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GENERALIZED MIXED CO-QUASI-VARIATIONAL INCLUSIONS IN BANACH SPACES

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Abstract: In this paper, we consider generalized mixed co-quasi-variational inclusions and propose an iterative algorithm for computing their approximate solutions by using definition of J-proximal mapping given in [9]. We prove that the approximate solutions obtained by proposed algorithm converge to the exact solution of our co-quasi-variational inclusion. Some special cases are also discussed.

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1. Introduction

It is worth mentioning that multivalued quasi-variational inclusions include mixed (quasi) variational inequalities, complementarity problems and nonlinear programming problems as special cases. Quasi-variational inclusions provide us a unified, natural, noval, innovative and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences.

Under Hilbert space setting, there is a substantial number of iterative algorithms for finding the approximate solutions of various variational inequalities, see [1, 2, 3, 11, 12, 13]. The most effective numerical technique is the projection method and its various variants, auxiliary principle technique, Newton and

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descent framework. The applicability of the projection method is limited due to the fact that it is not easy to find the projection except in a very special case and it strictly depends on the inner product property of Hilbert spaces. Due to the presence of nonlinear term, the project method cannot be applied to suggest any iterative algorithms for generalized mixed variational inequalities and generalized quasi-variational inclusions in Banach spaces. Recently, Chang [7], Chang et al [8], and Huang [14] introduced and studied some classes of set-valued variational inclusions in real Banach spaces. Alber and Yao [6] used a sunny nonexpansive retraction to construct the projection iterative method for finding the approximate solutions of a class of multivalued quasi-variational inequalities in Banach spaces and called them as co-quasi-variational inequalities.

Motivated and inspired by the research work going on in this field, we introduce and study a new class of generalized mixed co-quasi-variational inclusions in real Banach spaces. A noval and innovative iterative algorithm for finding the approximate solutions of generalized mixed co-quasi-variational inclusions is suggested and analysed. We also prove convergence result for the approximate solutions obtained by the proposed algorithm.

2. Formulation and Preliminaries

Let E be a Banach space with the dual space E^*, $\langle u, x\rangle$ be the dual pairing between $u \in E^*$ and $x \in E$ and $CB(E)$ be the family of all nonempty closed bounded subset of E, $D(.,.)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A,B) = \max\,\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(A,v)\} \text{ for all } A,B \in CB(E)\,,$$

where $d(u,B) = \inf_{v\in B} d(u,v)$ and $d(A,v) = \inf_{u\in A} d(u,v)$ and $J : E \to E^*$ is the normalized duality mapping defined by

$$\|Jx\|_{E^*} = \|x\| \text{ and } \langle x, J(x)\rangle = \|x\|^2, \text{ for all } x \in E.$$

Definition 2.1. Let $\phi : E \to R \cup \{+\infty\}$ be a proper functional. A vector $f^* \in E^*$ is called a *subgradient* of ϕ at $x \in \mathrm{dom}\phi$ if

$$\langle f^*, y - x\rangle \le \phi(y) - \phi(x), \text{ for all } y \in E.$$

The set of all subgradients of ϕ at x is denoted by $\partial\phi(x)$.

The mapping

$$\partial\phi(x) = \{f^* \in E^* : \phi(y) - \phi(x) \ge \langle f^*, y - x\rangle \text{ for all } y \in E\}$$

is said to be *subdifferential* of ϕ at x.

Definition 2.2. (see [9]) Let E be a Banach space with the dual space E^*, $\phi : E \to R \cup \{+\infty\}$ be a proper subdifferentiable (may not convex) functional and $J : E \to E^*$ be a mapping. If for any given point $x^* \in E^*$ and $\rho > 0$. There is a unique point $x \in E$ satisfying

$$\langle Jx - x^*, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \text{ for all } y \in E.$$

The mapping $x^* \to x$, denoted by $J_\rho^{\partial\phi}(x^*)$, is said to be *J-proximal mapping* of ϕ. We have $x^* - Jx \in \rho\partial\phi(x)$, it follows that $J_\rho^{\partial\phi}(x^*) = (J + \rho\partial\phi)^{-1}(x^*)$.

