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SOME GRAPHS WITH MAGICAL LABELLING

Ming Yao[1], Bing Yao[2 §], Zhongfu Zhang[3]
[1,3]Department of Mathematics
Lanzhou Vocational Technology College
Lanzhou, 730060, P.R. CHINA
[2,3]College of Mathematics and Information Science
Northwest Normal University
Lanzhou, 730070, P.R. CHINA
[2]e-mail: yybb918@163.com
[3]Institute of Applied Mathematics
Lanzhou Jiaotong University
Lanzhou, 730070, P.R. CHINA

Abstract: An edge-magic total labelling of a graph G with p vertices and q edges is a *bijective mapping* f from $V(G)\cup E(G)$ into $\{1,2,\cdots,p+q\}$ if there is a positive integer λ_f such that $f(u)+f(v)+f(uv)=\lambda_f$ whenever $uv\in E(G)$. We prove $p+q+3\leq\lambda_f\leq 2(p+q)$, and furthermore $\lambda_f\leq 2p+q$ if $1\leq f(u)\leq p$ for any $u\in V(G)$, G is called *semt*-magical in this case. If G is regular and *semt*-magical, then G has the uniquely magical constant. We construct several magical graphs of larger orders with other magical graphs of small orders.

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1. Introduction and Concepts

All graphs mentioned in this paper are simple, connected and finite. A *total labelling* f of a graph G with p vertices and q edges is to label every element of $V(G)\cup E(G)$ by a positive integer, we say f to be a *proper total labelling* if the labels of two elements of $V(G)\cup E(G)$ are completely different, namely, f is a

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§Correspondence author

bijective mapping from $V(G) \cup E(G)$ to a positive integer set. In 1970, Kotzig and Rosa defined the *edge-magic total labelling* (hereinafter *emt*-labelling) of a graph G with p vertices and q edges as a *bijective mapping* f from $V(G) \cup E(G)$ into $\{1, 2, \cdots, p+q\}$ if there is a positive integer λ such that

$$f(u) + f(v) + f(uv) = \lambda \text{ whenever } uv \in E(G). \tag{1}$$

In 1972, they proposed the *magical tree problem* as follows.

Whether do all trees have the edge-magic total labellings?

It is not hard to verify paths, stars and double stars have *emt*-labellings. A *supper edge-magic total labelling* (hereinafter *semt*-labelling) of a graph G with p vertices and q edges is an *emt*-labelling f of G with an additional requirement $\{f(u) \mid u \in V(G)\} = \{1, 2, \cdots, p\}$ in [3], thereby we have $\{f(uv) \mid uv \in E(G)\} = \{p+1, p+2, \cdots, p+q\}$. For the sake of convenience, we say G to be *emt-magical* if G has an *emt*-labelling, and G to be *semt-magical* if G has an *semt*-labelling. In [3] and [4], the author collected over 600 papers on different graph labellings, and pointed out that the problem of the magical trees is still open. We have known that there are a few examples and characteristics about the *emt*-magical and *semt*-magical graphs from [3] and [4].

We need some spacial labellings for our arguments. A *vertex bilabelling* f of a graph G with p vertices and q edges is defined by

$$f : V(G) \to \{1, 2, \cdots, p\} \text{ and } E(G) \to \{p+1, p+2, \cdots, p+q\},$$

and an *edge bilabelling* f of G is defined by

$$f : E(G) \to \{1, 2, \cdots, q\} \text{ and } V(G) \to \{q+1, q+2, \cdots, p+q\},$$

in general, we call them two as the *bilabellings* of G.

In general, that λ in the equation (1) is dependent on the labelling f of G, so we denote it by λ_f and call it a *magical constant* of G.

The *complementary labelling* h of a proper total labelling f of G is defined by $h(x) = p + q + 1 - f(x)$ for $x \in V(G) \cup E(G)$.

