International Journal of Pure and Applied Mathematics
Volume 34 No. 2 2007, 161-174

DETERMINANT EVALUATIONS FOR WEIGHING MATRICES

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Abstract: In the present paper we are interested in calculating the minors of weighing matrices $W(n, n-k)$ for $k \geq 1$. By using appropriate determinant manipulations we obtain formulas for the minors of order $n-1$, $n-2$ and $n-3$ of a $W(n, n-1)$ and show how these ideas can be similarly applied to determinant evaluations for $W(n, n-k)$ generally. We demonstrate the proof of the most generalized version of the Determinant Simplification Theorem and show how it can be used for the evaluation of minors of order $n-j$, $j \geq 1$, of weighing matrices. Application to numerical analysis connected with the growth problem is also given.

AMS Subject Classification: 15A15, 05B20, 65F40, 65F05, 65G50
Key Words: determinants, weighing matrices, symbolic computation, Gaussian elimination, complete pivoting, growth problem

1. Introduction

The problem of evaluating determinants is important and interesting in linear algebra due to their numerous applications. It is always challenging to derive analytical formulas for the determinant of an arbitrary given matrix, or for the principal determinants (minors) of it, but it seems more realistic to obtain such results for specially structured matrices. In this work we deal with minors

Received: October 6, 2006

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of weighing matrices $W(n, n-k)$ for n even and $k \geq 1$. A $(0, 1, -1)$ matrix $W = W(n, n-k)$, $k = 1, 2, \ldots$, of order n satisfying $WW^T = (n-k)I_n$ is called a *weighing matrix of order* n *and weight* $n-k$ or simply a *weighing matrix*. For $k = 0$ and $n \equiv 0 \,(\text{mod} 4)$, W is called a *Hadamard matrix*. Two important properties of the weighing matrices, which follow directly from the definition, are:

1. Every row and column of a $W(n, n-k)$ contains exactly k zeros;
2. Every two distinct rows and columns of a $W(n, n-k)$ are orthogonal to each other, which means that their inner product is zero.

Two matrices are said to be *Hadamard equivalent* or *H-equivalent* if one can be obtained from the other by a sequence of the operations:

1. Interchange any pairs of rows and/or columns;
2. Multiply any rows and/or columns through by -1.

By using appropriate determinant manipulations we provide theoretical proofs and formulas for the minors of order $n-1$, $n-2$ and $n-3$ of a $W(n, n-1)$. Although one cannot easily proceed with the minors of orders $n-4$, $n-5$, etc. due to the complicated calculations done by hand, the benefit of this presentation is that these ideas can be similarly applied to determinant evaluations for $W(n, n-k)$ generally. In order to overcome these difficulties, we demonstrate the most generalized version of the Determinant Simplification Theorem and illustrate how it can be used for the evaluation of minors of order $n-j$, $j \geq 1$, of weighing matrices. Finally, an application to numerical analysis connected with the growth problem is also given, where our motivation for studying the minors of weighing matrices lies.

Notation. Throughout this paper we assume, without loss of generality, that the first non zero entries of a row and column of a weighing matrix are always $+1$, because this can be achieved easily with the H-equivalent operation of multiplying columns and/or rows with -1 and leaves unaffected the magnitude of the determinant. For $k \geq 2$, the order n is considered to be even. The elements of a $(0, 1, -1)$ matrix will be denoted by $(0, +, -)$. I_n and J_n stand for the identity matrix of order n and the matrix with ones of order n, respectively. We write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper left corner of the matrix A. We denote with $x_{m \times n}$ the $m \times n$ block with elements x, x real, and with $X_{m \times n}$ the $m \times n$ block with the specific form of the matrix X.

Let $\underline{x}^T_{\beta+1}$ the vectors containing the binary representation of each integer $\beta + 2^{j-1}$ for $\beta = 0, \ldots, 2^{j-1} - 1$. Replace all zero entries of $\underline{x}^T_{\beta+1}$ by -1 and define the $j \times 1$ vectors $\underline{u}_k = \underline{x}_{2^{j-1}-k+1}$, $\;k = 1, \ldots, 2^{j-1}$. We write U_j for all the matrices with j rows and the appropriate number of columns, in which $\underline{u}_k$

occurs u_k times. So

$$U_j = \begin{array}{ccccc} \overbrace{+...+}^{u_1} & \overbrace{+...+}^{u_2} & \dots & \overbrace{+...+}^{u_{2^{j-1}-1}} & \overbrace{+...+}^{u_{2^{j-1}}} \\ +...+ & +...+ & \dots & -...- & -...- \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ +...+ & +...+ & \dots & +...+ & -...- \\ +...+ & -...- & \dots & +...+ & -...- \end{array} = \begin{array}{ccccc} u_1 & u_2 & \dots & u_{2^{j-1}-1} & u_{2^{j-1}} \\ + & + & \dots & + & + \\ + & + & \dots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ + & + & \dots & - & - \\ + & - & \dots & + & - \end{array}$$

