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ON THE LUSTERNIK-SCHNIRELMANN CATEGORY OF REAL STIEFEL MANIFOLDS

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Abstract: The purpose of this note is to study the Lusternik-Schnirelmann category of manifolds. In particular we investigate the Lusternik-Schnirelmann category of real Stiefel manifolds and the special orthogonal groups.

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1. Preliminaries

Given a space X. Let us say that a subset V of X is categorical if V is contractible in X. By a categorical covering of X we mean a finite numerable covering $V_1, V_2, , , , , V_k$ of X, for some k, by categorical subsets. We define the Lusternik-Schnirelmann category of X to be the least value of k for which such a covering exists. We denote this number by $\operatorname{cat} X$. If no such covering exists we say that the Lusternik-Schnirelmann category is infinite. For example, all compact manifolds and finite complexes have finite category.

We can also define the category by using subsets of X which are contractible in themselves, instead of contractible in X. The number thus defined is called as the strong category, and written $\operatorname{Cat} X$. Clearly $\operatorname{cat} X \leqq \operatorname{Cat} X$. However, this definition of $\operatorname{Cat} X$ is not a homotopy invariant, and so Ganea gave a homotopy

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invariant definition of the strong category. He takes the minimum of all $\mathrm{Cat}\, Y$, where Y has the same homotopy type of X. By Ganea's definition we have the following: If X has the homotopy type of CW-complex then

$$\mathrm{Cat}\, X = \mathrm{cat}\, X \text{ or } \mathrm{cat}\, X + 1.$$

Here we consider only the Lusternik-Schnirelmann category, so we call the Lusternik-Schnirelmann category L-S category or simply category. The computation of the L-S category of Lie groups and manifolds appears as the first problem in Ganea's list of unsolved problems.

The lower bound for the category can be obtained from the index of nilpotency of cohomology:

$$\mathrm{nil}\, \tilde{H}^*(X) \leqq \mathrm{cat}\, X.$$

This inequality holds for any cohomology theory with any coefficient ring.

On the otherhand, in the case of a manifold, the number of charts in a chart structure gives an upper bound for the category (indeed the strong category). We quote the following three inequalities to estimate the upper bound of the category of a space X.

Theorem 1.1. (see Fox [3]) *If X and Y are path-connected and paracompact then*

$$\mathrm{cat}\,(X \times Y) < \mathrm{cat}\, X \ + \mathrm{cat}\, Y.$$

Theorem 1.2. *Let X be a finite CW-complex. If X is $(r-1)$ connected $(r > 1)$ then*

$$\mathrm{cat}\, X \leqq 1 + \frac{\dim\ X}{r}.$$

Theorem 1.3. (see Singhof [7]) *Let G be a compact, connected Lie droup. If $\pi_1(G)$ is cyclic then*

$$\mathrm{cat}\, G \leqq \dim G - \mathrm{rank}\, G + 2.$$

We give some examples.

1. For the real projective $(n-1)$ space P_n $(n \geqq 1)$, $\mathrm{cat}\, P_n = n$.

2. For the complex Stiefel manifold $W_{m,k} = U(m)/U(m-k)$, $m > k$, $\mathrm{cat}\, W_{m,k} = k+1$. In particular, $\mathrm{cat}\, SU(m) = m$ and $\mathrm{cat}\, U(m) = m+1$ ((cf. Singhof [7, I, Theorems 1 and 2; II, Theorem 1]).

In these examples, the lower bound given by the nilpotency of cohomology ring coincides with the upper bound given by the number of charts. However, in the following example there is a gap between the lower bound and the upper bound.

3. For real and symplectic Stiefel manifolds, except a few special cases category is unknown. In particular, the category of special orthogonal groups and symplectic groups is unknown, except a few special cases.

2. L-S Category of $SO(n)$

Let $SO(n)$ denote the special orthogonal group. We recall the cohomology algebra of $SO(n)$ (see [1, Theorem 8.8]).

Theorem 2.1. (Borel) *We have*

$$H^*(SO(n);\mathbf{Z_2}) \cong \mathbf{Z_2}[x_1, x_2, , , , , x_{[\frac{n}{2}]}/I_n,$$

where I_n is the ideal generated by $x_i^{2^{s(i)}}$ $(1 \leqq i \leqq [\frac{n}{2}],\ \deg x_i = 2i-1)$. Here $s(i)$ is the smallest integer such that $2^{s(i)}(2i-1) \geqq n$ is satisfied.

From this theorem the lower bound of $\operatorname{cat} SO(n)$ is given as follows.

Theorem 2.2.

$$\operatorname{cat} SO(n) \geqq \sum_{i=1}^{[\frac{n}{2}]} 2^{s(i)} - [\frac{n}{2}] + 1,$$

where $s(i)$ is the smallest integer satisfying $2^{s(i)}(2i-1) \geqq n$ for $1 \leqq i \leqq [\frac{n}{2}]$.

We consider a few examples of $\operatorname{cat} SO(n)$. Since $SO(3) = P_4$ we have $\operatorname{cat} SO(3) = 4$. Since $SO(4) = SO(3) \times S^3$ we have $\operatorname{cat} SO(4) < \operatorname{cat} SO(3) + \operatorname{cat} S^3 = 6$. So, $\operatorname{cat} SO(4) = 5$ by Theorem 2.2.

By Theorem 2.2 and $\operatorname{cat} SO(5) \leqq \dim\ SO(5) - \operatorname{rank}\ SO(5) + 2 = 10$ we have $\operatorname{cat} SO(5) = 9$ or 10. In 1999, James and Singhof settled $\operatorname{cat} SO(5) = 9$ by the method of improved inequality for the category of a fibre space ([5], Section 4.(4.2), Corollary).