Definition 2.3. Let $M : E \to CB(E)$ be a set-valued mapping, $J : E \to E^*$ and $g : E \to E$ be two single valued mappings:

(i) M is said to be *Lipchitz continuous* if there exists a constant $\lambda_M \geq 0$ such that

$$D(M(x), M(y)) \leq \lambda_M \|x - y\|, \text{ for all } x, y \in E;$$

(ii) J is said to be α *strongly monotone* ($\alpha > 0$) if

$$\langle Jx - Jy, x - y\rangle \geq \alpha\|x - y\|^2, \text{ for all } x, y \in E;$$

(iii) g is said to be *strongly accretive* with constant $k > 0$ such that

$$\langle g(x) - g(y), J(x - y)\rangle \geq k\|x - y\|^2, \text{ for all } x, y \in E.$$

Definition 2.4. The J-proximal mapping $J_\rho^{\partial\phi}(x^*) = (J + \rho\partial\phi)^{-1}(x)$ is said to be *retraction* if

$$(J_\rho^{\partial\phi})^2 = J_\rho^{\partial\phi}.$$

Example 2.1. We will show that our Definition 2.4 of retraction J-proximal mapping is justified.

We take $\rho = 1$

$$J = \begin{bmatrix} a_{11}\ a_{12}\ \dots\ a_{1n} \\ a_{21}\ a_{22}\ \dots\ a_{2n} \\ \dots \\ a_{n1}\ a_{n2}\ \dots\ a_{nn} \end{bmatrix} \text{ an } n \times n \text{ matrix}$$

and

$$\partial\phi = \begin{bmatrix} b_{11}\ b_{12}\ \dots\ b_{1n} \\ b_{21}\ b_{22}\ \dots\ b_{2n} \\ \dots \\ b_{n1}\ b_{n2}\ \dots\ b_{nn} \end{bmatrix} \text{ also an } n \times n \text{ matrix.}$$

We define the following operations for matrix J and $\partial\phi$

(i) $a_{ij} + b_{ij} = 1 \quad$ if $i = j$

(ii) $a_{ij} + b_{ij} = 0 \quad$ if $i \neq j$

Then $(J + \rho\partial\phi)^{-1}(J + \rho\partial\phi)^{-1} = (J + \rho\partial\phi)^{-1}$, i.e. $(J_\rho^{\partial\rho})^2 = J_\rho^{\partial\phi}$.

Given single-valued mappings $P, f, h, g, m : E \to E,\ J : E \to E^*$ and multivalued mappings $M, S, T : E \to CB(E)$. Let $\phi : E \times E \to R \cup \{+\infty\}$ be such that for each fixed $x \in E,\ \phi(.,x)$ is a lower semicontinuous, subdifferentiable functional on E (may not be convex) satisfying $(g(x) - m(x)) \cap \mathrm{dom}\partial\phi(.,x) \neq \phi$, where $\partial\phi(.,x)$ is the subdifferentiable of $\phi(.,x)$. We consider the following *generalized mixed co-quasi-variational inclusion problem* in Banach spaces (for short GMCQVIP):

$$\begin{cases} \text{Find } x \in E,\ u \in M(x),\ v \in S(x) \text{ and } w \in T(x) \text{ such that} \\ g(x) - m(x) \in \ \mathrm{dom}\partial\phi(.,x) \text{ and} \\ \langle J(p(u) - (f(v) - h(w))), y - (g(x) - m(x))\rangle \\ \qquad\qquad \geq \phi((g(x) - m(x)), x) - \phi(y, x),\ \text{for all } y \in E. \end{cases}$$

Let us see some special cases of (GMCQVIP).

Special Cases. (i) If $E = H$ is a Hilbert space, $m \equiv 0$ and J is an identity mapping, then (GMCQVIP) reduces to the following *completely generalized nonlinear variational inclusion* problem considered and studied by Ahmad, Kazmi and Salahuddin [4]:

$$\text{(CGNVIP)} \quad \begin{cases} \text{Find } x \in H,\ u \in M(x),\ v \in S(x) \text{ and } w \in T(x) \text{ such that} \\ \langle p(u) - (f(v) - h(w)), y - g(x)\rangle \geq \phi((g(x), x) - \phi(y, x), \\ \text{for all } y \in H. \end{cases}$$

(ii) If $E = H$ is a Hilbert space, J, f, h, m are identity mappings; $m \equiv 0$; $P(x) = g(x)$ and if $\phi \equiv I_K$, the indicator function of a closed convex set K in H defined by

$$I_K = \begin{cases} 0, & x \in K\,, \\ +\infty, & \text{otherwise}, \end{cases}$$

then (GMCQVIP) becomes the following *generalized variational inequality problem* considered and studied by Verma [17]:

$$\text{(GVIP)} \quad \begin{cases} \text{Find } x \in H,\ v \in S(x)\ w \in T(x) \text{ such that} \\ \langle g(x) - (v - w), y - g(x)\rangle \geq 0, \quad \text{for all } y \in E. \end{cases}$$

It is clear that the (GMCQVIP) includes many kinds of quasi-variational inequality problems, variational inequality problems and complementarity problems as special cases, such as [10, 16, 17, 18].

