International Journal of Pure and Applied Mathematics

Volume 34 No. 2 2007, 195-199

OPERATORS SATISFYING $\mathrm{Re}\,\sigma_\alpha(T) = \sigma_\alpha(\mathrm{Re}\,T)$

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Abstract: The notion of weighing spectra is used to show that if T is a hyponormal operator, then $\mathrm{Re}\,\sigma_\alpha(T) = \sigma_\alpha(\mathrm{Re}\,T)$. The concept of approximate proper vectors is extended to prove the result for weighing spectra.

AMS Subject Classification: 47A53
Key Words: weighted spectrum, weighted approximate spectrum, α-closed set

1. Introduction

If T is any seminormal operator on a separable Hilbert space $\mathcal{H}$, then $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$ [3]. S.C. Arora and Pramod Arora [1] proved that if T is seminormal, then $\mathrm{Re}\,\sigma_\alpha(T) = \sigma_\alpha(\mathrm{Re}\,T)$, where $\sigma_\alpha(T)$ denotes the spectrum of weight α, α being any cardinal number lying between $\mathcal{N}_0$, the cardinality of the set of natural numbers and h, the dimension of the Hilbert space $\mathcal{H}$. In the present paper, this result is proved by an altogether different approach.

Throughout the paper, $\mathcal{H}$ is a fixed (complex) non separable Hilbert space of dimension $h \geq \mathcal{N}_0$ and $\mathcal{L}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. For each cardinal α, $\mathcal{N}_0 \leq \alpha \leq h$, ϑ_α denotes the two sided ideal, in

Received: October 27, 2006

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$\mathcal{L}(\mathcal{H})$ of all bounded linear operators of rank less than α and $\mathcal{T}_\alpha$ denotes the norm closure of ϑ_α in $\mathcal{L}(\mathcal{H})$. $\sigma_\alpha(T)$ and $\pi_\alpha(T)$ are used to denote the weighted spectrum and the weighted approximate spectrum of T of weight α. $R(T)$ and $N(T)$ denote the range and the null space of T respectively. $v(T)$ denotes the nullity of T.

For the definitions of α-Fredholm operators, α-closed sets, etc., we refer to [4]. The concept of approximate proper vectors (see [2]) is the main tool in developing the theory. We start with the following Lemma, the terminology of which conforms with that of [2].

Since every operator T in $\mathcal{L}(\mathcal{H})$ determines an operator T^0 in $\mathcal{L}(\mathcal{H}')$, we have the following lemma.

Lemma 1. *If T is an operator in $\mathcal{L}(\mathcal{H})$, then*

$$\pi_\alpha(T) = \pi_\alpha(T^0)\,.$$

Proof. A complex number λ is in $\pi_\alpha(T)$ if either λ is an eigenvalue of T of multiplicity at least α or the range of $T - \lambda I$ is not α-closed [4].

If λ is an eigenvalue of multiplicity at least α, then we claim that $v(T - \lambda I) = v(T^0 - \lambda I^0)$. To prove the claim, it is sufficient to prove that $x \in \ker(T - \lambda I)$ if and only if $\{x\}' \in \ker(T^0 - \lambda I^0)$. Thus, if x is in $\ker(T - \lambda I)$, we have $Tx = \lambda x$. It follows then by the definition that

$$T^0\{x\}' = (T_0\{x\})' = \{\lambda x\}' = \lambda\{x\}'\,.$$

Thus, $\{x\}'$ is in $\ker(T^0 - \lambda I^0)$.

Conversely, if $\{x\}'$ is in $\ker(T^0 - \lambda I^0)$, $T^0\{x\}' = \lambda\{x\}'$. Since $T^0\{x\}' = \{Tx\} + \eta$, and $\lambda\{x\}' = \{\lambda x\} + \eta$, it follows that $\{(T - \lambda I)x\}$ is in η. Therefore, $\phi\langle\{(T - \lambda I)x\}, \{(T - \lambda I)x\}\rangle = 0$. This gives that

$$\operatorname{glim}\langle\{(T - \lambda I)x\}, \{(T - \lambda I)x\}\rangle = 0$$

and hence, $\operatorname{glim}\|(T - \lambda I)x\|^2 = 0$. Hence, it follows that $Tx = \lambda x$ or x is in $\ker(T - \lambda I)$.

Also, since $x \mapsto \{x\}'$ a is an isomorphic linear mapping of $\mathcal{H}$ onto $\mathcal{H}'$, every closed subspace of $\mathcal{H}$ is mapped onto a closed subspace of $\mathcal{H}'$, which is contained in K.

