, and either $p, q \leq 4$ or $p, q \geq 4$. If $f : X \to Y$ satisfies*

$$\|Df(x,y)\| \leq \epsilon \left(\|x\|^p + \|y\|^q\right) \tag{4.10}$$

for all $x, y \in X$, then there exists a unique quartic function $Q : X \to Y$ which satisfies (1.2) and the inequality

$$\|f(y) - Q(y)\| \leq \frac{\epsilon}{2\,\|16 - 4^q\|}\,\|y\|^q \tag{4.11}$$

for all $y \in X$ The function Q is defined in (4.4). Moreover for each fixed $y \in X$ is continuous then, $f(ty) = t^4 f(y)$ for all $t \in \mathbb{R}$.

The proof of the corollary follows in the same way as given in S. Czerwik [4]. We note that p needs to be equal to q. When $p, q = 4$, we do not guarantee whether the quartic function is stable in the sense of Hyers-Ulam-Rassias assumed in (4.10). The following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.3. *Let X be a real normed space and Y be a Banach space. Let ϵ be real number. If $f : X \to Y$ satisfies*

$$\|Df(x,y)\| \leq \epsilon \tag{4.12}$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \to Y$ which satisfies (1.3) and the inequality

$$\|f(y) - Q(y)\| \leq \frac{\epsilon}{30} \tag{4.13}$$

for all $y \in X$ The function Q is defined in (4.4). Moreover for each fixed $y \in X$ is continuous, then $f(ty) = t^4 f(y)$ for all $t \in \mathbb{R}$.

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