Example 1.

$$U_3 = \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ + & + & + & + \\ + & + & - & - \\ + & - & + & - \end{array} \quad \text{and} \quad U_4 = \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \end{array}$$

Preliminary Results. 1. Let $A = (k-\lambda)I_v + \lambda J_v$, where $k,\ \lambda$ are integers. Then

$$\det A = [k + (v-1)\lambda](k-\lambda)^{v-1} \tag{1}$$

and

$$A^{-1} = \frac{1}{k^2 + (v-2)k\lambda - (v-1)\lambda^2}\{[k + (v-2)\lambda + \lambda]I - \lambda J\}\,. \tag{2}$$

2. Let $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$. Then

$$\det B = \det B_1 \cdot \det\left(B_4 - B_3 B_1^{-1} B_2\right). \tag{3}$$

2. Theoretical Calculations

In this section we will demonstrate results for the $n-1,\ n-2$ and $n-3$ minors of a $W(n, n-1)$ and we will explain how the ideas presented in the proof of Proposition 3 can be generalized for a $W(n, n-k)$.

In [3] has been proved the following result.

Proposition 1. *Let W be a $W(n, n-1)$. Then all possible $(n-1)\times(n-1)$ minors are: $W(n-1) = 0$ or $(n-1)^{\frac{n}{2}-1}$ and all possible $(n-2)\times(n-2)$ minors are: $W(n-2) = 0$ or $2(n-1)^{\frac{n}{2}-2}$.*

The next Proposition 2 computes the $n-3$ minors of a $W(n,n-1)$.

Proposition 2. *Let W be a $W(n,n-1)$. Then all possible $(n-3)\times(n-3)$ minors of W are: $W(n-3)=0, 2(n-1)^{\frac{n}{2}-3}$, or $4(n-1)^{\frac{n}{2}-3}$ for $n\equiv 0(\bmod 4)$ and $W(n-3)=2(n-1)^{\frac{n}{2}-3}$, or $4(n-1)^{\frac{n}{2}-3}$ for $n\equiv 2(\bmod 4)$.*

Proof. There are 48 possible cases, up to H-equivalence, for the upper left corner:

$$\begin{bmatrix} + & + & + \\ + & \pm & \pm \\ + & \pm & \pm \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & \pm & \pm \\ + & \pm & \pm \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & \pm \\ + & \pm & \pm \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & \pm \\ + & \pm & 0 \end{bmatrix}.$$

We will carry out the proof for a matrix of the second case, since the other cases can be handled similarly. Since we have that matrix W is a $W(n,n-1)$ let us suppose that it can be written in the following form:

$$W = \left[\begin{array}{ccc|cccccc}
 & & & & & \overbrace{\quad\quad}^{u} & \overbrace{\quad\quad}^{v} & \overbrace{\quad\quad}^{x} & \overbrace{\quad\quad}^{y} \\
+ & + & + & 0 & + & + \dots + & + \dots + & + \dots + & + \dots + \\
+ & - & 0 & + & \pm & + \dots + & + \dots + & - \dots - & - \dots - \\
+ & + & - & \pm & 0 & + \dots + & - \dots - & + \dots + & - \dots - \\ \hline
0 & + & \pm & & & & & & \\
+ & 0 & \pm & & & & & & \\
+ & + & + & & & & & & \\
 & \vdots & & & & & & & \\
+ & + & + & & & & & & \\
+ & + & - & & & & \mathrm{D} & & \\
 & \vdots & & & & & & & \\
+ & + & - & & & & & & \\
+ & - & + & & & & & & \\
 & \vdots & & & & & & & \\
+ & - & + & & & & & & \\
+ & - & - & & & & & & \\
 & \vdots & & & & & & & \\
+ & - & - & & & & & &
\end{array}\right],$$

where the first three columns contain also u $[+,+,+]$, v $[+,+,-]$, x $[+,-,+]$ and y $[+,-,-]$. First we will examine the case where the entry $(2,5)$ is $+1$ and $(3,4)$ is -1.