We say that h is a *total k-float labelling* of a proper total labelling f of G if $h(x) = f(x) + k$ for all $x \in V(G) \cup E(G)$, therefore, we have

$$h(u) + h(v) + h(uv) = 3k + f(u) + f(v) + f(uv) \quad \text{for} \quad uv \in E(G).$$

An *edge k-float labelling* of a proper total labelling f of G is ruled by $h(u) = f(u)$ for all $u \in V(G)$ and $h(uv) = f(uv) + k$ for all $uv \in E(G)$, it then gives us

$$h(u) + h(v) + h(uv) = k + f(u) + f(v) + f(uv) \quad \text{for} \quad uv \in E(G).$$

Suppose that f is an *emt*-labelling or a *semt*-labelling of G with p vertices and q edges, if we cannot add an edge uv to G to adjoin two nonadjacent vertices u and v of G, and label this edge by $p+q+1$ such that the resulting graph is also *emt*-magical or *semt*-magical, then we say G to be a *saturated emt-magical graph* or a *saturated semt-magical graph* by f. Let $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, we say f to be an α-labelling of G if $f(x) < f(y)$ for any edge xy of G with $x \in V_1$ and $y \in V_2$. Clearly, G must be bipartite if G has an α-labelling.

Let G and H be two disjoint graphs, the *join graph* $G + H$ is defined by adjoining each vertex of G to every vertex of H such that $V(G + H) = V(G) \cup V(H)$. For

$$\{u_1, u_2, \cdots, u_m\} \subseteq V(G) \text{ and } \{v_1, v_2, \cdots, v_m\} \subseteq V(H),$$

the *overlapping graph* $G \diamond H$ is obtained by overlapping (identifying) u_i onto v_i into a new vertex (u_i, v_i) for $1 \leq i \leq m$ and remove those possible duplicative edges. Hence, the graph $G \diamond H$ is still simple, connected.

2. Results and Proofs

Let h be the complementary labelling of an *emt*-labelling f of G which holds the equation (1), it turns out that

$$h(u) + h(v) + h(uv) = 3(p+q+1) - [f(u) + f(v) + f(uv)] = 3(p+q+1) - \lambda_f,$$

which shows that the complementary labelling h is an *emt*-labelling of G too with the magical constant $\lambda_h = 3(p+q+1) - \lambda_f$.

Lemma 1. *Let h be the complementary labelling of a proper total labelling f of a simple, connected graph G with p vertices and q edges.*

(i) If f is a vertex bilabelling, then its the complementary labelling h is also an edge bilabelling of G.

(ii) If f is an emt-labelling, so is its complementary labelling h, and $\lambda_h = 3(p+q+1) - \lambda_f$; and

(iii) If f is an semt-labelling, so is its complementary labelling h, and $\lambda_h = 3(p+q+1) - \lambda_f$.

Lemma 2. *Let G be a graph of order p and size q and let G be emt-magical or semt-magical.*

(i) If both p and q are even together or odd together, then G has at least two distinct magical constants.

(ii) For each labelling f that holds the equation (1)*, we have* $p+q+3 \leq \lambda_f \leq 2(p+q)$.

Proof. Let f be an *emt*-magical labelling of G, and we have its complementary labelling h which is also an *emt*-magical labelling of G. By Lemma 1, immediately, $\lambda_h = 3(p+q+1) - \lambda_f$ and then we have $\lambda_h \neq \lambda_f$ if both p and q are even together or odd together, i.e., we have proved the result (i).

Next, we consider the magical constant λ_f, obviously, $\lambda_f \geq p+q+3$ since $f(u) \geq 1$, $f(v) \geq 2$ and $f(uv) = p+q$ for a certain edge $uv \in E(G)$. Suppose that λ_f is the minimum of all magical constants of G which hold the equation (1), thus, it turns out that

$$\lambda_h = 3(p+q+1) - \lambda_f \leq 3(p+q+1) - (p+q+3) = 2(p+q),$$

as desired. □

Lemma 3. *Let G be a connected, semt-magical graph of order p and size q, we have the magical constant $\lambda_f \leq 2p+q$ for any semt-labelling f of G.*

Proof. Let G has p vertices and q edges. Let f be a *semt*-labelling, also a vertex bilabelling of G such that $1 \leq f(x) \leq p$ for any $x \in V(G)$. Suppose that the magical constant $\lambda_f \geq 2p+q+1$ in the following discussion.