Again, since $T^0\{x\}' = (T_0 x)' = \{Tx\}'$, for each x in $\mathcal{H}$, we get $\dim R(T^0) = \dim R(T)$. Hence, $R(T)$ is α-closed if and only if $R(T^0)$ is α-closed. □

Lemma 2. *If T is any operator in $\mathcal{L}(\mathcal{H})$,*

$$\pi_\alpha(T) = \pi_\alpha(T^0) \subset \pi_0(T^0)\,.$$

Proof. The relation $\pi_\alpha(T) = \pi_\alpha(T^0)$ has already been noted in Lemma 1. Suppose λ_0 is in $\pi_\alpha(T)$. Then, by definition, $\delta(T-\lambda_0 I) \geq \alpha$. Also, by definition, for $\varepsilon_0 > 0$, there exists a closed linear sub space K_{ε_0} containing $\ker(T-\lambda_0 I)$ with dimension $K_{\varepsilon_0} = \alpha$, such that

$$\begin{aligned} \|(T-\lambda_0 I)f\| &< \varepsilon\|f\|, \quad \text{for each } f \text{ in } K_\varepsilon,\ f \neq 0, \\ \|(T-\lambda_0 I)f\| &\geq \varepsilon\|f\|, \quad \text{for each } f \text{ in } K_\varepsilon^\perp. \end{aligned}$$

Since $\alpha = \delta_{\varepsilon_0} = \min_{\varepsilon>0} \delta_\varepsilon$, each K_ε corresponding to $\varepsilon > 0$ has dimension greater than or equal to α or in other words, K_ε is infinite dimensional. Thus, there exists a sequence $\{x_n\}$ of unit vectors in K_ε such that $x_n \to 0$ weakly and $\|(T-\lambda_0 I)x_n\| \to 0$. Let $u = \{x_n\}$. Then, $\|u\| = 1$ and $\|(T^0 - \lambda_0 I)u\|^2 = \operatorname{glim} \|Tx_n - \lambda_0 x_n\|^2$. Hence, $T^0 u = \lambda_0 u$. Therefore, λ_0 is in $\pi_0(T^0)$. □

The motivation for the results that follows now comes from [3], where Berberian has proved that for a hyponormal operator, $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$. For the terminology used in the following results, we refer to [4]. To begin with, we have the following theorem.

Theorem 1. *Let T be a hyponormal operator on $\mathcal{H}$. Then,*

$$\operatorname{Re}\pi_\alpha(T) \subset \sigma_\alpha(\operatorname{Re} T).$$

Proof. For a complex number λ_0 in $\pi_\alpha(T)$, $\delta(T-\lambda_0 I) \geq \alpha$. Since $\delta(T-\lambda_0 I) = \min_{\varepsilon>0} \delta_\varepsilon(T-\lambda_0 I)$, there exists an $\varepsilon_0 > 0$ such that $\delta(T-\lambda_0 I) = \delta_{\varepsilon_0}(T-\lambda_0 I)$. Corresponding to this $\varepsilon_0 > 0$, there exists a closed linear subspace K_{ε_0} such that $\ker(T-\lambda_0 I) \subset K_{\varepsilon_0}$ with $\dim K_{\varepsilon_0} = \delta_{\varepsilon_0}(T-\lambda_0 I)$ and such that

$$\begin{aligned} \|(T-\lambda_0 I)f\| &< \varepsilon_0\|f\|, \quad \text{for each } f \text{ in } K_{\varepsilon_0},\ f \neq 0, \\ \|(T-\lambda_0 I)f\| &\geq \varepsilon_0\|f\|, \quad \text{for each } f \text{ in } K_{\varepsilon_0}^\perp. \end{aligned} \tag{1}$$

It follows from (1) that $\|(T-\lambda_0 I)f\| < \varepsilon_0$, for each f in K_{ε_0} with $\|f\| = 1$. In fact, it holds for each $\varepsilon > 0$ and each corresponding K_ε has dimension $(\geq \delta_{\varepsilon_0}) \geq \alpha_0$.

Since T is hyponormal, $\|(T^* - \bar{\lambda}_0 I)f\| \leq \|(T-\lambda_0 I)f\| < \varepsilon$ for each f in K_{ε_0} with $\|f\| = 1$ and $\dim K_\varepsilon \geq \alpha$. Thus, it follows that $\|(\operatorname{Re} T - \alpha_0 I)f\| < \varepsilon$, for each f in K_ε with $\|f\| = 1$ and $\dim K_\varepsilon \geq \alpha$, where $\operatorname{Re}\lambda_0 = \alpha_0$. Since α is infinite, there exists a sequence $\langle x_n \rangle$ of unit vector in K_ε such that $(\operatorname{Re} T - \alpha_0 I)x_n \to 0$ as $n \to \infty$. Hence, α_0 is in $\pi(\operatorname{Re} T)$.

Since by changing the Hilbert space (see [2]), one can assume that $\pi(\operatorname{Re} T) = \pi_0(\operatorname{Re} T)$, α_0 is in $\pi_0(\operatorname{Re} T)$. Thus, α_0 is an eigenvalue of $\operatorname{Re}(T)$.