From the order of the matrix W and the orthogonality of its three first rows we get the following system of four equations

$$\begin{cases} u+v+x+y=n-5\,, \\ u+v-x-y=-1\,, \\ u-v+x-y=-1\,, \\ u-v-x+y=1\,. \end{cases}$$

which has the exact solution $(u,v,x,y)=\frac{1}{4}(n-6,n-6,n-6,n-2)$ and in this case we have obviously $n\equiv 2(\bmod 4)$.

According to the properties of a $W(n,n-1)$, the $(n-3)\times(n-3)$ matrix DD^T has the form $DD^T=\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, where

$$A=\begin{bmatrix} n-3 & -1 \\ -1 & n-3 \end{bmatrix},\quad B=\begin{bmatrix} \overbrace{0}^{u} & \overbrace{-2}^{v} & \overbrace{2}^{x} & \overbrace{0}^{y} \\ -2 & -2 & 0 & 0 \end{bmatrix},$$

$$C=\begin{bmatrix} C_{1u\times u} & -1_{u\times v} & -1_{u\times x} & 1_{u\times y} \\ -1_{v\times u} & C_{1v\times v} & 1_{v\times x} & -1_{v\times y} \\ -1_{x\times u} & 1_{x\times v} & C_{1x\times x} & -1_{x\times y} \\ 1_{y\times u} & -1_{y\times v} & -1_{y\times x} & C_{1y\times y} \end{bmatrix},$$

$$C_1=\begin{bmatrix} n-4 & -3 & \dots & -3 \\ -3 & n-4 & \dots & -3 \\ \vdots & \vdots & \ddots & \vdots \\ -3 & -3 & \dots & n-4 \end{bmatrix}.$$

From this point and on, the idea of the proof is to apply consecutively formula (3) appropriately for the appearing block matrices and carry out the calculations with help of (1) and (2). According to (3), we have

$$\det DD^T=\det A\cdot\det(C-B^TA^{-1}B). \qquad (4)$$

After the necessary calculations we have

$$\det A=(n-4)(n-2)\,, \qquad (5)$$

$$\text{and}\quad C-B^TA^{-1}B=\begin{bmatrix} E_{u\times u} & F \\ F^T & G \end{bmatrix},$$

where E is a matrix of the form $E=(n-1)I-\dfrac{3n^2-14n+12}{(n-4)(n-2)}J$,

$$F=\begin{bmatrix} -n_{u\times v} & -n^2+6n-4_{u\times x} & 1_{u\times y} \end{bmatrix}\quad\text{and}$$

$$G=\begin{bmatrix} H_{v\times v} & n_{v\times x} & -1_{v\times y} \\ n_{x\times v} & E_{x\times x} & -1_{x\times y} \\ -1_{y\times v} & -1_{y\times x} & C_{1y\times y} \end{bmatrix},$$

where H is a matrix of the form $H=(n-1)I-\dfrac{3n-4}{(n-4))}J$.

So, according to (3),

$$\det(C - B^T A^{-1} B) = \det E_{u\times u} \cdot \det(G - F^T E_{u\times u}^{-1} F). \qquad (6)$$

From (1) we have

$$\det E_{u\times u} = \frac{(n^3+4n^2-40n+40)(n-1)^{\frac{n-10}{4}}}{4(n-2)(n-4)} \qquad (7)$$

and from (2) $E_{u\times u}^{-1} = (n-1)I_u - \frac{4(3n^2-14n+12)}{4(n-4)(n-2)} J_u$.