We have a vertex v_1 of G such that $f(v_1) = 1$. If the neighbor set $N(v_1)$ of all vertices that are adjacent to v_1 contains two vertices u_{11} and u_{12}, thus,

$$p+1+f(v_1u_{11}) \geq 1+f(u_{11})+f(v_1u_{11}) = f(v_1)+f(u_{11})+f(v_1u_{11}) \geq 2p+q+1$$

and

$$p+1+f(v_1u_{12}) \geq 1+f(u_{12})+f(v_1u_{12}) = f(v_1)+f(u_{12})+f(v_1u_{12}) \geq 2p+q+1,$$

it turns out that $f(v_1u_{11}) = f(v_1u_{12}) = p+q$, a contradiction. Therefore, $N(v_1) = \{u_{11}\}$ and $f(v_1) = 1$, $f(u_{11}) = p$ and $f(v_1u_{11}) = p+q$ by the assumption of $\lambda_f \geq 2p+q+1$.

Suppose that $f(v_2) = 2$. Let u_{21} be a vertex of the neighbor set $N(v_2)$ of all vertices that are adjacent to v_2. We have

$$\begin{aligned} 2+(p-1)+f(v_2u_{21}) &\geq 2+f(u_{21})+f(v_2u_{21}) \\ &= f(v_2)+f(u_{21})+f(v_2u_{21}) \geq 2p+q+1, \end{aligned}$$

it shows us that $f(v_2u_{21}) = p+q$, but this label conflicts with $f(v_1u_{11}) = p+q$. □

It is natural to ask for what structure does G have if this graph G has an *semt*-magical labelling f such that the magical number $\lambda_f = p+q+3$? We, immediately, have a result there is a vertex u of G such that the graph $G_1 = G - v_1$ is also *semt*-magical, in fact, $f(v_1) = 1$ and $f(u_i) = i+1$ for $u_i \in N(v_1) = \{u_1, u_2, \cdots, u_{\mathrm{d}(v_1)}\}$ and $f(v_1u_i) = p+q-i+1$ for $1 \leq i \leq \mathrm{d}(v_1)$. If G_1 has the same property as that of G, we have the second graph $G_2 = G_1 - v_2$, and we may have a sequence $G_k \subset G_{k-1} \subset \cdots \subset G_1 \subset G$ such that $G_i = G_{i-1} - v_i$ is connected and each G_i has an *semt*-magical labelling f_i such that the magical constant $\lambda_{f_i} = p_i + q_i + 3$ for $1 \leq i \leq k$. Then we can deduce that each v_i is a vertex of degree two in G_{i-1} for $1 \leq i \leq k-1$ since $(p_i + q_i + 3) + 3 = p_{i-1} + q_{i-1} + 3$. Furthermore, $|E(G)| = |E(G_k)| + 2k$ and $|V(G)| = |V(G_k)| + k$.

Theorem 4. *If G is regular and semt-magical, then G has the uniquely magical constant.*

Proof. Let G be a k-regular graph with p vertices and q edges. Suppose that G has a *semt*-magical labelling f, then

$$\sum_{uv \in E(G)} [f(u) + f(v) + f(uv)] = q\lambda_f \quad \text{for} \quad 2q = kp$$

and furthermore

$$\begin{aligned} &\sum_{uv \in E(G)} [f(u) + f(v) + f(uv)] \\ &= k(1 + 2 + \cdots + p) + [(p+1) + (p+2) + \cdots + (p+q)] \\ &= k\frac{p(p+1)}{2} + \frac{q(2p+q+1)}{2} = 2q \cdot \frac{p+1}{2} + \frac{q(2p+q+1)}{2}. \end{aligned}$$