Also, by changing the Hilbert space, one can have $\pi_\alpha(T) \subset \pi_0(T)$ (Lemma 2). Since λ_0 is in $\pi_\alpha(T)$, it follows that λ_0 is an eigenvalue of multiplicity atleast α. Thus,

$$v(T - \lambda_0 I) \geq \alpha\,. \tag{2}$$

As T is hyponormal, $N(T^* - \bar{\lambda}_0 I) \supset N(T - \lambda_0 I)$ and thus,

$$v(T^* - \bar{\lambda}_0 I) \geq v(T - \lambda_0 I) \geq \alpha\,. \tag{3}$$

Hence, from (2) and (3), it follows that $v(\operatorname{Re} T - \alpha_0 I) \geq \alpha$. Thus, α_0 is in $\pi_{0\alpha}(ReT) \subset \sigma_\alpha(\operatorname{Re} T)$. Hence,

$$\operatorname{Re} \pi_\alpha(T) \subset \sigma_\alpha(\operatorname{Re} T)\,. \qquad \square$$

Remark 1. The above result also implies that

$$\operatorname{Re} \pi_\alpha(T) \subset \pi_{0\alpha}(\operatorname{Re} T)\,.$$

Theorem 2. *If T is hyponormal, then*

$$\operatorname{Re} \sigma_\alpha(T) \subset \sigma_\alpha(\operatorname{Re} T)\,.$$

Proof. Let λ_0 be in $\sigma_\alpha(T)$. Then, the vertical line $\operatorname{Re} \lambda_0$ cuts the weighted spectrum $\sigma_\alpha(T)$ at a boundary point μ of $\sigma_\alpha(T)$. Since $\partial\sigma_\alpha(T) \subset \pi_\alpha(T)$, see [2], it follows that $\operatorname{Re} \mu = \operatorname{Re} \lambda_0$ which is in $\operatorname{Re} \pi_\alpha(T) \subset \sigma_\alpha(\operatorname{Re} T)$ (Theorem 1). Hence, $\operatorname{Re} \sigma_\alpha(T) \subset \sigma_\alpha(\operatorname{Re} T)$. $\square$

Theorem 3. *For a hyponormal operator T,*

$$\sigma_\alpha(\operatorname{Re} T) = \operatorname{Re} \sigma_\alpha(T)$$

Proof. In view of Theorem 2, it suffices to show that $\sigma_\alpha(\operatorname{Re} T) \subset \operatorname{Re} \sigma_\alpha(T)$. Let $T = H + iJ$, where H and J are self adjoint and let $D = T^*T - TT^*$. Clearly, $D \geq 0$. Then,

$$HJ - JH = -\frac{1}{2} iD\,. \tag{1}$$

Changing the Hilbert space, one can suppose that $\sigma_\alpha(H) \subset \pi_0(H)$.

Assuming $0 \neq \alpha_0$ in $\sigma_\alpha(H)$, it is to be shown that $0 \neq \alpha_0$ is in $\operatorname{Re} \sigma_\alpha(T)$.

Let $M = N(H - \alpha_0 I)$. Since α_0 is an eigenvalue of H, $M \neq \{0\}$. We claim now that M is invariant under J.

Let x be in M. This implies that $(H-\alpha_0 I)x = 0$. From (1), it follows that

$$(H-\alpha_0 I)J - J(H-\alpha_0 I) = -\frac{1}{2}iD\,.$$

Therefore,

$$-\frac{1}{2}i\langle Dx, x\rangle = \langle Jx, (H-\alpha_0 I)x\rangle - \langle J(H-\alpha_0 I)x, x\rangle = 0\,.$$

Hence, $\langle Dx, x\rangle = 0 \Rightarrow \|D^{\frac{1}{2}}x\|^2 = 0 \Rightarrow D^{\frac{1}{2}}x = 0$, $Dx = 0$.

Thus, (1) implies that

$$\begin{aligned} 0 &= H(Jx) - J(Hx) = H(Jx) - J(\alpha_0 x) \\ &= H(Jx) - \alpha_0(Jx)\,. \end{aligned}$$

This gives that Jx is in M. Let $J_1 = J|_M$. Also, M is invariant under H with $H|_M = \alpha_0 I$. Thus, M is invariant under $T = H + iJ$ and

$$T|_M = \alpha_0 I + iJ_1\,.$$

Since J_1 is self adjoint, $T|_M$ is normal and since T is hyponormal, M reduces T. Let $T_1 = T|_M$ and $T_2 = T|_{M^\perp}$. Therefore, $T = T_1 \oplus T_2$ and hence, $\sigma_\alpha(T) = \sigma_\alpha(T_1) \cup \sigma_\alpha(T_2)$, see [1]. Thus,

$$\operatorname{Re}\sigma_\alpha(T) = \operatorname{Re}\sigma_\alpha(T_1) \cup \operatorname{Re}\sigma_\alpha(T_2)\,.$$

Since T_1 is normal, $\operatorname{Re}\sigma_\alpha(T_1) = \operatorname{Re}\pi_\alpha(T_1) = \{\alpha_0\}$. Thus, $\{\alpha_0\}$ is in $\operatorname{Re}\sigma_\alpha(T_1) \subset \operatorname{Re}\sigma_\alpha(T)$. Hence, $\sigma_\alpha(\operatorname{Re} T) \subset \operatorname{Re}\sigma_\alpha(T)$. Thus, $\sigma_\alpha(\operatorname{Re} T) = \operatorname{Re}\sigma_\alpha(T)$. □

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