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ON THE WELL-POSEDNESS PROBLEM FOR THE GENERALIZED b-FAMILY OF EQUATIONS

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Abstract: In this paper, we study the local well posedness of the Cauchy problem for the generalized b-family equations. By applying some Sobolev's inequalities, semigroup theorem and related knowledge of PDE and using Kato's theory, we prove that there is a unique local solution of this problem which is continuously depending on the initial value.

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1. Introduction

Recently, Holm and Staley [8], [9] studied a one-dimensional version of active fluid transport that is described by the following family of 1+1 evolutionary equations

$$m_t + um_x + bu_x m = 0, \quad u = g * m\,, \tag{1.1}$$

where the fluid velocity $u(t;x)$ is defined on the real line vanishing at spatial infinity and $u = g * m$ denotes the convolution (or filtering)

$$u(x) = \int_R g(x-y)m(y)dy\,.$$

The family of equations (1.1) is characterized by the kernel g and the real dimensionless constant b, which is the ratio of stretching to convective transport. The parameter b is also the number of covariant dimensions associated with the momentum density m, The function $g(x)$ will determine the travelling wave

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shape and length scale for equation (1.1), while the constant b will provide a balance or bifurcation parameter for the nonlinear solution behavior. Here we take $g(x) = \frac{1}{2}e^{-|x|}$, this means $m = u - u_{xx}$, so equation (1.1) becomes the following form:

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}\,, \quad t > 0,\ x \in R \tag{1.2}$$

It was proved by Degasperis and Procesi using the method of asymptotic integrability that equation (1.2) cannot be completely integrable unless $b = 2$ or $b = 3$.

If $b = 2$, equation (1.2) becomes the Camassa-Holm (CH) equation of the form

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}\,, \quad t > 0,\ x \in R \tag{1.3}$$

modeling the unidirectional propagation of shallow water waves over a flat bottom. Again $u(t; x)$ stands for the fluid velocity at time t in the spatial x direction [1], [5], equation (1.3) has a bi-Hamiltonian structure [6] and is completely integrable [1], [2]. It has been shown that this equation is locally well-posed [3], [19] with the initial data $u_0 \in H^s(R)$ for $s > \frac{3}{2}$. The Camassa-Holm equation (1.3) possesses the peak-on solitons of the form

$$u(x,t) = ce^{-|x-ct|}, \quad c > 0\,. \tag{1.4}$$

If $b = 3$, equation (1.2) becomes the Degasperis-Procesi (DP) equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in R \tag{1.5}$$

Degasperis, Holm and Hone [4] proved the formal integrability of the Degasperis-Procesi equation by constructing a Lax pair. They also showed that the equation has bi-Hamiltonian structure and an infinite sequence of conserved quantities. Recently Lundmark [17] showed that the Degasperis- Procesi equation (1.5) has not only peaked solitons (1.4), but also shock peakons of the form

$$u(x,t) = -\frac{1}{t+k}\operatorname{sgn}(x)e^{-|x|}, \quad k > 0\,.$$

Equation (1.5) can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation. Dullin, Gottwald and Holm showed that the Degasperis-Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation.

After the Degasperis-Procesi equation was derived, many papers were devoted to its study. For example, Yin proved local well-posedness to equation

(1.5) with initial data $u_0 \in H^s(R), s > \frac{3}{2}$ on the line [21] and the precise blow-up scenario and a blow-up result were derived. The global existence of strong solutions and global weak solutions to equation (1.5) are also investigated in [22]. Recently, Lenells [15] classified all weak travelling wave solutions. Matsuno [18] studied multisoliton solutions and their peak-on limits.

For equation (1.2) some papers were also devoted to its study. In [7], Guilong Gui, Yue Liu and Lixin Tian investigated its local well-posedness, existence and uniqueness of the global solution and the blow-up phenomena. Here we consider the generalized form of equation (1.2) as follows:

$$u_t - u_{xxt} + (b+1)u^m u_x = bu_x u_{xx} + uu_{xxx} + \gamma(u - u_{xx}), \quad t > 0,\ x \in R \quad (1.6)$$

Obviously, equation (1.2) is the special case of equation (1.6) when $m = 1, \gamma = 0$. In this paper our goal is to study the local well-posedness problem of equation (1.6).

