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EXPONENTIAL BOUNDEDNESS FOR SOLUTIONS OF LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY

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Abstract: Exponential bounds are obtained for the solutions of a new class of linear impulsive differential equations with distributed delay. The coefficients and the powers of these estimates are derived explicitly from the parameters of the equations and independent of the initial data.

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1. Introduction

Impulsive differential equations with distributed delay may express the motion of some real world simulation processes which depend on their history and are subject to short time disturbances. The historical dependence may cause the presence of the delays through the differential equation as well as through the impulsive conditions and this often turns out to be the cause of phenomena substantially affecting the motions. Such processes occur in the theory of optimal control, theoretical physics, population dynamics, ecological competition and neural networks, see in particular [1, 3, 7, 11, 13, 15, 17, 18] and the references cited therein.

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An estimation of the solution by an exponential function with positive power is not interesting by itself. It may be useful only to prove global existence in case one has local existence. However, an estimation by an exponential function with negative power is an essential property for studying differential equations. In particular, it is often used in stability theory and oscillation analysis. Under certain conditions such estimates have been obtained for delay differential equations, integro-differential equations and impulsive differential equations [5, 9, 10, 14].

Most existing impulsive delay differential equations considered in the literature have involved delays only through the state variable and have considered linear impulsive conditions where the solution value in the impulses is independent of history, we name here [4, 6, 12, 16]. However, it is natural to ask what happens if this is not the case. We will give an affirmative answer to this question by considering a new class of impulsive delay differential equations in which the impulsive conditions allow distributed delays of the jumps. These impulsive conditions are naturally possible since at any discontinuity point the solution value is also determined by its history.

The purpose of this paper is to obtain exponential bounds for the solutions of such class of equations which generalizes those considered in [5, 9, 10, 14] and thus our results could improve the results previously obtained. It is to be noted that the coefficients and the powers of these estimates are derived explicitly from the parameters of the equations and independent of the initial data. We provide, moreover, a sharper bound.

2. Preliminaries

Let $\eta : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^{n\times n}$ be a kernel function (cf. [9]) satisfying the following conditions:

(a) $\eta(t,s)$ is normalized so that $\eta(t,s) = 0$ for $s \in [0,\infty)$, $\eta(t,s) = \eta(t,-\tau)$ for $s \in (-\infty,-\tau]$;

(b) $\eta(t,s)$ is continuous in t on $[0,\infty)$ uniformly for $s \in [-\tau,0]$;

(c) There exists a function $V(t) > 0$ bounded for $t \geq 0$ such that the total variation of $\eta(t,s)$ in s on $[-\tau,0]$ for $t \geq 0$ is not larger than $V(t)$.

We shall consider impulsive differential equation with distributed delay of the form

$$\begin{cases} x'(t) &= \int_{-\tau}^{0} \mathrm{d}_s \eta(t,s)x(t+s), \quad t \neq \theta_i, \\ \Delta x(\theta_i) &= A_{i0}x(\theta_i) + \sum_{-j\leq k<0} A_{ik}x(\theta_{i+k}), \quad i \in \mathbb{N}, \end{cases} \tag{1}$$

where:

(d) $A_{i0}, A_{ik} \in \mathbb{R}^{n\times n}$ such that $I + A_{i0}^T$ is invertible (I is the $n \times n$ identity matrix);

(e) $\theta_i < \theta_{i+1}$ such that $\lim_{i\to\infty} \theta_i = \infty$, $\theta_0 = 0$ and $\theta_{-1} > \theta_{-2} > \ldots > \theta_{-j}$ belong to the interval $[-\tau, 0)$, $j \in \mathbb{N}$ is fixed.