Hence,

$$\begin{aligned} G - F^T E_{u\times u}^{-1} F &= \frac{n-1}{n^3+4n^2-40n+40} \begin{bmatrix} K_{1v\times v} & L_{1v\times x} & -L_{2v\times y} \\ L_{1x\times v} & K_{2x\times x} & -L_{3x\times y} \\ -L_{2y\times v} & -L_{3y\times x} & K_{3y\times y} \end{bmatrix} \\ &\equiv \frac{n-1}{n^3+4n^2-40n+40} \begin{bmatrix} K_{1v\times v} & N_2 \\ N_2^T & N_1 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} &K_1 = (k_1-\lambda_1)I + \lambda_1 J, \\ &k_1 = n^3 - 60n + 80, \quad \lambda_1 = -4(n^2+5n-10), \\ &K_2 = (k_2-\lambda_2)I + \lambda_2 J, \quad k_2 = (n-2)(n^2+2n-44), \\ &\lambda_2 = -4(n^2+2n-12), \quad K_3 = (k_3-\lambda_3)I + \lambda_3 J, \\ &k_3 = (n-1)(n-4)(n^2+4n-28), \quad \lambda_3 = -4(n^2+n-18), \\ &L_1 = \lambda_4 J, \quad \lambda_4 = 16n, \quad L_2 = \lambda_5 J, \quad \lambda_5 = 3n-10\,, \\ &L_3 = \lambda_6 J, \quad \lambda_6 = -16(n-4). \end{aligned}$$

The matrices I, J in the above assignments are of appropriate order.

So, according to (3),

$$\det(G - F^T E_{u\times u}^{-1} F) = \det K_{1v\times v} \cdot \det(N_1 - N_2^T {K_{1v\times v}}^{-1} N_2). \qquad (8)$$

We proceed in an absolute similar way like before in order to calculate $\det K_{1v\times v}$ and $\det(N_1 - N_2^T K_{v\times v}^{-1} N_2)$, by making use of (1), (2) and (3). We have

$$\det K_{1v\times v} = \frac{5(n-1)(n-2)(n+2)(n-1)^{\frac{n-10}{4}}}{n^3+4n^2-40n+40} \qquad (9)$$

$$N_1 - N_2^T K_{1v\times v}{}^{-1} N_2 = \frac{n-1}{n^2-4}\begin{bmatrix} P_{1x\times x} & Q_{1x\times y} \\ Q_{1y\times x} & P_{2y\times y}\end{bmatrix},$$

where

$$P_1 = (p_1 - q_1)I + q_1 J, \quad p_1 = \frac{5n^2-20n-44}{5}, \quad q_1 = \frac{-4(5n+6)}{5},$$

$$P_2 = (p_2 - q_2)I + q_2 J, \; p_2 = \frac{5n^5+252n^3+5120n-1736n^2-4640-36n^4}{5(n^3+4n^2-40n+40)},$$

$$q_2 = \frac{-4(5n-6)}{5}, \quad Q_1 = q_3 J, \text{ and } q_3 = \frac{-32}{5}.$$

According to (3),

$$\det(N_1 - N_2^T K_{1v\times v}{}^{-1} N_2)$$
$$= \det P_{1x\times x} \cdot \det(P_{2y\times y} - Q_{1y\times x} P_{1x\times x}{}^{-1} Q_{1x\times y}), \quad (10)$$

$$\det P_{1x\times x} = \frac{8(3n+2)(n-1)(n-1)^{\frac{n-10}{4}}}{5(n^2-4)}, \quad (11)$$

$$P_{2y\times y} - Q_{1y\times x} P_{1x\times x}{}^{-1} Q_{1x\times y} = R_{3y\times y}, \quad (12)$$

where $R_{3y\times y} = (r_1 - s_1)I + s_1 J$, $r_1 = \dfrac{(n-1)(3n-10)}{3n+2}$, $s_1 = \dfrac{-12(n-1)}{3n+2}$,

$$\det R_{3y\times y} = \frac{8(n-1)^{\frac{n-2}{4}}}{3n+2}. \quad (13)$$

Finally, from (4), (5), (6), (7), (8), (9), (10), (11), (12) and (13) we have

$$\det DD^T = \det A \det E_{u\times u} \det K_{1v\times v} \det P_{1x\times x} \det R_{3y\times y} = 16(n-1)^{n-6}.$$

Hence, $\det D \equiv W(n-3) = 4(n-1)^{\frac{n}{2}-3}$.

Similarly we handle all possible cases for the entries $(2,5)$ and $(3,4)$ and we obtain the results $W(n-3) = 0, 4(n-1)^{\frac{n}{2}-3}$.

Similar calculations for the remaining cases of the upper left corner complete the proof. □

The idea in the above demonstrated proof can be utilized for the analytical computations of $n-j$ minors of a general $W(n, n-k)$. For this purpose, one should take into account the following lemma.