2. Notation and Some Useful Lemmas

As above and henceforth, we denote the norm of the Lebesgue space L^p by $\|\cdot\|_{L^p}$, $1 \le p \le \infty$ and the norm in the Sobolev space H^s, $s \in R$ by $\|\cdot\|_s$. We also use $\langle\cdot,\cdot\rangle$ to represent the standard inner product in $L^2(R)$, and$\langle\cdot,\cdot\rangle_s$ the standard inner product in $H^s(R)$. Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $S : Y \mapsto X$ be a topological isomorphism. $\mathbf{L}(Y;X)$ denotes the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $\mathbf{L}(X)$. $[A,B] = AB - BA$ denote the communicator of linear operators A, B.

Lemma 2.1. *Let $s,t \in R$ such that $-s < t \le s$. Then*

$$\| fg \|_t \le c \| f \|_s \| g \|_t \quad \text{if } s > \frac{1}{2}$$

and

$$\|fg\|_{s+t-m/2} \le c\|f\|_s\|g\|_t \quad \text{if } s < \frac{1}{2},$$

where c is a positive constant depending on $s,\ t$.

Lemma 2.2. (Kato's) *Let $f \in H^r,\ r > \frac{3}{2},\ M_f$ is the multiplication operator by f. Then*

$$\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}} \in L(L^2(R)) \quad \text{if } |\tilde{t}|,\ |\tilde{s}| \le r - 1 .$$

Moreover

$$\|\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}}\omega\|_0 \le c\|f'\|_{r-1}\|\omega\|_0 ,$$

where $c > 0$ is a constant.

Lemma 2.3. *Let $f, g \in H^s$ and $s > \frac{1}{2}$, then*

$$\|fg\| \le c\|f\|_s\|g\|_s .$$

That is because H^s is a Banach algebra for $s > \frac{1}{2}$.

Lemma 2.4. *Let $s > \frac{3}{2}$, then $\|u_x\|_{L^\infty} \le \|u\|_s$.*

This lemma follows directly from the Sobolev Embedding Theorem.

3. Kato's Theory

Consider the Cauchy problem associated to a quasilinear evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u)u = f(u) \in X,\ t \ge 0\,, \\ u(0) = u_0 \in Y\,, \end{cases} \tag{3.1}$$

where $A(u)$ is a linear operator depending on the unknown, and the initial value. To study the Cauchy problem (local in the time) associated to (3.1) we will make the following assumptions:

(X) X and Y are reflexive Banach spaces where $X \subset Y$ with the inclusion continuous and dense, and there is an isomorphism S from Y onto X such that $\|\phi\|_Y = \|S\phi\|_X$ for all $\phi \in Y$.

(A_1) Let W be an open ball centered in 0 and contained in Y. The linear operator$A(u) \in G(X, 1, \beta)$, where β is a real number, i.e., $-A(u)$ generates a c_0-semigroup such that

$$\|e^{-sA(u)}\|_{\mathbf{L}(X)}$$

Note that if X is a Hilbert space, then $A(u) \in G(X, 1, \beta)$ if and only if:

$(a)\langle A\phi, \phi\rangle_X \ge -\beta\|\phi\|_X^2,\ \forall\phi \in \mathbf{D}(A)$.

$(b)(A + \lambda)$ is onto for some (all) $\lambda > \beta$. Under these conditions $A(u)$ is said to be quasi-m-accretive.

(A_2) The map

$$w \in W \to B(w) = [S, A(w)]S^{-1} \in L(X)$$

is uniformly bounded and Lipchitz continuous, that is, there exist constants $\lambda_1, \mu_1 > 0$, such that

$$\begin{aligned} &\|B(w)\|_{\mathbf{L}(X)} \le \lambda_1 \,, \\ &\|B(w) - B(y)\|_{\mathbf{L}(X)} \le \mu_1 \|w - y\|_y \,, \end{aligned} \tag{3.2}$$

for all $w, y \in W$.

(A_3) $Y \subseteq \mathbf{D}(A(w))$ for each $w \in W$ (so that $A|_Y \in \mathbf{L}(X, Y)$ by the Closed Graph theorem). Moreover, the map $w \in W \to A(w) \in \mathbf{L}(Y, X)$ satisfies the following Lipschitz condition:

$$\|A(w) - A(y)\|_{\mathbf{L}(Y,X)} \le \mu_2 \|w - y\|_X \,, \tag{3.3}$$

for all $w, y \in W$, where μ_2 is a non-negative constant.