The notation $\Delta x(\theta_i)$ means the difference $x(\theta_i^+) - x(\theta_i)$ such that $x(\theta_i^+) := \lim_{t\to\theta_i^+} x(t)$. By a solution of (1) on an interval J, we mean a function x defined on J such that x is continuous everywhere on J except possibly at $\theta_i \in J$ for $i \in \mathbb{N}$, where $x(\theta_i^+)$ and $x(\theta_i^-) := \lim_{t\to\theta_i^-} x(t)$ exist, $x(\theta_i) := x(\theta_i^-)$ and that x satisfies (1). Let $PLC([\sigma-\tau,\sigma],\mathbb{R}^n)$ denote the set of piecewise left continuous functions $\phi : [\sigma-\tau,\sigma] \to \mathbb{R}^n$ having a finite number of discontinuity points of the first kind at θ_i. For given $\sigma \geq 0$ and $\phi \in PLC([\sigma-\tau,\sigma],\mathbb{R}^n)$, the initial value problem of (1) is to find a solution $x(t)$ of (1) such that

$$x(t) = \phi(t), \quad t \in [\sigma-\tau,\sigma]. \tag{2}$$

Under the above conditions, one can easily show that the initial value problem has a unique solution which belongs to the set $PLC([\sigma-\tau,\infty),\mathbb{R}^n)$.

Definition 1. A matrix solution $X(t,\alpha)$ of (1) satisfying $X(\alpha,\alpha) = I$ and $X(t,\alpha) = 0$ for $t < \alpha$ is called a fundamental matrix of equation (1).

Lemma 2. (see [2]) *Let $X(t,\alpha)$ be a fundamental matrix of (1) and $\sigma \geq 0$ be a real number. If $x(t)$ is a solution of (1), then*

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} \mathrm{d}_s\Big[\int_{\sigma}^{s+\tau} X(t,\alpha)\eta(\alpha,s-\alpha)\,\mathrm{d}\alpha\Big]x(s) \\ &+ \sum_{n(\sigma)-j\leq m<n(\sigma)} \Big[\sum_{n(\sigma)\leq i<m+j} X(t,\theta_i^+)A_{i(m-i)}\Big]x(\theta_m), \end{aligned} \tag{3}$$

where $n(t) = \min\{i \in \mathbb{N} : \theta_i \geq t\}$.

It is worth mentioning here that the above formula is obtained using the non classical inner product

$$< x(t), y(t) > \;=\; x^T(t)y(t) + \int_{t-\tau}^{t} x^T(s)\,\mathrm{d}_s \int_{t}^{s+\tau} \eta^T(\alpha,s-\alpha)y(\alpha)\,\mathrm{d}\alpha$$

$$+\sum_{n(t)-j\leq m<n(t)} x^T(\theta_m) \sum_{n(t)\leq i<m+j} A^T_{i(m-i)} y(\theta_i^+),$$

which is constructed in [2] by following the same techniques used by Halanay in [8].

The following technical lemma proves to be helpful.

Lemma 3. *Let $\lambda \in \mathbb{R}$. Then the following inequalities hold:*

$$\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda k} \leq \frac{e^{2\lambda} e^{-\lambda\sigma}}{e^{\lambda}-1}, \quad \lambda \geq 0 \quad \textit{and} \quad \sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda k} \leq \frac{e^{\lambda} e^{-\lambda\sigma}}{e^{\lambda}-1}, \quad \lambda < 0 \tag{4}$$

and

$$\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda k} \geq \frac{(1-e^{-\lambda\tau})e^{-\lambda\sigma}e^{\lambda}}{e^{\lambda}-1}, \quad \lambda \geq 0, \tag{5}$$

where $[\sigma]$ denotes the greatest integer not exceeding σ.

Proof. We observe that

$$\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda k} = e^{-\lambda[\sigma]}\left[1+\nu+\nu^2+\ldots+\nu^{[\sigma+\tau]-[\sigma]}\right], \quad \text{where} \quad \nu = e^{-\lambda}.$$

Further,

$$\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda k} = \frac{e^{-\lambda[\sigma]}(1-\nu^{[\sigma+\tau]-[\sigma]+1})}{1-\nu} \leq \frac{e^{\lambda}e^{-\lambda[\sigma]}}{e^{\lambda}-1}.$$

Using that $[\sigma] \leq \sigma$ and $[\sigma]+1 \geq \sigma$ we respectively have the inequalities (4). To prove inequality (5), it suffices to note that since $[\sigma+\tau]+1 \geq \sigma+\tau$, we have

$$1-e^{-\lambda([\sigma+\tau]-[\sigma]+1)} \geq 1-e^{\lambda\sigma}e^{-\lambda([\sigma+\tau]+1)} \geq 1-e^{-\lambda\tau}. \qquad \square$$

3. The Main Result

We begin by the following theorem which provides an exponential bound for the fundamental matrix $X(t,s)$.

