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A NOTE ON INTEGRAL INEQUALITIES ON TIME SCALES

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Abstract: In this article we study new integral inequalities involving two functions and their derivatives on time scales.

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1. Introduction

The theory of dynamic equations on time scales (aka measure chains) was introduced by Hilger [3] with the motivation of providing a unified approach to continuous and discrete analysis. The generalized derivative or Hilger derivative $f^{\triangle}(t)$ of a function $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is a so-called "time scale" (an arbitrary closed nonempty subset of $\mathbb{R}$) becomes the usual derivative when $\mathbb{T} = \mathbb{R}$, that is $f^{\triangle}(t) = f'(t)$. On the other hand, if $\mathbb{T} = \mathbb{Z}$, then $f^{\triangle}(t)$ reduces to the usual forward difference, that is $f^{\triangle}(t) = \triangle f(t)$. This theory not only brought equations leading to new applications, But also, this theory allows one to get some insight into and better understanding of the subtle differences between discrete and continuous systems [1, 2].

In an elegant note [5], A.M. Ostrowski proved the following interesting and useful inequality (see also [4, p. 468]):

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$$\left|f(x)-\frac{1}{b-a}\int_b^a f(t)dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty, \qquad (1.1)$$

for all $x \in [a,b]$, where $f:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{x\in(a,b)}\left|f^{\triangle}\right| < \infty$.

B.G. Pachpatte [5] have established new inequalities of the type (1.1) involving two functions and their derivatives.

In this paper, we study new integral inequalities involving two functions and their derivatives on time scales.

Now, first we mention without proof several fundamental definitions and result from the calculus on time scales in an excellent introductory text by Bohner and Peterson [2].

2. General Definitions

Definition 1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

Definition 2. We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t\in\mathbb{T}: s<t\}$ for the points $a<b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 3. The mappings $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$$

and $\rho(t)=\sup\{s\in\mathbb{T}:\ s<t\}$ are called the jump operators.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ in the following way.

Definition 4. A nonmaximal element $t\in\mathbb{T}$ is said to be *right-dense if* $\sigma(t)=t$, *right-scattered* if $\sigma(t)>t$, *left-dense if* $\rho(t)=t$, *left-scattered* if $\rho(t)<t$.

In the case $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t$, and if $\mathbb{T}=h\mathbb{Z}$, $h>0$, then $\sigma(t)=t+h$.

Definition 5. The mapping $\mu:\mathbb{T}\to\mathbb{R}^+$ defined by $\mu(t)=\sigma(t)-t$ is called the *graininess function.*

If $\mathbb{T}=\mathbb{R}$, then $\mu(t)=0$, and when $\mathbb{T}=\mathbb{Z}$, we have $\mu(t)=1$.

Definition 6. Let $f:\mathbb{T}\to\mathbb{R}$. f is called differentiable at $t\in\mathbb{T}^k$, with (delta) derivative $f^{\triangle}(t)\in\mathbb{R}^n$ if given $\varepsilon>0$ there exists a neighborhood U of t such that, for all $s\in U$,

$$\left\| f^{\sigma}(t)-f(s)-f^{\triangle}(t)[\sigma(t)-s]\right\| \leq \varepsilon \left\|\sigma(t)-s\right\|,$$

where $f^{\sigma}=f\circ\sigma$.

If $\mathbb{T}=\mathbb{R}$, then $f^{\triangle}(t)=\frac{df(t)}{dt}$, and if $\mathbb{T}=\mathbb{Z}$, then $f^{\triangle}(t)=f(t+1)-f(t)$.

Some basic properties of delta derivatives are the following, see [2].

Theorem 1. *Assume that $f:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}^k$.*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is differentiable at t and t is right-scattered, then f is differentiable at t with

$$f^{\triangle}(t)=\frac{f^{\sigma}(t)-f(t)}{\sigma(t)-t}.$$

(iii) If f is differentiable at t and t is right-dense, then

$$f^{\triangle}(t)=\lim_{t\to s}\frac{f(t)-f(s)}{t-s}.$$

(iv) If f is differentiable at t, then

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\triangle}(t)\,.$$

Example 1. (i) If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=\alpha$ for all $t\in\mathbb{T}$, where $\alpha\in\mathbb{R}$ is constant, then $f^{\triangle}(t)\equiv 0$.

(ii) If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=t$ for all $t\in\mathbb{T}$, then $f^{\triangle}(t)\equiv 1$.