(f_1) The function $f : W \to Y$ is bounded, i.e. there is a constant $\lambda_2 > 0$ such that

$$\|f(w)\|_Y \le \lambda_2 \,, \tag{3.4}$$

for all $w \in W$, and the function $w \in W \to f(w)$ is Lipschitz in $X(Y)$, i.e.

$$\|f(w) - f(y)\|_X \le \mu_3 \|w - y\|_X \,, \quad \forall w, y \in W \,, \tag{3.5}$$

$$\|f(w) - f(y)\|_Y \le \mu_4 \|w - y\|_Y \,, \quad \forall w, y \in W \,, \tag{3.6}$$

where μ_3, μ_4 is non-negative constant.

In practice, as we will see, the value of $\beta, \lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3, \mu_4$ are functions of r, the radius of ball W.

Theorem 3.1. (Kato) *Assume conditions* $(X), (A_1) - -(A_3), (f_1)$. *Given* $u_0 \in Y$; *there is* $T > 0$ *and unique solution*

$$u \in C([0, T), Y) \bigcap C^1([0, T), X)$$

to (3.1) with $u(0) = u_0$. *Moreover, the map*

$$u_0 \in Y \to u \in C([0, T), Y) \bigcap C^1([0, T), X)$$

is a continuous map from Y *to* $C([0, T), Y) \bigcap C^1([0, T), X)$.

4. Local Theory

In this section we will apply Kato's theory to establish local well posedness for the Cauchy problem associated to the generalized b-family equations. The equation (1.6) can be rewritten in the following way

$$\begin{cases} u_t + uu_x - \gamma u = \partial_x (1-\partial_x^2)^{-1}(\frac{1}{2}u^2 + \frac{b-3}{2}u_x^2 - \frac{b+1}{m+1}u^{m+1})\,, \\ u(0,x) = u_0(x)\,. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $u_0 \in H^s, s > \frac{3}{2}$, then for any constant b there exists $T > 0$ depending on $\|u_0\|_s$, and unique solution u to (1.6) (or (4.1)) such that $u \in C([0,T], H^s)\bigcap C^1([0,T], H^{s-1})$. Moreover, the mapping $u_0 \in H^s \to u \in C([0,T], H^s)\bigcap C^1([0,T], H^{s-1})$ is continuous.*

To prove this theorem, we will apply Kato' theory with $X = H^{\frac{1}{2}}$ $Y = H^s$, $S = \Lambda^{s-1/2}$, $\Lambda = (1-\partial_x^2)^{1/2}$, $A(u) = (u\partial_x - \gamma)$, $f(u) = \partial_x(1-\partial_x^2)^{-1}(\frac{1}{2}u^2 + \frac{b-3}{2}u_x^2 - \frac{b+1}{m+1}u^{m+1})$. Obviously, S is an isomorphism of H^s onto $H^{1/2}$. Thus, in order to derive Theorem 4.1 from Theorem 3.1, we only need to verify that $A(u)$ and $f(u)$ satisfy the conditions $(X), (A_1) - -(A_3), (f_1)$. We break the argument into several lemmas.

Lemma 4.1. (see [23]) *The operator $A(u) = (u\partial_x - \gamma)$ with $u \in H^s$, $s > \frac{3}{2}$ belongs to $G(H^{\frac{1}{2}}, 1, \beta)$.*

Lemma 4.2. *(i) $\forall u \in H^s, s > \frac{3}{2}, B(u) = [\Lambda^{s-1/2}, (u\partial_x - \gamma)]\Lambda^{1/2-s} \in \mathbf{L}(H^{1/2})$.*

(ii) $\|(B(u) - B(v))w\|_{1/2} \le \|w\|_{1/2}\|u-v\|_s,\ s > \dfrac{3}{2}$.

Proof. (i) Note that

$$\begin{aligned} [\Lambda^{s-1/2}, (u\partial_x - \gamma)]\Lambda^{1/2-s} &= [\Lambda^{s-1/2}, u\partial_x]\Lambda^{1/2-s} - [\Lambda^{s-1/2}, \gamma]\Lambda^{1/2-s} \\ &= [\Lambda^{s-1/2}, u]\Lambda^{1/2-s}\partial_x - 0\,, \end{aligned}$$

therefore

$$\begin{aligned} \|B(u)w\|_{1/2} &= \|\Lambda^{1/2}[\Lambda^{s-1/2}, u]\Lambda^{1-s}\Lambda^{-1/2}\partial_x w\|_0 \\ &\le \|u\|_s\|\Lambda^{-1/2}\partial_x w\|_0 \le \|u\|_s\|w\|_{1/2}\,. \end{aligned}$$