Lemma 1. *We can always have two rows of a $W(n, n-k)$, n even, $k \geq 2$, containing a 2×2 block with zeros. Actually this can be achieved by performing at most one appropriate row and column exchange.*

Hence, it follows that two rows of a $W(n, n-k)$ will have the form:

$$\begin{array}{cccc} \overbrace{\begin{matrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & + & \dots & + \end{matrix}}^{k} & \overbrace{\begin{matrix} + & \dots & + \\ 0 & \dots & 0 \end{matrix}}^{k-2} & \overbrace{\begin{matrix} + & \dots & + \\ + & \dots & + \end{matrix}}^{u} & \overbrace{\begin{matrix} + & \dots & + \\ - & \dots & - \end{matrix}}^{u} \end{array}$$

The significance of Lemma 1 lies in the fact that with appropriate application of it, we can determine the form of the first few j rows and columns of a $W(n, n-k)$, $j \geq 2$. So, in order to calculate the $n-j$ minor, we write the first j rows and columns f a $W(n, n-k)$ according to Lemma 1 and carry out the calculations similarly to the proof of Proposition 2.

3. The Determinant Simplification Theorem

Notation. We write $J_{b_1, b_2, \cdots, b_z}$ for the all ones matrix with diagonal blocks of sizes $b_1 \times b_1$, $b_2 \times b_2$ $\cdots$ $b_z \times b_z$, and $a_{ij} J_{b_1, b_2, \cdots, b_z}$ for the matrix, for which the elements of the block with corners $(i + b_1 + b_2 + \cdots + b_{j-1}, i + b_1 + b_2 + \cdots + b_{i-1})$, $(i + b_1 + b_2 + \cdots + b_{j-1}, b_1 + b_2 + \cdots + b_i)$, $(b_1 + b_2 + \cdots + b_j, i + b_1 + b_2 + \cdots + b_{i-1})$, $(b_1 + b_2 + \cdots + b_j, b_1 + b_2 + \cdots + b_i)$ are a_{ij} an integer.

We write $(k_i - a_{ii}) I_{b_1, b_2, \cdots, b_z}$ for the direct sum $(k_1 - a_{11}) I_{b_1} + (k_2 - a_{22}) I_{b_2} + \cdots + (k_z - a_{zz}) I_{b_z}$.

Example 2. According to this notation we have that the matrix

$$A = \begin{bmatrix} a & k & b & b & b \\ k & a & b & b & b \\ & & & & \\ b & b & k & a & a \\ b & b & a & k & a \\ b & b & a & a & k \end{bmatrix},$$

can be written as $A = (k - a_{ii}) I_{2,3} + a_{ij} J_{2,3}$, where $(a_{ij}) = \left[\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right]$.

Theorem 1. (The most generalized version of the Determinant Simplification Theorem) *Let* $A = (k_i - a_{ii}) I_{b_1, b_2, \cdots, b_z} + a_{ij} J_{b_1, b_2, \cdots, b_z}$, $i, j = 1, \dots, z$. *Then*

$$\det A = \prod_{i=1}^{z} (k_i - a_{ii})^{b_i - 1} \det D$$

where

$$D = \begin{bmatrix} k_1 + (b_1 - 1)a_{11} & b_2 a_{12} & b_3 a_{13} & \cdots & b_z a_{1z} \\ b_1 a_{21} & k_2 + (b_2 - 1)a_{22} & b_3 a_{23} & \cdots & b_z a_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ b_1 a_{z1} & b_2 a_{z2} & b_3 a_{z2} & \cdots & k_z + (b_z - 1)a_{zz} \end{bmatrix}.$$