Thereby, we have

$$2q\lambda_f = 2q(p+1) + q(2p+q+1) \quad \text{or} \quad 2\lambda_f = 4p + q + 3,$$

it implies that two *semt*-magical labellings of a *semt*-magical regular graph have the same magical constant. □

Theorem 5. *Let two simple, connected graphs G_1 and G_2 be semt-magical, then the join graph $G_1 + G_2$ contains at least a semt-magical graph H with $|V(H)| = |V(G_1)| + |V(G_2)|$.*

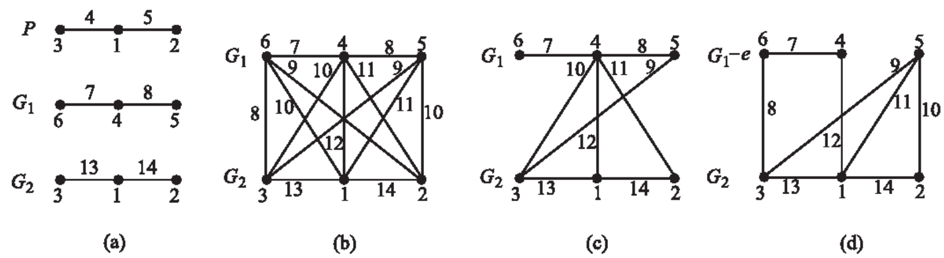

Figure 1: An example for Theorem 5

Proof. Let G_i have p_i vertices and q_i edges and let f_i be an *semt*-labelling of G_i for $i = 1, 2$, also, f_i is a vertex *bi*-labelling of G_i for $i = 1, 2$. Thereby, we have $f_i(u) + f_i(v) + f_i(uv) = \lambda_{f_i}$ for all $uv \in E(G_i)$ and $i = 1, 2$. We have a total p_2-float labelling h of f_1 which holds

$$h(u) + h(v) + h(uv) = 3p_2 + f_1(u) + f_1(v) + f_1(uv) = 3p_2 + \lambda_{f_1},$$

and we can construct an edge $(3p_2 + \lambda_{f_1} - \lambda_{f_2})$-float labelling g of f_2. Next, we define a total labelling of $G_1 + G_2$ as follows. We color each edge (u_1, u_2) of the join graph $(G_1 + G_2)$ with the color number $3p_2 + \lambda_{f_1} - [h(u_1) + g(u_2)]$, where $u_i \in V(G_i)$ $(i = 1, 2)$, and next remove those edges with the same color from the join graph $(G_1 + G_2)$ except one, the remainder graph just is *semt*-magical. □

In (a) of Figure 1, we show the original graph P with an *semt*-labelling f, and G_1 with the total 3-float labelling of f, and G_2 with the edge 9-float labelling of f. In (b) of Figure 1, it gives the join graph $G_1 + G_2$, and we remove an edge subset of some edges from $G_1 + G_2$ such that $(G_1 + G_2) - E^*$ is *semt*-magical, see (c) and (d) in Figure 1.

As known, all of paths and stars are *semt*-magical, so is the cycle C_3, but the cycle C_4. The Figure 2 shows a graphical illustrations of the proof of Corollary 6 below.