Now we recall some definitions, notations and results which will be used throughout the paper.

The *uniform convexity* of the Banach space E means that for any given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E,\ \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| = \epsilon$ ensure the following inequality

$$\|x + y\| \leq 2(1 - \delta).$$

The function

$$\delta_B(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| = 1,\ \|y\| = 1,\ \|x - y\| = \epsilon\right\}$$

is called the *modulus of the convexity* of the space E.

The *uniform smoothness* of the space E means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \leq \epsilon\|y\| \text{ holds.}$$

The function

$$\tau_B(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1,\ \|y\| = t\right\}$$

is called the *modulus of the smoothness* of the space E.

We remark that the space E is uniformly convex if and only if $\delta_B(\epsilon) > 0$ for all $\epsilon > 0$, and it is uniformly smooth if and only if $\lim_{t\to 0} t^{-1}\tau_B(t) = 0$.

Theorem 2.1. (see [9]) *Let E be a reflexive Banach space with the dual space E^* and $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional which may not be convex. Let $J : E \to E^*$ be an α-strongly monotone continuous mapping. Then for any $\rho > 0$, and any $x^* \in E^*$, there exist a unique $x \in E$ such that*

$$\langle Jx - x^*, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \text{for all } y \in E.$$

That is $x = J_\rho^{\partial\phi}(x^*)$ and so the J-proximal mapping of ϕ is well defined.

Theorem 2.2. (see [9]) *Let E be a reflexive Banach space with the dual space E^*, $J : E \to E^*$ be a α-strongly monotone continuous mapping, $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous subdifferentiable proper functional and $\rho > 0$, be an arbitrary constant. Then the J-proximal mapping $J_\rho^{\partial\phi}$ of ϕ is $1/\alpha$-Lipschitz continuous.*

The following proposition plays an important role in proving our main results.

Proposition 2.1. (see [5]) *Let E be a uniformly smooth Banach space and $J : E \to E^*$ be a normalized duality mapping. Then, for all $x, y \in E$*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle$,

(ii) $\langle x-y, J(x)-J(y)\rangle \le 2D^2\tau_B(4\|x-y\|/D)$, *where*

$$D = \sqrt{(\|x\|^2 + \|y\|^2)/2}.$$

3. An Iterative Algorithm and Convergence Result

We first transfer (GMCQVIP) into a fixed point problem.

Theorem 3.1. *(x, u, v, w) is a solution of (GMCQVIP) if and only if (x, u, v, w) satisfies the following relation:*

$$g(x) = m(x) + J_\rho^{\partial\phi(.,x)}\{J((g(x)-m(x)) - \rho(J(p(u)) - (f(v)-h(w)))\}, \quad (3.1)$$

where $x \in E$, $u \in M(x)$ $v \in S(x)$ $w \in T(x)$, $\rho > 0$ and $J_\rho^{\partial\phi(.,x)} = (J + \rho\partial\phi(.,x))^{-1}$ is the J-proximal mapping of $\phi(.,x)$.

Proof. Assume that (x, u, v, w) satisfies relation (3.1), i.e.

$$g(x) - m(x) = J_\rho^{\partial\phi(.,x)}\{J((g(x)-m(x)) - \rho(J(p(u)) - (f(v)-h(w)))\}.$$

Since $J_\rho^{\partial\phi(.,x)} = (J + \rho\partial\phi(.,x))^{-1}$, the above equality holds if an only if

$$J((g(x)-m(x)) - \rho(J(p(u) - (f(v)-h(w))) \in J((g(x)-m(x)) + \rho\partial\phi((g(x)-m(x),x).$$

By the definition of subbdifferential of $\phi(.,x)$, the above relation holds if and only if

$$\phi(y,x) - \phi((g(x)-m(x),x) \ge \langle -J(p(u) - (f(v)-h(w))), y - (g(x)-m(x))\rangle.$$

Hence we have

$$\langle J(p(u) - (f(v)-h(w))), y - (g(x)-m(x))\rangle \ge \phi((g(x)-m(x),x) - \phi(y,x),$$

$$\forall y \in E\,,$$

i.e. (x, u, v, w) is a solution of (GMCQVIP). □

To compute the approximate solutions of (GMCQVIP), we propose the following iterative algorithm.