Proof. The matrix A is of the form

$$A = \left[\begin{array}{cccc} \overbrace{\begin{array}{cccc} k_1 & a_{11} & \cdots & a_{11} \end{array}}^{b_1} & \overbrace{\begin{array}{cccc} a_{12} & a_{12} & \cdots & a_{12} \end{array}}^{b_2} & \cdots & \overbrace{\begin{array}{cccc} a_{1z} & a_{1z} & \cdots & a_{1z} \end{array}}^{b_z} \\ a_{11} \; k_1 \; \cdots \; a_{11} & a_{12} \; a_{12} \; \cdots \; a_{12} & \cdots & a_{1z} \; a_{1z} \; \cdots \; a_{1z} \\ \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots \\ a_{11} \; a_{11} \; \cdots \; k_1 & a_{12} \; a_{12} \; \cdots \; a_{12} & \cdots & a_{1z} \; a_{1z} \; \cdots \; a_{1z} \\ & & & \\ a_{21} \; a_{21} \; \cdots \; a_{21} & k_2 \; a_{22} \; \cdots \; a_{22} & \cdots & a_{2z} \; a_{2z} \; \cdots \; a_{2z} \\ a_{21} \; a_{21} \; \cdots \; a_{21} & a_{22} \; k_2 \; \cdots \; a_{22} & \cdots & a_{2z} \; a_{2z} \; \cdots \; a_{2z} \\ \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots \\ a_{21} \; a_{21} \; \cdots \; a_{21} & a_{22} \; a_{22} \; \cdots \; k_2 & \cdots & a_{2z} \; a_{2z} \; \cdots \; a_{2z} \\ & & & \\ \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots & \vdots \; \vdots \quad \vdots \end{array}\right],$$

of order $(b_1 + b_2 + \ldots + b_z) \times (b_1 + b_2 + \ldots + b_z)$ and with diagonal blocks of sizes $b_1 \times b_1$, $b_2 \times b_2$, $\ldots$ and $b_z \times b_z$.

We denote the matrix A as $[\underline{c}_1 \; \underline{c}_2 \; \ldots \; \underline{c}_{b_1+b_2+\ldots b_z}]$ in column form and as $[\underline{r}_1^T \; \underline{r}_2^T \; \ldots \; \underline{r}_{b_1+b_2+\ldots b_z}^T]$ in row form.

We replace $\underline{r}_2^T, \ldots, \underline{r}_{b_1}^T$ with $\underline{r}_2^T - \underline{r}_1^T, \ldots, \underline{r}_{b_1}^T - \underline{r}_1^T$, respectively, $\underline{r}_{b_1+2}^T, \ldots, \underline{r}_{b_1+b+2}^T$ with $\underline{r}_{b_1+2}^T - \underline{r}_{b_1+1}^T, \ldots, \underline{r}_{b_1+b_2}^T - \underline{r}_{b_1+1}^T$, respectively, etc. and so we obtain

$$\left[\begin{array}{cccccccccccc} k_1 & a_{11} & a_{11} & \cdots & a_{11} & a_{12} & a_{12} & a_{12} & \cdots & a_{12} & \cdots & \\ a_{11} - k_1 & k_1 - a_{11} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \\ \vdots & & & & & \vdots & & & & & & \\ a_{11} - k_1 & 0 & 0 & \cdots & k_1 - a_{11} & 0 & 0 & 0 & \cdots & 0 & \cdots & \\ & & & & & & & & & & & \\ a_{21} & a_{21} & a_{21} & \cdots & a_{21} & k_2 & a_{22} & a_{22} & \cdots & a_{22} & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & a_{22} - k_2 & k_2 - a_{22} & 0 & \cdots & 0 & \cdots & \\ \vdots & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & & a_{22} - k_2 & 0 & 0 & \cdots & k_2 - a_{22} & \cdots \\ & & & & & & & & & & & \\ \vdots & & & & & \vdots & & & & & \ddots & \end{array}\right].$$

For the sake of simplicity and better presentation we omit writing the rest of the blocks.

In order to zero the entries below k_1, k_2, etc. in the same block, we replace $\underline{c}_1, \underline{c}_{b_1+1}$, etc. with $\underline{c}_2 + \ldots \underline{c}_{b_1}, \underline{c}_{b_1+1} + \ldots \underline{c}_{b_1+b_2}$, etc., respectively, and get

$$\begin{bmatrix} k_1+(b_1-1)a_{11} & a_{11} & \cdots & a_{11} & b_2a_{12} & a_{12} & \cdots & a_{12} & \cdots \\ 0 & k_1-a_{11} & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \vdots & & & \\ 0 & 0 & \cdots & k_1-a_{11} & 0 & 0 & \cdots & 0 & \cdots \\ b_1a_{21} & a_{21} & \cdots & a_{21} & k_2+(b_2-1)a_{22} & a_{22} & \cdots & a_{22} & \cdots \\ 0 & 0 & \cdots & 0 & 0 & k_2-a_{22} & \cdots & 0 & \cdots \\ \vdots & & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & k_2-a_{22} & \cdots \\ \vdots & & & & & \vdots & & & \ddots \end{bmatrix}.$$