Corollary 6. *The join $P_n + H_3$ of a path P_n and the complete graph K_3 is always semt-magical.*

Theorem 7. *For $i = 1, 2$ let G_i be a semt-magical graph with p_i vertices and q_i edges, thereby we have $f_i(u) + f_i(v) + f_i(uv) = \lambda_{f_i}$ for $uv \in E(G_i)$. For $2k \leq \lambda_{f_1} - \lambda_{f_2} - p_1 - q_1 + 3p_2$ is even, then there is a overlapping graph $G_1 \diamond G_2$ that is also semt-magical, and*

$$p_1 + p_2 - k = |V(G_1 \diamond G_2)| \text{ and } |E(G_1 \diamond G_2)| = q_1 + q_2.$$

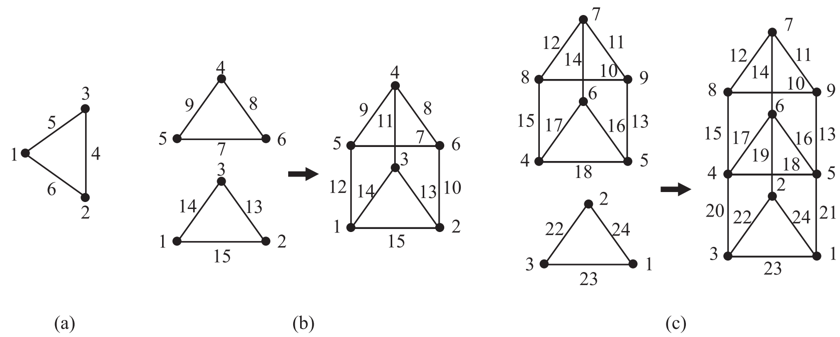

Figure 2: A graphical illustration of Theorem 5 from (a) to (c)

Proof. Let G_i have p_i vertices and q_i edges and let f_i be an *semt*-labelling of G_i for $i = 1, 2$, also, f_i is a vertex *bi*-labelling of G_i for $i = 1, 2$. For an integer $k > 0$, we have a total $(p_2 - k)$-float labelling h of f_1 which holds

$$h(u) + h(v) + h(uv) = 3(p_2 - k) + f_1(u) + f_1(v) + f_1(uv) = 3(p_2 - k) + \lambda_{f_1},$$

the maximum of labels of edges in G_1 is $p_1 + q_1 + q_2 - k$. We can construct an edge M-float labelling g of f_2 such that

$$\lambda_{f_2} + M = 3(p_2 - k) + \lambda_1 \quad \text{or} \quad M + 3k = 3p_2 + \lambda_{f_1} - \lambda_{f_2}.$$

There is $M \geq p_1 + q_1 - k$ since the minimum of edge labels of G_2 under the labelling g is $p_2+1+M \geq p_1+q_1+p_2-k+1$, we have $2k \leq \lambda_{f_1} - \lambda_{f_2} - p_1 - q_1 + 3p_2$. Without loss of generalization, suppose that $f_1(u_i) = i$ for $1 \leq i \leq p_1$, and $f_2(v_j) = j$ for $1 \leq j \leq p_2$. Notice that

$$h(u_i) = f_1(u_i) + p_2 - k = p_2 - k + i = f_2(v_{p_2-k+i}) \text{ for } 1 \leq i \leq k.$$

We, hence, overlap u_i onto v_{p_2-k+i} into a new vertex $w_i = (u_i, v_{p_2-k+i})$ for $1 \leq i \leq k$, the overlapping graph $G_1 \diamond G_2$ is *semt*-magical with the vertex set

$$V^* = \{v_1, v_2, \cdots, v_{p_2-k}, w_1, w_2, \cdots, w_k, u_{k+1}, u_{k+2}, \cdots, u_{p_1}\}.$$

This overlapping graph $G_1 \diamond G_2$ has a *semt*-labelling f^* as the form: $f^*(u_i) = h(u_i)$ for $u_i \in V^*$, $f^*(v_j) = g(v_j)$ for $v_j \in V^*$ and $f^*(w_i) = p_2 - k + i$ for $w_i \in V^*$ and $1 \leq i \leq k$; for each edge uv of $G_1 \diamond G_2$ we have $f^*(uv) = h(uv)$ if $uv \in E(G_1)$ and $f^*(uv) = g(uv)$ if $uv \in E(G_2)$. □

In the proof of the above Theorem 7, we can take $M < p_1 + q_1 - k$, but $|E(G_1 \diamond G_2)| < q_1 + q_2$, namely, there are some duplicate edges after applying the overlapping operation on two graphs.