(iii) If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=t^2$ for all $t\in\mathbb{T}$, then $f^{\triangle}(t)\equiv t+\sigma(t)$

(iv) If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)=t^3$ for all $t\in\mathbb{T}$, then $f^{\triangle}(t)\equiv t^2+t\sigma(t)+[\sigma(t)]^2$.

Definition 7. The function $f:\mathbb{T}\to\mathbb{R}$ is said to be *rd-continuous* (denote $f\in C_{rd}(\mathbb{T},\mathbb{R})$) *if, at all* $t\in\mathbb{T}$:

(i) f is continuous at every right-dense point $t\in\mathbb{T}$,

(ii) $\lim_{s\to t^-}f(s)$ exists and is finite at every left-dense point $t\in\mathbb{T}$.

Definition 8. Let $f\in C_{rd}(\mathbb{T},\mathbb{R})$. Then $g:\mathbb{T}\to\mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\triangle}(t)=f(t)$ for any $t\in\mathbb{T}^k$. In this case, we define

$$\int_a^t f(s)\triangle s=g(t)-g(a),\qquad t\in\mathbb{T}.$$

Theorem 2. *If f is $\triangle$-integrable on $[a,b]$, then so is $|f|$, and*

$$\left|\int_a^b f(t)\triangle t\right| \le \int_a^b |f(t)|\triangle t.$$

We assume that $\mathbb{T}=[a,b]$ is an arbitrary interval on time scale. For notational purposes, the intersection of a real interval $[a,b]$ with a time scale $\mathbb{T}$ is denoted by $[a,b]\cap\mathbb{T}$: $[a,b]_{\mathbb{T}}$. Our main results are given in the following theorems.

3. Main Result

Theorem 3. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be rd continuous functions on $\mathbb{T}$ and differentiable on $\mathbb{T}^k$, whose derivatives $f^{\triangle},g^{\triangle}:\mathbb{T}^k\to\mathbb{R}$ are bounded on $\mathbb{T}^k$, i.e., $\left\|f^{\triangle}\right\|_\infty=\sup_{x\in\mathbb{T}^k}\left|f^{\triangle}(x)\right|<\infty$, $\left\|g^{\triangle}\right\|_\infty=\sup_{x\in\mathbb{T}^k}\left|g^{\triangle}(x)\right|<\infty$. Then*

$$\begin{aligned}\left|f(x)g(x)-\frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)\triangle y+f(x)\int_a^b g(y)\triangle y\right]\right| \\ \le \frac{1}{2(b-a)}\left\{|g(x)|\left\|f^{\triangle}\right\|_\infty+|f(x)|\left\|g^{\triangle}\right\|_\infty\right\} \\ \times\left[x(b-a)-(b^2-a^2)+\int_a^b\sigma(y)\triangle y\right],\end{aligned} \tag{3.1}$$

for all $t\in\mathbb{T}$.

Proof. For any $x,y\in\mathbb{T}$ we have the following identities:

$$f(x)-f(y)=\int_y^x f^{\triangle}(\tau)\triangle\tau\,, \tag{3.2}$$

$$g(x)-g(y)=\int_y^x g^{\triangle}(\tau)\triangle\tau. \tag{3.3}$$

Multiplying both sides of (3.2) and (3.3) by $g(x)$ and $f(x)$ respectively and adding we get

$$\begin{aligned}2f(x)g(x)-[g(x)f(y)+f(x)g(y)] \\ =g(x)\int_y^x f^{\triangle}(\tau)\triangle\tau+f(x)\int_y^x g^{\triangle}(\tau)\triangle\tau\,.\end{aligned} \tag{3.4}$$

Integrating both sides of (3.4) with respect to y over $\mathbb{T}$ and rewriting we have

$$f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)\triangle y + f(x)\int_a^b g(y)\triangle y\right]$$
$$= \frac{1}{2(b-a)}\int_a^b \left\{g(x)\int_y^x f^{\triangle}(\tau)\triangle\tau + f(x)\int_y^x g^{\triangle}(\tau)\triangle\tau\right\}\triangle y. \quad (3.5)$$