Theorem 4. *For the fundamental matrix $X(t,s)$ of equation (1), the following estimate holds:*

$$\|X(t,s)\| \leq (j+1)^{i(t,s)}\Big[\max\{1,b\}\Big]^{i(t,s)} \exp\Big\{\int_s^t V(z)\mathrm{d}z\Big\}, \tag{6}$$

where $b = \max_{-j\leq k<0, i\in\mathbb{N}}\{\|A_{ik}\|, \|I + A_{i0}\|\}$ and $i(t,s)$ denotes the number of discontinuity points belonging to the interval (s,t).

Proof. Without loss of generality, let s be fixed such that $\theta_0 < s \leq \theta_1$. First, consider the case $t \in (s,\theta_1]$, that is, $i(t,s) = 0$. Then, $w(t) = X(t,s)$ is the solution of the differential equation in (1) on $(s,\theta_1]$, hence

$$w(t) = I + \int_s^t\int_{-\tau}^0 \mathrm{d}_r\eta(\xi,r)w(\xi+r)\mathrm{d}\xi.$$

Thus,

$$z(t) \leq 1 + \int_s^t V(\xi)z(\xi)\mathrm{d}\xi,$$

where $z(t) = \sup_{s<\zeta\leq t}\|w(\zeta)\|$. Applying Gronwall-Bellman inequality, we obtain

$$z(t) \leq \exp\Big\{\int_s^t V(\xi)\mathrm{d}\xi\Big\}.$$

Therefore

$$\|X(t,s)\| \leq \exp\Big\{\int_s^t V(\xi)\mathrm{d}\xi\Big\}, \quad t \in (s,\theta_1]. \tag{7}$$

Assume that inequality (6) holds for $i(t,s) = n < k$. Then

$$\|X(t,s)\| \leq (j+1)^n\Big[\max\{1,b\}\Big]^n \exp\Big\{\int_s^t V(z)\mathrm{d}z\Big\}, \quad \theta_n < t \leq \theta_{n+1}.$$

For $n = k$, let $\theta_0 < s < \ldots < \theta_k < t \leq \theta_{k+1}$. Then

$$w(t) = w(\theta_k^+) + \int_{\theta_k}^t\int_{-\tau}^0 \mathrm{d}_r\eta(\xi,r)w(\xi+r)\mathrm{d}\xi.$$

Using the impulsive conditions in (1), we have

$$w(t) = (I + A_{i0})w(\theta_k) + \sum_{0<m\leq j} A_{i(-m)}w(\theta_{k-m}) + \int_{\theta_k}^t\int_{-\tau}^0 \mathrm{d}_r\eta(\xi,r)w(\xi+r)\mathrm{d}\xi.$$

It follows that,

$$z(t) \leq \|(I + A_{i0})\| \|w(\theta_k)\| + \sum_{0<m\leq j} \|A_{i(-m)}\| \|w(\theta_{k-m})\| + \int_{\theta_k}^{t} V(\xi) z(\xi) \mathrm{d}\xi,$$

where $z(t) = \sup_{\theta_k<\zeta\leq t} \|w(\zeta)\|$. Then

$$\begin{aligned} z(t) \quad &\leq \quad \max\{1,b\}(j+1)^{k-1}\Big[\max\{1,b\}\Big]^{k-1} \exp\Big\{\int_s^{\theta_k} V(\xi)\mathrm{d}\xi\Big\} \\ &+ \quad \max\{1,b\} \sum_{0<m\leq j} (j+1)^{k-m-1}\Big[\max\{1,b\}\Big]^{k-m-1} \\ &\qquad \exp\Big\{\int_s^{\theta_{k-m}} V(\xi)\mathrm{d}\xi\Big\} + \int_{\theta_k}^{t} V(\xi) z(\xi)\mathrm{d}\xi. \end{aligned}$$

Applying Gronwall-Bellman inequality, one obtains

$$\begin{aligned} \|X(t,s)\| \quad &\leq \quad (j+1)^{k-1}\Big[\max\{1,b\}\Big]^{k} \exp\Big\{\int_s^{t} V(\xi)\mathrm{d}\xi\Big\} \\ &+ \quad j(j+1)^{k-1}\Big[\max\{1,b\}\Big]^{k} \exp\Big\{\int_s^{t} V(\xi)\mathrm{d}\xi\Big\}. \end{aligned}$$

Then, we get

$$\|X(t,s)\| \leq (j+1)^{k}\Big[\max\{1,b\}\Big]^{k} \exp\Big\{\int_s^{t} V(\xi)\mathrm{d}\xi\Big\},$$

which finishes the proof. □

We are now in a position to state and prove the main result of this paper.