For (ii) we only have to replace u by $u - v$ into (i), we can obtain the following result easily

$$\|(B(u) - B(v))w\|_{1/2} \le \|w\|_{1/2}\|u - v\|_s\,.$$

This will complete the proof of Lemma 4.2. □

Lemma 4.3. *(i)* $H^s \subseteq D(u\partial_x - \gamma) = \{f \subseteq H^{1/2} | (u\partial_x - \gamma)f \in H^{1/2}\},\ s > \frac{3}{2}$.
(ii) $A(u) \in L(H^s, H^{1/2}),\ s > \frac{3}{2}$.
(iii) $\|A(u) - A(v)\|_{L(H^s,H^{1/2})} \leq \mu\|u - v\|_{1/2}$.

Proof. Let $\omega \in H^s, s > \frac{3}{2}$ then

$$\|(u\partial_x - \gamma)w\|_{1/2} \leq \|u\partial_x - \gamma\|_{1/2}\|\partial_x w\|_{s-1} \leq \|u - \gamma\|_{1/2}\|w\|_s\,.$$

This proves (i) and (ii). Since

$$\|(A(u) - A(v))w\|_{1/2} = \|(u - v)\partial_x w\|_{1/2} \leq \|u - v\|_{1/2}\|w\|_s$$

So Lemma 4.3 is proved. □

Lemma 4.4. *Let* $f(u) = \partial_x(1 - \partial_x^2)^{-1}(\frac{1}{2}u^2 + \frac{b-3}{2}u_x^2 - \frac{b+1}{m+1}u^{m+1})$ *then:*
(i) $\|f(u)\|_s \leq \mu,\ s > \frac{3}{2}$.
(ii) $\|f(u) - f(v)\|_{1/2} \leq c\|u - v\|_{1/2}$.
(iii) $\|f(u)-f(v)\|_s \leq c\|u-v\|_s,\ s > \frac{3}{2}$, *where* μ *and* c *are positive constants.*

Proof. We prove (ii) first

$$\begin{aligned}\|f(u) - f(v)\|_{1/2} \leq \|\Lambda^{-1/2}(u^2 - v^2)\|_0 + \frac{b-3}{2}\|\Lambda^{-1/2}\partial_x(u - v)\partial_x(u + v)\|_0 \\ + \frac{b+1}{m+1}\|\Lambda^{-1/2}(u^{m+1} - v^{m+1})\|_0 \\ \leq \|(u + v)(u - v)\|_0 + \frac{b-3}{2}\|\partial_x(u + v)\partial_x(u - v)\|_{-1/2} \\ + \frac{3}{m+1}\|u^{m+1} - v^{m+1}\|_0\,,\end{aligned}$$

where

$$\|(u - v)(u + v)\|_0 \leq \|u + v\|_{L^\infty}\|u - v\|_0 \leq \|u + v\|_s\|u - v\|_{1/2}\,, \tag{4.2}$$

$$\begin{aligned}&\|\partial_x(u + v)\partial_x(u - v)\|_{-1/2} \leq \|\partial_x(u - v)\|_{-1/2}\|\partial_x(u + v)\|_{s-1} \\ &\leq \|u + v\|_s\|u - v\|_{1/2}\,,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\|u^{m+1} - v^{m+1}\|_0 = \|(u - v)(u^m + u^{m-1}v + \cdots + v^m)\|_0 \\ &\leq \|u - v\|_0\|u^m + u^{m-1}v + \cdots + v^m\|_s\,.\end{aligned}$$

Note that $s > \frac{1}{2}, H^s$ is a Banach algebre

$$\|u^m + u^{m-1}v + \cdots + v^m\|_s \leq \|u\|_s^m\|v\|_s + \cdots + \|v\|_s^m \triangleq K < \infty\,,$$

then

$$\|u^{m+1} - v^{m+1}\| \leq K\|u - v\|_{1/2}\,. \tag{4.4}$$

Combining (4.2)-(4.4) we completed the proof of (ii)