Since $SO(8) = S^7 \times SO(7)$, $\operatorname{cat} SO(8)$ can be determined by $\operatorname{cat} SO(7)$. Recently, Japanese topologists computed $\operatorname{cat} SO(n)$, for $6 \leqq n \leqq 9$, as follows: $\operatorname{cat} SO(6) = 10$, $\operatorname{cat} SO(7) = 12$, $\operatorname{cat} SO(8) = 13$, $\operatorname{cat} SO(9) = 21$.

The upper bound obtained for $\operatorname{cat} SO(n)$ does not always coincide with the actual values of $\operatorname{cat} SO(n)$. However, from these examples it seems plausible to conjecture the following.

Conjecture 2.1.

$$\operatorname{cat} SO(n) = \sum_{i=1}^{[\frac{n}{2}]} 2^{s(i)} - [\frac{n}{2}] + 1.$$

Here, $s(i)$ is the smallest integer satisfying $2^{s(i)}(2i-1) \geqq n$ for $1 \leqq i \leqq [\frac{n}{2}]$.

3. L-S Category of $V_{n,k}$

The real Stiefel manifold is defined as the factor space $V_{n,k} = SO(n)/SO(n-k)$. We recall the cohomology algebra of $V_{n,k}$(see [6, Theorem 5.3 and Theorem 7.1]).

Theorem 3.1. (Miller) *Let $n-k$ =even. Then we have*

$$H^*(V_{n,k}; \mathbf{Z_2}) \cong \mathbf{Z_2}[x_{[\frac{n-k}{2}]+1}, x_{[\frac{n-k}{2}]+2}, ,,,, x_{[\frac{n}{2}]}, x]/I_{n,k}\ ,$$

where $I_{n,k}$ is the ideal generated by $x_i^{2^{s(i)}}$, $([\frac{n-k}{2}]+1 \leqq i \leqq [\frac{n}{2}],\ \deg x_i = 2i-1)$, and x^{2^s} $(\deg x = n-k)$. Here $s(i)$ (resp. s) is the smallest integer such that $2^{s(i)}(2i-1) \geqq n$ (resp. $2^s(n-k) \geqq n$) is satisfied.

In the case $n-k$ =odd, we omit the element x in the above.

Theorem 3.2. (Miller) *The cohomology algebra $H^*(V_{n,k}; R)$ with rational coefficients is a free algebra over R generated by unit and all x_{2i-1}, x_{2i}, with $n-k < \deg x_{2i} = 2i < n$, x_{n-1} if $n-1$ is odd, and x_{n-k} if $n-k$ is even, subject to the relation; if $n-k$ is even $x_{n-k}^2 = 0$. In particular, if $n-k$ is odd $H^*(V_{n,k}; R)$ is an exterior algebra on these generators.*

Since $V_{n,k}$ is $(n-k-1)$ connected and dim $V_{n,k} = \frac{k(k-1)}{2} + k(n-k)$ we have the following lemma by Theorem 1.2.

Lemma 3.1. *Let $k \geqq 2$. If $k(k+2\ell-1) < 2\ell n$, then $\operatorname{cat} V_{n,k} \leqq k+\ell$ for $\ell = 1, 2, 3, \ldots$.*

Theorem 3.3. *Let $k \geqq 2$. Then $\operatorname{cat} V_{n,k} = k+1$ for $n \geqq [\frac{k(k+1)}{2}] + 1$.*

Proof. In fact, by Lemma 3.1 $\operatorname{cat} V_{n,k} \leqq k+1$ for $n \geqq [\frac{k(k+1)}{2}]+1$. On the other hand, by Theorem 3.2 the cohomology algebra with rational coefficients $H^*(V_{n,k}; R)$ is an algebra with k generators. Thus we conclude that $\operatorname{cat} V_{n,k} = k+1$ for $n \geqq [\frac{k(k+1)}{2}] + 1$. □

We call the case $\ell = 1$ in Lemma 3.1 the stable range. Here we give some examples outside the stable range. In case $k = n-1$, obviously $\operatorname{cat} V_{n,n-1}$ is given by $\operatorname{cat} SO(n)$.

James and Whitehead showed that; $V_{n,k}$ is trivial as a fibre bundle over S^{n-1} if and only if the associated principal bundle $V_{n,n} = O(n)$ admits a cross-section, i.e, if and only if $n = 2, 4$ or 8. Thus $V_{4,k}(k \leqq 4)$ and $V_{8,k}(k \leqq 8)$

are trivial as fibre bundle. So, $\operatorname{cat} V_{n,k} < \operatorname{cat} S^{n-1} + \operatorname{cat} V_{n-1,k-1}$. By these facts and Theorems 1.2, 3.1 and 3.2 we can compute some categories of $V_{n,k}$ as follows.

Theorem 3.4. (i) $\operatorname{cat} V_{3,2} = 4$,
(ii) $\operatorname{cat} V_{4,3} = 5$, $\operatorname{cat} V_{5,3} = 5$, $\operatorname{cat} V_{6,3} = 4$ *or* 5,
(iii) $\operatorname{cat} V_{5,4} = 9$, $\operatorname{cat} V_{6,4} = 6, 7$ *or* 8, $\operatorname{cat} V_{7,4} = 5, 6$ *or* 7,
$\operatorname{cat} V_{8,4} = \operatorname{cat} V_{9,4} = \operatorname{cat} V_{10,4} = 5$.

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