CONSTRUCTION OF 2-CONNECTED $k$-REGULAR SIMPLE GRAPHS

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Abstract: The construction of 2-connected $k$-regular simple graphs is obtained in this paper. According to two situations of $n = 5l$ ($l \geq 1$) and $n = 5l + t$ ($t = 1, 2, 3, 4$), we give the construction procedures when $k = 4$, where $n$ denotes the vertex number of graph $G_n$. Then the methods are extended to $k = 2m$ and $k = 2m + 1$ ($m \geq 1$).

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1. Introduction

We take the basic terminology from [1]. The graphs considered in this paper will be finite simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and $|V(G)| = n$. For a vertex $v$ of $G$, the degree of $v$ is denoted by $d_G(v)$. The union $G_1 \cup G_2$ of $G_1$ and $G_2$ is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If $G_1$ and $G_2$ are disjoint, we sometimes denote their union by $G_1 + G_2$. Suppose that $E'$ is a nonempty subset of $E(G)$, the spanning subgraph of $G$ with edge set $E(G) \setminus E'$ is written simply as $G - E'$, which is the subgraph obtained from $G$ by deleting the edges in $E'$.

Two graphs $G$ and $H$ are said to be isomorphic (written $G \cong H$) if there are bijections $\theta: V(G) \rightarrow V(H)$ and $\phi: E(G) \rightarrow E(H)$ such that $uv \in E(G)$ if and only if $\phi(uv) = \theta(u)\theta(v)$. Such a pair $(\theta, \phi)$ of mappings is called an
isomorphism between $G$ and $H$. A graph $G$ is called $k$-regular if $d(v) = k$ for all $v \in V(G)$. Particularly, if $k = 4$, then we get 4-regular graphs discussed in this paper. For some new results about regular graphs, we can see [4], [2] and [3]. A 2-connected graph is the graph which has no cut-vertex. $G_n$ denotes a 2-connected $k$-regular simple graph with $n$ vertices.

2. Construction Procedures

Let us give the construction procedures for 2-connected 4-regular graphs $G_n$ in the following.

Case 1. $n = 5l$, $l$ is an integral number and $l \geq 1$.

Case 1.1. $l = 1$. The 4-regular graph with least vertices is $K_5$, denoted by $G_5$.

Case 1.2. $l = 2$. Let $H_1 \cong K_5$ and $H_1 \cong K_5$. For any edge $v_1v_2 \in H_1$ and $u_1u_2 \in H_2$, we have

$$G_{10} = (H_1 - v_1v_2) \cup (H_2 - u_1u_2) \cup (\{v_1u_1, v_2u_2\} \text{ or } \{v_1u_2, v_2u_1\}).$$

Then $G_{10}$ are 4-regular graphs with 10 vertices.

Case 1.3. $l \geq 3$. Suppose we already have the graphs $G_{5(l-1)}$. For any edge $v_1v_2 \in G_{5(l-1)}$ and $u_1u_2 \in G_5$, let

$$G_{5l} = (G_{5(l-1)} - v_1v_2) \cup (G_5 - u_1u_2) \cup (\{v_1u_1, v_2u_2\} \text{ or } \{v_1u_2, v_2u_1\}).$$

So 2-connected 4-regular graphs $G_{5l}$ are obtained.

Case 2. $n \neq 5l$.

Case 2.1. $n = 5l + 1$. Let $v_1v_2$ and $v_3v_4$ are two independent edges of $G_{5l}$, and a vertex $v$ disjointed from $G_{5l}$. Then

$$G_{5l+1} = (G_{5l} - \{v_1v_2, v_3v_4\}) \cup \{v_i|i = 1, 2, 3, 4\}.$$  

Case 2.2. $n = 5l + 2$. For any three edges $v_1v_2, v_3v_4$ and $v_5v_6$ of $G_{5l}$, suppose at least two edges of them have no same ends. Let $K_2 = u_1u_2$, then

$$G_{5l+2} = (G_{5l} - \{v_1v_2, v_3v_4, v_5v_6\}) \cup \{v_iu_j|i = 1, 2, 3, 4, 5, 6; j = 1, 2\}.$$  