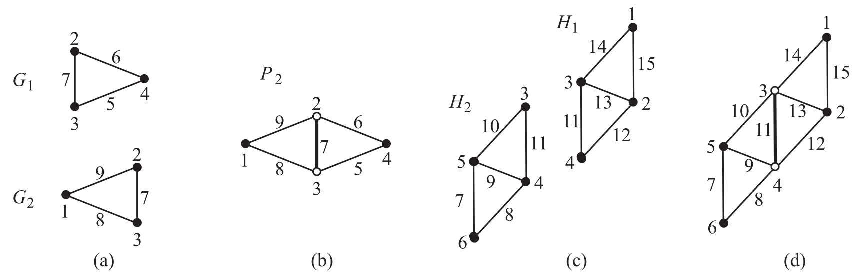

Figure 3:

In Figure 1 (b) and Figure 3 (c), we give two different *semt*-labellings of $K_4 - e$ but they have the same magical number 12.

Theorem 8. *Let u be a vertex apart from a connected and semt-magical graph G, then we have an semt-magical graph obtained by adjoining u to a certain vertex of G with an edge.*

Proof. Let f be a *semt*-labelling, also a vertex bilabelling of G of order p and size q such that $1 \leq f(x) \leq p$ for any $x \in V(G)$, and the magical constant $\lambda_f \leq 2p+q$ by Lemma 3. We adjoin u to a vertex u_0 of G, the resulting graph, denoted by H, has $p_1 = p+1$ vertices and $q_1 = q+1$ edges. We give directly a labelling h of H as this: $h(u) = 1$, $h(uu_0) = p+q+2$, and $h(x) = f(x)+1$ whenever $x \in V(H)\backslash\{u\}$ and $h(xy) = f(xy)+1$ for any edge $xy \in E(H)\backslash\{uu_0\}$, thereby, $\lambda_h = \lambda_f + 3$ and

$$\begin{aligned}\lambda_h = h(u) + h(uu_0) + h(u_0) &= 1 + p + q + 2 + f(u_0) + 1 = \lambda_f + 3 \\ &\leq 2p + q + 3 = 2(p+1) + (q+1) = 2p_1 + q_1,\end{aligned}$$

also, we have $f(u_0) = \lambda_f - (p+q+1) \leq 2p+q-(p+q+1) = p-1$, thereby h is a *semt*-labelling of H since f is a *semt*-labelling of G. □

Obviously, we are not able to prove the magical trees problem by Theorem 8 and the mathematical induction. A graph may be *semt*-magical or *emt*-magical (see Figure 4 (a), (b), (c)), noticeable, $K_{1,2}$ has an *emt*-labelling f with magical constant λ_f with respect to the inequality $\lambda_f \leq 2p+q$. Thereby, we cannot conjecture: If G has a magical labelling f such that the magical constant $\lambda_f \geq 2p+q+1$, then G is *emt*-magical. But, we add an additional condition that if G is a graph with the minimal degree great than 2 to our conjecture, we guess it may be available. A star $K_{1,n}$ could be *semt*-magical and *emt*-magical, and α-labelling.

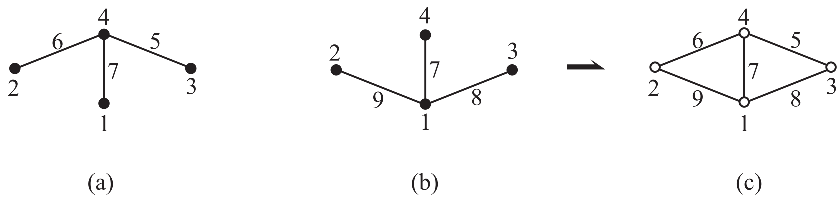

Figure 4:

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