Theorem 5. *Assume that there exist positive numbers ρ and M such that $\theta_{i+1} - \theta_i \geq \rho$ and*

$$M = \sup_{t>0} \int_t^{t+1} V(\xi)\mathrm{d}\xi < \infty. \tag{8}$$

Then, the solution of (1) has the following estimate:

$$\|x(t)\| \leq K\|\phi\| e^{\lambda(t-\sigma)}, \tag{9}$$

where

$$K = \max\{K_0, \frac{2K_0Me^{2\lambda}}{e^{\lambda}-1}, \frac{j(j-1)bK_0}{2}\}, \quad \|\phi\| = \sup_{\sigma-\tau\leq z\leq\sigma} \|\phi(z)\|,$$

and

$$\lambda = \begin{cases} M + \frac{\ln(j+1)}{\rho}, & b \le 1\,, \\ M + \frac{\ln b(j+1)}{\rho}, & b > 1\,, \end{cases} \qquad K_0 = \begin{cases} (j+1)e^M, & b \le 1\,, \\ (j+1)be^M, & b > 1\,. \end{cases}$$

Proof. Equation (8) implies that $\int_a^b V(\xi)\mathrm{d}\xi \le M(b-a+1)$, where $a, b > 0$. Moreover, from the condition $\theta_{i+1} - \theta_i \ge \rho$, we deduce that $i(t,s) \le \frac{t-s}{\rho} + 1$. By virtue of Theorem 4 and the assumptions given above, we obtain

$$\|X(t,s)\| \le K_0 e^{\lambda(t-s)}, \tag{10}$$

where K_0 and λ are given in the theorem statement. Indeed, for $b \le 1$, we have

$$\|X(t,s)\| \le (j+1)^{\frac{t-s}{\rho}+1} e^{(t-s+1)M} = (j+1)e^M e^{(M+\frac{\ln(j+1)}{\rho})(t-s)}.$$

For $b > 1$, we have

$$\|X(t,s)\| \le (j+1)^{\frac{t-s}{\rho}+1} b^{\frac{t-s}{\rho}+1} e^{(t-s+1)M} = (j+1)be^M e^{(M+\frac{\ln b(j+1)}{\rho})(t-s)}.$$

In view of (3), we obtain

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) + \int_\sigma^{\sigma+\tau} X(t,\nu)\eta(\nu,-\tau)x(s)\,\mathrm{d}s \\ &+ \int_\sigma^{\sigma+\tau}\int_{\alpha-\tau}^{\sigma} X(t,\alpha)\eta_s(\alpha,s-\alpha)x(s)\,\mathrm{d}s\,\mathrm{d}\alpha \\ &+ \sum_{n(\sigma)-j\le m<n(\sigma)} \Big[\sum_{n(\sigma)\le i<m+j} X(t,\theta_i^+)A_{i(m-i)}\Big]x(\theta_m), \end{aligned}$$

where η_s denotes the derivative of η with respect to s. Since $\theta^+_{n(\sigma)} \ge \sigma$, it follows that

$$\begin{aligned} \|x(t)\| &\le \|\phi\|\Big[K_0 e^{\lambda(t-\sigma)} + K_0\int_\sigma^{\sigma+\tau} e^{\lambda(t-\nu)}\|\eta(\nu,-\tau)\|\mathrm{d}\nu \\ &+ \int_\sigma^{\sigma+\tau}\|X(t,\alpha)\|\int_{\sigma-\tau}^{\sigma}\|\eta_s(\alpha,s-\alpha)\|\mathrm{d}s\mathrm{d}\alpha \\ &+ \frac{1}{2}j(j-1)bK_0e^{\lambda(t-\sigma)}\Big]. \end{aligned} \tag{11}$$