From (3.5) and using the properties of modulus we have

$$\left|f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)\triangle y + f(x)\int_a^b g(y)\triangle y\right]\right|$$
$$\leq \frac{1}{2(b-a)}\int_a^b \left\{|g(x)|\left\|f^{\triangle}\right\|_\infty |x-y| + |f(x)|\left\|g^{\triangle}\right\|_\infty |x-y|\right\}\triangle y$$
$$= \frac{1}{2(b-a)}\left\{|g(x)|\left\|f^{\triangle}\right\|_\infty + |f(x)|\left\|g^{\triangle}\right\|_\infty\right\}\int_a^b |x-y|\triangle y$$
$$= \frac{1}{2(b-a)}\left\{|g(x)|\left\|f^{\triangle}\right\|_\infty + |f(x)|\left\|g^{\triangle}\right\|_\infty\right\}$$
$$\times\left[x(b-a) - (b^2-a^2) + \int_a^b \sigma(y)\triangle y\right].$$

The proof is complete. □

Corollary 1. *If we take $\mathbb{T} = \mathbb{R}$, it is clear that we can have the same as in Theorem 2.1 in [6].*

Remark 1. We note that, by taking $g(t) = 1$ and hence $g^{\triangle}(t) = 0$ in Theorem 3 we have

$$\left|f(x) - \frac{1}{(b-a)}\int_a^b f(y)\triangle y\right|$$
$$\leq \frac{1}{(b-a)}\left\|f^{\triangle}\right\|_\infty\left[x(b-a) - (b^2-a^2) + \int_a^b \sigma(y)\triangle y\right].$$

Conclusion 1. If we take $\mathbb{T} = \mathbb{R}$, we recapture the well known Ostowkski's inequality in (1.1).

Theorem 4. *Let $f, g,\ f^{\triangle}, g^{\triangle}$ Theorem 3. Then*

$$\left|f(x)g(x) - \frac{1}{(b-a)}\left[g(x)\int_a^b f(y)\triangle y\right.\right. \quad (3.6)$$

$$+f(x)\int_a^b g(y)\triangle y\Bigg] + \frac{1}{(b-a)}\int_a^b f(y)g(y)\triangle y\Bigg|$$
$$\leq \frac{1}{(b-a)}\left\|f^{\triangle}\right\|_\infty \left\|g^{\triangle}\right\|_\infty \Bigg[x^2(b-a) - 2x(b^2-a^2) + (b^3-a^3)$$
$$+\int_a^b [2x - y - \sigma(y)]\,\sigma(y)\triangle y\Bigg],$$

for all $x \in \mathbb{T}$.

Proof. From the hypotheses, the identities (3.2) and (3.3) hold. Multiplying the left and right sides (3.2) and (3.3) we get

$$f(x)g(x) - [g(x)f(y) + f(x)g(y) + f(y)g(y)] = \left\{\int_y^x f^{\triangle}(\tau)\triangle\tau\right\}\left\{\int_y^x g^{\triangle}(\tau)\triangle\tau\right\}. \tag{3.7}$$

Integrating both sides of (3.7) with respect to y over $\mathbb{T}$ and rewriting we have

$$f(x)g(x) - \frac{1}{(b-a)}\left[g(x)\int_a^b f(y)\triangle y + f(x)\int_a^b g(y)\triangle y\right] + \frac{1}{(b-a)}\int_a^b f(y)g(y)\triangle y$$
$$= \frac{1}{(b-a)}\int_a^b \left\{\int_y^x f^{\triangle}(\tau)\triangle\tau\right\}\left\{\int_y^x g^{\triangle}(\tau)\triangle\tau\right\}\triangle y. \tag{3.8}$$

From (3.8) and using the properties of modulus we obtain

$$\Bigg|f(x)g(x) - \frac{1}{(b-a)}\left[g(x)\int_a^b f(y)\triangle y + f(x)\int_a^b g(y)\triangle y\right]$$
$$+\frac{1}{(b-a)}\int_a^b f(y)g(y)\triangle y\Bigg|$$
$$\leq \frac{1}{(b-a)}\left\|f^{\triangle}\right\|_\infty \left\|g^{\triangle}\right\|_\infty \int_a^b |x-y|^2\triangle y$$
$$= \frac{1}{(b-a)}\left\|f^{\triangle}\right\|_\infty \left\|g^{\triangle}\right\|_\infty [x^2(b-a) - 2x(b^2-a^2) + (b^3-a^3)$$
$$+\int_a^b [2x - y - \sigma(y)]\,\sigma(y)\triangle y].$$

The proof is complete. □

Corollary 2. *If we take* $\mathbb{T} = \mathbb{R}$, *it is clear that we can have the same as in Theorem 2.2 in [6].*

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