For (iii) we can directly apply Lemma 4.3 and get the following conclusion

$$\begin{aligned}
\|f(u) - f(v)\|_s &\leq \|\partial_x(1-\partial_x^2)^{-1}(\frac{1}{2}(u^2 - v^2) + \frac{b-3}{2}(u_x^2 - v_x^2))\|_s \\
&+ \|\partial_x(1-\partial_x^2)^{-1}\frac{b+1}{m+1}(u^{m+1} - v^{m+1})\|_s \\
&\leq \|\frac{b-3}{2}(u_x^2 - v_x^2)\|_{s-1} + \|\frac{b+1}{m+1}(u^{m+1} - v^{m+1})\|_{s-1} \\
&+ \|\frac{1}{2}(u^2 - v^2)\|_{s-1}\,,
\end{aligned}$$

where

$$\begin{aligned}
&\|u_x^2 - v_x^2\|_{s-1} \leq \|(u+v)(u-v)\|_s \leq \|u+v\|_s\|u-v\|_s \\
&\|u^2 - v^2\|_{s-1} \leq \|(u-v)(u+v)\|_s \leq \|u+v\|_s\|u-v\|s \\
&\|u-v\|_{s-1} \leq \|u-v\|_s \\
&\|u^{m+1} - v^{m+1}\|_{s-1} \leq K\|u-v\|_s
\end{aligned}$$

Therefore, there is constant $c > 0$ satisfies (iii). We can easily find that (i) is an immediate consequence of (iii), since we choose $v = 0$. So far, we have verified conditions of Theorem 3.1, that is to say the proof of Theorem 4.1 is completed. □

Theorem 4.2. *The existence time for equation (1.6)(or (4.1)) may be chosen independently of s in the following sense. If*

$$u \in C([0,T), H^s)\bigcap C^1([0,T), H^{s-1})$$

is the solution of equation (1.6) with $u(0) = u_0 \in H^{s'}$ for some $s' \neq s, s' > \frac{3}{2}$ then, $u \in C([0,T), H^{s'})\bigcap C^1([0,T), H^{s'-1})$ with the same time T. In particular, if $u_0 \in H^\infty$ then $u \in C([0,T), H^\infty)$.

It suffices to consider the case $s' > s$, since the case $s' < s$ is obvious from the uniqueness of solutions which is guaranteed by Theorem 4.1. Also we can suppose that $s < s \leq s+1$. Since if $s' > s+1$, we obtain the result by iterated application of the below argument.

If we apply operator Λ^2 to (4.1), we obtain the following evolution equation for $m(t) = \Lambda^2 u(t) = u - uxx$:

$$\begin{cases} \dfrac{d}{dt}m(t) + A(t)m + B(t)m = f(t)\,, \\ m(0) = \Lambda^2 u(0)\,, \end{cases} \tag{4.5}$$

where $A(t)m = \partial_x(um)$, $B(t)m = [(b+1)u_x - \gamma]m, f(t) = (b+1)(u - u^m)u_x$. And $u \in C([0,T), H^s)$ is viewed as a known function. Note also that $m \in C([0,T), H^{s-2})$ and $m(0) = \Lambda^2 u(0) \in H^{s'-2}$. Our objective is to prove that $m \in C([0,T), H^{s'-2})$ which will imply $u \in C([0,T), H^{s'})$ because $(1-\partial_x^2)$ is an isomorphism from $H^{s'}$ to $H^{s'-2}$. Note that $u \in C([0,T), H^s)$, $u_x \in H^{s-1}$, and that H^{s-1} is a Banach algebra. Then we obtain $B(t) \in \mathbf{L}(H^{s-1})$, $f(t) \in C([0,T), H^{s-1})$.

We also break the proof into several lemmas.

Lemma 4.5. *There exists a unique propagator $\{U(t,s)\}$ associate to the family $\{A(t)\}$ with the corresponding spaces $X = H^h$, $Y = H^k$, where*

$$-s \le h \le s-2,\; 1-s \le k \le s-1 \text{ and } k \ge h+1\,. \tag{4.6}$$

In particular, $U(t,s)$ maps H^r into itself for $-s \le r \le s-1$.

Proof. The proof is similar to the proof of Theorem 4.1. We only have to verify three conditions.