In virtue of Lemma 3, we observe that

$$\int_\sigma^{\sigma+\tau} e^{\lambda(t-\nu)}\|\eta(\nu,-\tau)\|\mathrm{d}\nu \le \sum_{k=[\sigma]}^{[\sigma+\tau]}\int_k^{k+1} e^{\lambda(t-\nu)}V(\nu)\mathrm{d}\nu$$

$$\leq \quad Me^{\lambda t}\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda k} \leq \frac{Me^{2\lambda}}{e^{\lambda}-1}e^{\lambda(t-\sigma)},$$

and thus,

$$\begin{aligned}\int_{\sigma}^{\sigma+\tau}\|X(t,\alpha)\|\int_{\sigma-\tau}^{\sigma}\|\eta_s(\alpha,s-\alpha)\|\mathrm{d}s\mathrm{d}\alpha \quad &\leq \quad \int_{\sigma}^{\sigma+\tau}\|X(t,\alpha)\|V(\alpha)\mathrm{d}\alpha \\ &\leq \quad \frac{Me^{2\lambda}}{e^{\lambda}-1}e^{\lambda(t-\sigma)}.\end{aligned}$$

Hence, (11) becomes

$$\|x(t)\| \leq K_0 e^{\lambda(t-\sigma)}\|\phi\| + 2\|\phi\|K_0\frac{Me^{2\lambda}}{e^{\lambda}-1} + \frac{1}{2}j(j-1)\|\phi\|bK_0e^{\lambda(t-\sigma)}.$$

Therefore (9) is obtained. □

Remark 6. The coefficient K and the power of the estimate (9) are derived explicitly from the parameters of the equation and are independent of the initial data.

Remark 7. To avoid the case $I + A_{i0} \equiv 0$ and $\sum_{-j\leq k<0} A_{ik} \equiv 0$, for some i. We can use the estimate

$$\|X(t,s)\| \leq (j+1)^{i(t,s)}\Big[1+b\Big]^{i(t,s)}\exp\Big\{\int_s^t V(z)\mathrm{d}z\Big\},$$

instead of (6) in Theorem 4. Thus, for instance, (10) becomes

$$\|X(t,s)\| \leq Te^{\gamma(t-s)},$$

where $T = (j+1)(1+b)e^M$ and $\gamma = M + \frac{\ln(1+b)(j+1)}{\rho}$.

4. A Sharper Bound

It is possible to obtain a sharper bound for the solutions of (1) in which the exponent depends also on a positive number l where $\theta_{i+1} - \theta_i \leq l$. We shall show this for the following particular case of equation (1)

$$\begin{cases} x'(t) &= \int_{-\tau}^{0}\mathrm{d}_s\eta(t,s)x(t+s), \quad t \neq \theta_i, \\ \Delta x(\theta_i) &= C_i x(\theta_i) + D_i x(\theta_{i-j}), \quad i \in \mathbb{N}, \end{cases} \tag{12}$$

where we set $k = -j$. The solution representation of (12), therefore, takes the form

$$\begin{aligned} x(t) &= X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} \mathrm{d}_s \Big[\int_{\sigma}^{s+\tau} X(t,\alpha)\eta(\alpha, s-\alpha)\,\mathrm{d}\alpha \Big] x(s) \\ &+ \sum_{n(\sigma)-j \le k < n(\sigma)} X(t,\theta_{k+j}^{+}) D_{k+j}\phi(\theta_k). \end{aligned} \tag{13}$$

Theorem 8. *For the fundamental matrix $X(t,s)$ of (12), the following estimate holds:*

$$\begin{aligned} \|X(t,s)\| &\le (2^{i(t,s)}-1)\Big[\max\{1,b\}\Big]^{i(t,s)} \exp\Big\{ \int_s^t V(z)\mathrm{d}z \Big\} \\ &+ \Big[\max\{1,b\}\Big]^{i(t,s)} \exp\Big\{ \int_{\theta_{m(t)}}^t V(z)\mathrm{d}z + \int_{\theta_{m(t)-(j+1)}}^{\theta_{m(t)-j}} V(z)\mathrm{d}z \\ &+ \int_{\theta_{m(t)-2(j+1)}}^{\theta_{m(t)-2(j+1)+1}} V(z)\mathrm{d}z + \ldots + \int_{\theta_{r_j(t,s)}}^{\theta_{r_j(t,s)+1}} V(z)\mathrm{d}z \\ &+ \int_s^{\theta_{n(s)}} V(z)\mathrm{d}z \Big\}, \end{aligned} \tag{14}$$

where $m(t) = \max\{k : \theta_k \le t\}$ *and* $r_j(t,s) = \min\{m(t) - k(j+1) : m(t) - k(j+1) > s, k \in \mathbb{N}\}$.