We give an explanation to the conditions that the three edges have no same ends. If $v_1v_2, v_3v_4$ and $v_5v_6$ have same ends, then there are adjacent each other. Without loss of generality, assume that $v = v_1 = v_3 = v_5$ and $G'_{5l} = G_{5l} - \{vu_i|i = 2, 4, 6\}$. So we have $d_{G'}(v) = 1$. Therefore, one of vertices
from \(v_2, v_4\) and \(v_6\) must be connected with \(v\), suppose such vertex is \(v_2\). Let \(G''_5 = G_5 - \{vv_4, vv_6\}\), then \(d_{G''}(v) = 2\), \(d_{G''}(v_4) = 3\) and \(d_{G''}(v_6) = 3\). In order to make the degree of all vertices to be 4, the vertices \(v, v_4\) and \(v_6\) need 4 edges, but the vertices \(u_1\) and \(u_2\) need 6. This is a contraction.

**Case 2.3.** \(n = 5l + 3\). Let \(e_1, e_2\) and \(e_3\) be any three edges of \(G_5\) and \(G'_5 = G_5 - \{e_1, e_2, e_3\}\). For a vertex \(v\) in \(G'_5\) with degree below 4, we connect \(v\) to a vertex of \(K_3\). Do repeatedly, until the degree of all vertices in \(G'_5 + K_3\) are 4. So we get \(G_{5l+3}\).

**Case 2.4.** \(n = 5l + 4\). Deleting any two edges of \(G_5\), we get the result graphs \(G'_{5l}\). For two graphs \(G'_{5l}\) and \(K_4\), using same methods as Case 2.3. So 2-connected 4-regular simple graphs \(G_n\) \((n \geq 5)\) are obtained.

Now we extended the methods of above to \(k = 2m\) and \(k = 2m + 1\). \(m\) is an integer number and \(m \geq 1\).

Firstly for \(k = 2m\). The 2\(k\)-regular graph with least vertices is \(K_{2m+1}\), denoted by \(G_{2m+1}\). If \(n = (2m + 1)l\), then

\[
G_{(2m+1)l} = (G_{(2m+1)l-1} - v_1v_2) \cup (G_{2m+1} - u_1u_2) \cup \{v_1u_1, v_2u_2\}
\]

or \(\{v_1u_2, v_2u_1\}\),

where \(v_1v_2 \in G_{(2m+1)l-1}\) and \(u_1u_2 \in G_{2m+1}\). If \(n = (2m + 1)l + 1\), for \(m\) independent edges \(e_1, e_2, \cdots, e_m\) of \(G_{(2m+1)l}\) and a vertex \(v\) disjoined from \(G_{(2m+1)l}\), we can get \(G_{(2m+1)l+1}\) by deleting \(e_i\) \((i = 1, 2, \cdots, m)\) from \(G_{(2m+1)l}\) and connecting every ends of \(e_i\) \((i = 1, 2, \cdots, m)\) to \(v\). If \(n = (2m + 1)l + 2\), by deleting all the edges of a \((2m - 1)\)-cycle of \(G_{(2m+1)l}\) and connecting all vertices of the cycle to \(u_1\) and \(u_2\) respectively, then \(G_{(2m+1)l+2}\) are constructed, where \(k_2 = u_1u_2\). Do repeatedly of above procedures, until \(n = (2m + 1)(l + 1) - 1\). So 2-connected \(2m\)-regular graphs are obtained.

Secondly for \(k = 2m + 1\). The \((2m + 1)\)-regular graph with least vertices is \(K_{2m+2}\), denoted by \(G_{2m+2}\). If \(n = (2m + 2)l\), same methods of above can be used to get \(G_{(2m+2)l}\). If \(n = (2m + 2)l + 2\), by deleting all the edges of a \(2m\)-cycle of \(G_{(2m+1)l+2}\) and connecting all vertices of the cycle to \(u_1\) and \(u_2\) respectively, then \(G_{(2m+1)l+2}\) are constructed, where \(k_2 = u_1u_2\). Do repeatedly of above procedures, until \(n = (2m + 1)(l + 1) - 2\). So 2-connected \((2m + 1)\)-regular graphs are gotten.

Over all the procedures of above, we obtain 2-connected \(k\)-regular simple graphs \(G_n\).
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References