Then we can zero the entries a_{ii} in the first row of every diagonal block by replacing $\underline{r}_1^T, \underline{r}_{b_1+1}^T$, etc. with $\underline{r}_1^T - \frac{a_{11}}{k_1-a_{11}}\underline{r}_2^T - \ldots - \frac{a_{11}}{k_1-a_{11}}\underline{r}_{b_1}^T$, $\underline{r}_{b_1+1}^T - \frac{a_{22}}{k_2-a_{22}}\underline{r}_{b_1+2}^T - \ldots - \frac{a_{22}}{k_2-a_{22}}\underline{r}_{b_1+b_2}^T$, etc., respectively.

We obtain $A' \equiv$

$$\begin{bmatrix} k_1+(b_1-1)a_{11} & 0 & \cdots & 0 & b_2a_{12} & a_{12} & \cdots & a_{12} & \cdots \\ 0 & k_1-a_{11} & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \vdots & & & \\ 0 & 0 & \cdots & k_1-a_{11} & 0 & 0 & \cdots & 0 & \cdots \\ b_1a_{21} & a_{21} & \cdots & a_{21} & k_2+(b_2-1)a_{22} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & k_2-a_{22} & \cdots & 0 & \cdots \\ \vdots & & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & k_2-a_{22} & \cdots \\ \vdots & & & & & \vdots & & & \ddots \end{bmatrix}.$$

We note that this matrix A' derived from A with application of elementary column and row operations (specifically, addition of a multiple of a column/row to an other) has the same determinant as A, according to the known property of determinants.

Next, we expand the determinant of A' by the second row. Taking into account the fact that the second row contains only zero entries apart from $k_1 - a_{11}$ implies that det$A$ equals to the product of $k_1 - a_{11}$ and the determinant of the remaining $(b_1+b_2+\ldots+b_z-1) \times (b_1+b_2+\ldots+b_z-1)$ matrix A'', which is actually A' with its second row and column removed. Similarly, we expand A'' by its second row, which contains again $k_1 - a_{11}$ and the other entries are zero. After we have expanded in the same manner the determinant by all the rows in the first diagonal block, we proceed similarly with the entries $k_2 - a_{22}$

in the second diagonal block etc., until we reach the last (the z-th) diagonal block.

After all these steps we obtain

$$\det A = \det A' = \prod_{i=1}^{z}(k_i - a_{ii})^{b_i - 1}\det D,$$

with D given as in the enunciation of the theorem. □

The above illustrated Theorem 1 can be used for specifying minors of order $n-j$, $j \geq 1$, of a weighing matrix. Any matrix $W = W(n, n-1)$ can be written in the structure $W = \begin{bmatrix} M & U_j \\ U_j^T & C \end{bmatrix}$.

M, C are $j \times j$ and $(n-j) \times (n-j)$ matrices respectively with zeros on the diagonal. Obviously, M has j zeros on the diagonal and C has $n-j$. The elements in the $(n-j) \times (n-j)$ matrix CC^T obtained by removing the first j rows and columns of the weighing matrix W can be permuted to appear in the form

$$CC^T = (n - 1 - j - a_{ii})I_{u_1, u_2, \cdots, u_{2^{j-1}}} + a_{ik}J_{u_1, u_2, \cdots, u_{2^{j-1}}} \,,$$

where $(a_{ik}) = (-\underline{u}_i \cdot \underline{u}_k)$, $a_{ii} = (-\underline{u}_i \cdot \underline{u}_i) = -j$, with $\cdot$ the inner product. By the Determinant Simplification Theorem

$$\det \; CC^T = (n-1)^{n-2^{j-1}-j}\det D,$$

where D, of order 2^{j-1}, is given by

$$D = \begin{bmatrix} n-1-ju_1 & u_2a_{12} & u_3a_{13} & \cdots & u_za_{1z} \\ u_1a_{21} & n-1-ju_2 & u_3a_{23} & \cdots & u_za_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ u_1a_{z1} & u_2a_{z2} & u_3a_{z2} & \cdots & n-2-ju_z \end{bmatrix},$$

where $z = 2^{j-1}$.