Proof. We prove this inequality by employing the same arguments followed in the proof of Theorem 4. Indeed, for $\theta_0 < s \le \theta_1$, $t \in (s, \theta_1]$, that is, $i(t,s) = 0$, we have

$$\|X(t,s)\| \le \exp\Big\{ \int_s^t V(z)\mathrm{d}z \Big\} \le \exp\Big\{ \int_{\theta_0}^t V(z)\mathrm{d}z + \int_s^{\theta_1} V(z)\mathrm{d}z \Big\}. \tag{15}$$

Assume that inequality (14) is true for $i(t,s) = n < k$. For $n = k$, let $\theta_0 < s < \ldots < \theta_k < t \le \theta_{k+1}$. Then

$$w(t) = w(\theta_k^+) + \int_{\theta_k}^t \int_{-\tau}^0 d_r\eta(\xi, r) w(\xi + r)\mathrm{d}\xi.$$

Using the impulsive conditions in (12), we obtain

$$w(t) = (I + C_k)w(\theta_k) + D_k w(\theta_{k-j}) + \int_{\theta_k}^t \int_{-\tau}^0 d_r\eta(\xi, r) w(\xi + r)\mathrm{d}\xi.$$

Therefore,

$$
\begin{aligned}
z(t) \le \|(I+C_k)\| \Bigg[&(2^{k-1}-1)\Big[\max\{1,b\}\Big]^{k-1} \exp\Big\{ \int_s^{\theta_k} V(\xi)\mathrm{d}\xi \Big\} \\
&+ \Big[\max\{1,b\}\Big]^{k-1} \exp\Big\{ \int_{\theta_{k-1}}^{\theta_k} V(\xi)\mathrm{d}\xi \\
&+ \int_{\theta_{k-1-(j+1)}}^{\theta_{k-1-j}} V(\xi)\mathrm{d}\xi + \int_{\theta_{k-1-2(j+1)}}^{\theta_{k-2(j+1)}} V(\xi)\mathrm{d}\xi + \ldots + \int_s^{\theta_{m(s)}} V(\xi)\mathrm{d}\xi \Big\} \Bigg] \\
&+ \|D_k\| \Bigg[(2^{k-j-1}-1)\Big[\max\{1,b\}\Big]^{k-j-1} \exp\Big\{ \int_s^{\theta_{k-j}} V(\xi)\mathrm{d}\xi \Big\} \\
&+ \Big[\max\{1,b\}\Big]^{k-j-1} \exp\Big\{ \int_{\theta_{k-j-1}}^{\theta_{k-j}} V(\xi)\mathrm{d}\xi \\
&+ \int_{\theta_{k-2j-2}}^{\theta_{k-1-2j}} V(\xi)\mathrm{d}\xi + \int_{\theta_{k-3j-3}}^{\theta_{k-3j-2}} V(\xi)\mathrm{d}\xi + \ldots + \int_s^{\theta_{m(s)}} V(\xi)\mathrm{d}\xi \Big\} \Bigg] + \int_{\theta_k}^t V(\xi) z(\xi)\mathrm{d}\xi,
\end{aligned}
$$

where $z(t) = \sup_{\theta_k \le \zeta \le t} \|w(\zeta)\|$. Applying Gronwall-Bellman inequality, we obtain

$$
\begin{aligned}
\|X(t,s)\| \le &\Big[(2^{k-1}-1) + 1 + (2^{k-j-1}-1)\Big]\Big[\max\{1,b\}\Big]^k \exp\Big\{ \int_s^t V(\xi)\mathrm{d}\xi \Big\} \\
&+ \Big[\max\{1,b\}\Big]^k \exp\Big\{ \int_{\theta_k}^t V(\xi)\mathrm{d}\xi \\
&+ \int_{\theta_{k-(j+1)}}^{\theta_{k-j}} V(\xi)\mathrm{d}\xi + \int_{\theta_{k-2(j+1)}}^{\theta_{k-2j-1}} V(\xi)\mathrm{d}\xi + \ldots + \int_s^{\theta_{m(s)}} V(\xi)\mathrm{d}\xi \Big\},
\end{aligned}
$$

and this gives the desired result since we have $\theta_{n(t)} = \theta_k$ for $i(t,s) = k$. □