(i) $A(t) \in G(X, 1, \beta)$, that is, we will show that $\langle A(t)m, m\rangle_h \ge -\beta\|m\|_h^2$, i.e.

$$\langle A(t)m, m\rangle_h \ge -\beta\|m\|_h^2\,. \tag{4.7}$$

We begin estimating the term on the left-hand side of this inequality, which can be written in the following way:

$$\begin{aligned}
\langle A(t)m, m\rangle_h &= \langle \Lambda^h A(t)m, \Lambda^h m\rangle_0 = \langle \Lambda^h \partial_x(um), \Lambda^h m\rangle_0 \\
&= -\langle \Lambda^h(um), \partial_x\Lambda^h m\rangle_0 = -\langle \Lambda^h(u\Lambda^{-h}(\Lambda^h m)), \partial_x \Lambda^h m\rangle_0 \\
&= -\langle u\Lambda^h m, \partial_x\Lambda^h m\rangle_0 + \langle \Lambda^{h+1}[\Lambda^{-h}, u]\Lambda m, \partial_x \Lambda^{h-1} m\rangle_0 \\
&= \langle \partial_x u, (\Lambda^h m)^2\rangle_0 + \langle \Lambda^{h+1}[\Lambda^{-h}, u]\Lambda^h m, \partial_x\Lambda^{h-1}m\rangle_0\,.
\end{aligned}$$

Therefore

$$\langle \partial_x u, (\Lambda^h m)^2\rangle_0 \le \|\partial_x u\|_{L^\infty}\|\Lambda^h m\|_0^2 \le \|u\|_k\|m\|_h^2$$

$$\langle \Lambda^{h+1}[\Lambda^{-h}, u]\Lambda^h m, \partial_x \Lambda^{h-1} m\rangle_0 \le \|u\|_k \|\Lambda^h m\|_0 \|\Lambda^{h-1}\partial_x m\|_0 \le \|u\|_k\|m\|_h^2$$

Thus (4.7) is proved.

(ii) $\Lambda^h \partial_x[\Lambda^{k-h}, u(t)]\Lambda^{-k}$ is uniformly L^2-bounded, by applying Lemma 3.2 we obtain the following estimates

$$\|\Lambda^h \partial_x[\Lambda^{k-h}, u]\Lambda^{-k}\omega\|_0 \leq c\|u\|_s\|\omega\|_0 .$$

Here we have used that Λ^{k-h} is the isomorphism of Y to X.

(iii) $A(t) \in \mathbf{L}(Y, X)$ is strongly continuous in t, but this is a consequence of the continuity of u and of the following inequality:

$$\begin{aligned}
\|(A(t+\Delta t) - A(t))m\|_h &= \|\partial_x((u(t+\Delta t) - u(t)))\omega\|_h \\
&\leq \|(u(t+\Delta t) - u(t))\omega\|_{h+1} \\
&\leq \|u(t+\Delta t) - u(t)\|_{s-1}\|\omega\|_{h+1} \\
&\leq \|u(t+\Delta t) - u(t)\|_s\|w\|_k .
\end{aligned}$$

Lemma 4.6.

$$m(t) = U(t,0)m(0) + \int_0^t U(t,\tau)[-B(\tau)m(\tau) + f(\tau)]d\tau . \qquad (4.8)$$

Proof. In Lemma 4.5, choose $h = s-3,\ k = s-2$ (which satisfy (4.6)). Since $m \in C([0,T), H^{s-2}) \bigcap C^1([0,T), H^{s-3})$, as is easily verified by $u \in C([0,T]; H^s)$, we can carry out the standard computation

$$\frac{dU(t,\tau)m(\tau)}{d\tau} = U(t,\tau)[\frac{dm(\tau)}{d\tau} + A(\tau)m(\tau)] = U(t,\tau)[-B(\tau)m(\tau) + f(\tau)].$$

And obtain (4.8) on integration $\tau \in [0,t]$. This is essentially a uniqueness proof for the solution m of (4.5) in the class $C([0,T), H^{s'-2})$.

Lemma 4.7. $m \in C([0,T), H^{s'-2})$

Proof. Note that if $s < s' \leq s+1$, then we have that

$$f(t) \in C([0,T), H^{s-1}) \subset C([0,T), H^{s'-2})$$

and $B(t) \in \mathbf{L}(H^{s'-2})$ is strongly continuous in $[0,t)$ and that $H^{s-1}H^{s'-2} \subset H^{s'-2}$ by $s-1 > \frac{1}{2}$. Due to $-s < s-2 \leq s'-2 < s-1$ the family $U(t,\tau) : H^{s'-2} \mapsto H^{s'-2}$ is strongly continuous on $H^{s'-2}$to itself. Observe that $m(0) \in H^{s'-2}$. We have only see (4.8) as an integral equation of Volterra type, which can be solved for m by successive approximation. This completes the proof of Theorem.

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