The $(n-j) \times (n-j)$ minor of W is the determinant of C, for which we have

$$\det C = ((n-1)^{n-2^{j-1}-j}\det D)^{1/2} \,.$$

Remark 1. In order to find all possible values of the $(n-j)$ minors, one should insert all possible matrices M in the upper left $j \times j$ corner of W. It is sufficient to consider the elements of the first row and column as $+1$ and to perform exhaustive searches over the rest of the entries.

Remark 2. We described the simplest case of a $W(n, n-1)$ just for an overview. In this case we can work with a less general version of the Determinant Simplification Theorem, in which k_i of the enunciation are equal to a constant $c \equiv k_i$, but for a $W(n, n-k)$, $k > 1$, the most generalized version of it is necessary. The advantage of the above described method-algorithm lies in the fact that it can be easily modified appropriately to work for every $k > 1$.

4. Application to the Growth Problem

Traditionally, backward error analysis for GE (Gaussian Elimination) on a matrix $A = (a_{ij}^{(0)})$ is expressed in terms of the *growth factor*

$$g(n, A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}^{(0)}|},$$

which involves all the elements $a_{ij}^{(k)}$, $k = 0, 1, 2, \ldots, n-1$ that occur during the elimination. For a completely pivoted (CP, no exchanges are needed during GE with complete pivoting) matrix A we have

$$g(n, A) = \frac{\max\{p_1, p_2, \ldots, p_n\}}{|a_{11}^{(0)}|},$$

where $p_1, p_2, \ldots, p_n$ are the pivots of A.

Cryer [1] defined $g(n) = \sup\{\, g(n, A) \mid A \in \mathbb{R}^{n\times n}, CP\}$. The problem of determining $g(n)$ for various values of n is called the *growth problem*. The determination of $g(n)$ in general remains a mystery. Wilkinson in [5] proved that $g(n) \leq [n\, 2\, 3^{1/2} \ldots n^{1/(n-1)}]^{1/2}$ and that this bound is not attainable.

In [1] Cryer conjectured "$g(n, A) = n$, iff A is a Hadamard matrix". This conjecture became one of the most famous open problems in numerical analysis, has been investigated by many mathematicians and is still an open problem. Experiments on the computer have led us to a similar formulation for the growth conjecture for weighing matrices, cf. [3].

It can be proved (see [2]) that the magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = \frac{W(j)}{W(j-1)}, \quad j = 1, 2, \ldots, n, \quad W(0) = 1. \tag{14}$$

So, it is obvious that the calculation of minors is important in order to study pivot structures, and moreover the growth problem for CP weighing matrices.

Our above described study is intended to calculate $(n-j)$ minors, so with help of relation (14) we can calculate actually pivots from the end of the pivot pattern of a matrix. In [4] was given an algorithmic counting technique for specifying pivots from the beginning. So, the combination of these two methods can result to the calculation of the whole pivot pattern for a $W(n, n-k)$.

With help of these techniques, we could calculate the pivot pattern for $W(6,5)$, $W(8,7)$, $W(10,9)$, $W(12,11)$, $W(6,4)$, $W(8,6)$, $W(10,8)$ and $W(12,10)$ and finally establish that their growth factors are equal to 5, 7, 9, 11, 4, 6, 8 and 10, respectively. This statement confirms the growth conjecture for these specific matrices.

5. Conclusions

The subject of our research is to find an efficient way for calculating the minors of weighing matrices $W(n, n-k)$. For achieving this purpose we derived first the values of the $n-j$ minors, $j = 1, 2, 3$ for a $W(n, n-1)$ by making use of appropriate determinant manipulations and taking into account the special properties of these matrices. Next, we demonstrated the most generalized version of the Determinant Simplification Theorem and discussed its usefulness for a more sophisticated way for calculating minors of weighing matrices. Finally, we present briefly the connection of our work with an approach to the growth problem for this class of matrices.

The ideas presented in this work can be used as the fundamental basis, on which the calculation of the pivot patterns of further weighing matrices of small orders, such as $W(14,12)$ and $W(16,14)$, can be developed. Another benefit of the proposed technique is that the methods presented here can be very easily modified for the specification of minors of other weighing matrices $W(n, n-k)$, $k = 2, 3, \ldots$. A computational approach of the proof of Proposition 2, which would eliminate the difficulties occurring from the calculations by hand, is under investigation.

Acknowledgments

This research was financially supported by PENED 03ED 740 of the Greek General Secretariat for Research and Technology.

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