Define

$$
L_1(t,s) = K_0' e^{\lambda'(t-s)} + K_2 e^{\lambda_1(t-s) + M\delta(t,s)[(l+1) - \rho(j+1)]},
$$

and

$$
L(t,s) = L_1(t,s) - K_1 e^{\lambda_1(t-s)}, \quad h(\mu) = \begin{cases} 0, & \mu \le 0\,, \\ 2\mu, & \mu > 0\,, \end{cases}
$$

$$
M_1 = \sup_{t>0} \int_t^{t+1} \|\eta(z,-\tau)\| dz,
$$

where

$$\lambda_1 = \begin{cases} M & , b \le 1, \\ M + \frac{\ln b}{\rho}, & b > 1, \end{cases} \quad K_1 = \begin{cases} e^M & , b \le 1, \\ be^M, & b > 1, \end{cases} \quad K_2 = \begin{cases} e^{2M} & , b \le 1, \\ be^{2M}, & b > 1, \end{cases}$$

and

$$\delta(t,s) = \frac{m(t) - n(s)}{j+1}, \quad \mu = \frac{M((l+1) - \rho(j+1))}{j+1},$$

$$K_0^{'} = \begin{cases} 2e^M & , b \le 1, \\ 2be^M, & b > 1, \end{cases} \quad \lambda^{'} = \begin{cases} M + \frac{\ln 2}{\rho} & , b \le 1\,, \\ M + \frac{\ln 2b}{\rho}, & b > 1\,. \end{cases}$$

Theorem 9. *In addition to the assumptions of Theorem 5, assume that there exists a positive number l such that $\theta_{i+1} - \theta_i \le l$. Then, the solution of (12) has the following estimate:*

$$\|x(t)\| \le \|\phi\| \Big[L(t,\sigma) + jb\big(L_1(t,\sigma) - K_1 e^{\lambda_1 (t - \theta^+_{n(\sigma)+j})}\big) + \Big(\frac{K_0^{'} e^{2\lambda^{'}} e^{\lambda^{'}(t-\sigma)}}{e^{\lambda^{'}} - 1} - \frac{K_1(1-e^{-\lambda_1 \tau}) e^{\lambda_1(t-\sigma)}}{e^{\lambda_1} - 1} + \frac{K_2 e^{h(\mu)}}{e^{\mu} - 1} e^{\mu(t-\sigma)} \Big)(M_1 + M) \Big]. \quad (16)$$

Proof. By virtue of Theorem 8 and the assumptions of the theorem, we get for $b \le 1$,

$$\|X(t,s)\| \le \Big[2^{\frac{t-s}{\rho}+1} - 1 \Big] e^{M(t-s+1)} + e^{M(t-\theta_{m(t)}+1) + M(\theta_{n(s)} - s + 1) + M(l+1)\frac{m(t) - r_j(t,s)}{j+1}}$$
$$\le 2e^M 2^{\frac{t-s}{\rho}} e^{M(t-s)} - e^M e^{M(t-s)} + e^{2M} e^{M[(t-s) + (l+1)\delta(t,s) - (\theta_{m(t)} - \theta_{n(s)})]},$$

or

$$\|X(t,s)\| \le 2e^M e^{(\frac{\ln 2}{\rho} + M)(t-s)} - e^M e^{M(t-s)} + e^{2M} e^{M(t-s)} \times e^{M[(l+1)\delta(t,s) - \rho(j+1)\frac{m(t)-n(s)}{j+1}]} = 2e^M e^{(\frac{\ln 2}{\rho}+M)(t-s)} - e^M e^{M(t-s)} + e^{2M} e^{M(t-s) + M\delta(t,s)[(l+1) - \rho(j+1)]}.$$

For $b > 1$, we obtain

$$\|X(t,s)\| \le 2be^M 2^{\frac{(t-s)}{\rho}} b^{\frac{(t-s)}{\rho}} e^{M(t-s)} - e^M b b^{\frac{(t-s)}{\rho}} e^{M(t-s)} + b b^{\frac{(t-s)}{\rho}} e^{2M} e^{M(t-s) + M\delta(t,s)[(l+1) - \rho(j+1)]},$$

or

$$\|X(t,s)\| \leq 2be^{M}e^{(M+\frac{\ln 2b}{\rho})} - be^{M}e^{(M+\frac{\ln b}{\rho})(t-s)} + be^{2M}e^{(M+\frac{\ln b}{\rho})(t-s)+M\delta(t,s)[(l+1)-\rho(j+1)]}.$$

Combining the two cases, we conclude that

$$\|X(t,s)\| \leq L(t,s).$$

In view of the representation of solutions (13), we have

$$\|x(t)\| \leq L(t,\sigma)\|\phi\| + jb\|\phi\|\big(L_1(t,\sigma) - K_1 e^{\lambda_1(t-\theta^+_{n(\sigma)+j})}\big) + \|\phi\|\int_\sigma^{\sigma+\tau} L(t,s)\|\eta(s,-\tau)\|\mathrm{d}s + \|\phi\|\int_\sigma^{\sigma+\tau} L(t,s)V(s)\mathrm{d}s. \quad (17)$$

We observe that

$$\int_\sigma^{\sigma+\tau} L(t,s)\|\eta(s,-\tau)\|\mathrm{d}s \leq \sum_{k=[\sigma]}^{[\sigma+\tau]}\int_k^{k+1} K_0^{'}e^{\lambda^{'}(t-s)}\|\eta(s,-\tau)\|\mathrm{d}s - K_1\sum_{k=[\sigma]}^{[\sigma+\tau]}\int_k^{k+1} e^{\lambda_1(t-s)}\|\eta(s,-\tau)\|\mathrm{d}s + K_2\sum_{k=[\sigma]}^{[\sigma+\tau]}\int_k^{k+1} e^{[\lambda_1+\frac{M((l+1)-\rho(j+1))}{j+1}](t-s)}\|\eta(s,-\tau)\|\mathrm{d}s,$$

or

$$\begin{aligned}\int_\sigma^{\sigma+\tau} L(t,s)\|\eta(s,-\tau)\|\mathrm{d}s &\leq K_0^{'}e^{\lambda^{'}t}\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda^{'}k}\int_k^{k+1}\|\eta(s,-\tau)\|\mathrm{d}s\\ &- K_1e^{\lambda_1 t}\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\lambda_1(k+1)}\int_k^{k+1}\|\eta(s,-\tau)\|\mathrm{d}s\\ &+ K_2e^{\mu t}\sum_{k=[\sigma]}^{[\sigma+\tau]} e^{-\mu k_\mu}\int_k^{k+1}\|\eta(s,-\tau)\|\mathrm{d}s,\end{aligned}$$

where

$$k_\mu = \begin{cases} k+1\,, & \mu < 0\,,\\ k\,, & \mu \geq 0\,.\end{cases}$$

By the help of Lemma 3 we obtain the inequalities

$$\int_{\sigma}^{\sigma+\tau} L(t,s)\|\eta(s,-\tau)\|\mathrm{d}s \leq \frac{K_0' M_1 e^{2\lambda'} e^{\lambda'(t-\sigma)}}{e^{\lambda'}-1} - \frac{K_1 M_1 (1-e^{-\lambda_1 \tau}) e^{\lambda_1 (t-\sigma)}}{e^{\lambda_1}-1} + \frac{K_2 M_1 e^{h(\mu)} e^{\mu(t-\sigma)}}{e^{\mu}-1},$$

and

$$\int_{\sigma}^{\sigma+\tau} L(t,s)V(s)\mathrm{d}s \leq \frac{K_0' M e^{2\lambda'} e^{\lambda'(t-\sigma)}}{e^{\lambda'}-1} - \frac{K_1 M (1-e^{-\lambda_1 \tau}) e^{\lambda_1 (t-\sigma)}}{e^{\lambda_1}-1} + \frac{K_2 M e^{h(\mu)} e^{\mu(t-\sigma)}}{e^{\mu}-1}.$$

In view of (17), (16) is obtained. □

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