Algorithm 3.1. Let $M, S, T : E \to CB(E)$ be set-valued mappings, $J : E \to E^*$ and $P, f, h, m, g : E \to E$ be single-valued mappings. Let $\phi : E \times E \to R \cup \{+\infty\}$ be a lower semicontinuous, subdifferentiable (may not be convex) proper functional on E such that $(g(x) - m(x)) \in \text{dom}\partial\phi(., x)$. For any $x_0 \in E,\ u_0 \in M(x_0),\ v_0 \in S(x_0),\ w_0 \in T(x_0)$ and $0 < \epsilon < 1$, let

$$\begin{aligned} x_1 = x_0 &- g(x_0) \\ &+ m(x_0) + J_\rho^{\partial\phi(.,x_0)}\{J(g(x_0) - m(x_0)) - \rho(J(p(u_0) - (f(v_0) - h(w_0)))\}. \end{aligned}$$

Since $u_0 \in M(x_0) \in CB(E)$, $v_0 \in S(x_0) \in CB(E)$, and $w_0 \in T(x_0) \in CB(E)$, by Nadler [15], there exist $u_1 \in M(x_1),\ v_1 \in S(x_1)$ and $w_1 \in T(x_1)$ such that

$$\|u_0 - u_1\| \le (1+1)D(M(x_0), M(x_1)) + \epsilon\|x_0 - x_1\|,$$

$$\|v_0 - v_1\| \le (1+1)D(S(x_0), S(x_1)) + \epsilon\|x_0 - x_1\|,$$

$$\|w_0 - w_1\| \le (1+1)D(T(x_0), T(x_1)) + \epsilon\|x_0 - x_1\|.$$

Let

$$\begin{aligned} x_2 = x_1 &- g(x_1) \\ &+ m(x_1) + J_\rho^{\partial\phi(.,x_1)}\{J(g(x_1) - m(x_1)) - \rho(J(p(u_1) - (f(v_1) - h(w_1)))\}. \end{aligned}$$

Continuing the above process inductively, we can define the following iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ for solving (GMCQVIP) as follows:

$$\begin{aligned} x_{n+1} = x_n &- g(x_n) + m(x_n) \\ &+ J_\rho^{\partial\phi(.,x_n)}\{J(g(x_n) - m(x_n)) - \rho(J(p(u_n) - (f(v_n) - h(w_n)))\} \end{aligned} \quad (3.2)$$

$$u_n \in M(x_n), \quad \|u_n - u_{n+1}\| \le (1 + (n+1)^{-1})D(M(x_n), M(x_{n+1})),$$

$$v_n \in S(x_n), \quad \|v_n - v_{n+1}\| \le (1 + (n+1)^{-1})(S(x_n), S(x_{n+1})),$$

$$w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \le (1 + (n+1)^{-1})(T(x_n), T(x_{n+1})).$$

$n = 0, 1, 2, 3, \ldots.$

If $E=H$ is a Hilbert space, $m\equiv 0$ and J is an identity mapping, then Algorithm 3.1 reduces to the Algorithm 3.1 of Ahmad, Kazmi and Salahuddin [4] for solving (CGNVIP).

Theorem 3.2. *Let E be a real uniformly smooth Banach space with the module of smoothness $\tau_B(t)\le Ct^2$ for $C>0$. Let $M,S,T:E\to CB(E)$ be Lipschitz continuous mappings with Lipschitz constants $\lambda_M,\ \lambda_S$ and λ_T, respectively. Let g,P,f,m and h be Lipschitz continuous mappings with Lipschitz constants $\lambda_g,\ \lambda_P,\ \lambda_f,\ \lambda_m$ and λ_h, respectively and g is strongly accretive with constant k satisfying $g(E)=E$. Let $J:E\to E^*$ is α-strongly monotone and Lipschitz continuous with constant λ_j. Let $\phi:E\times E\to R\cup\{+\infty\}$ be such that for each fixed $x\in E$, $\phi(.,x)$ is a lower semicontinuous subdifferentiable proper functional satisfying $g(x)-m(x)\in\ \mathrm{dom}\partial\phi(.,x)$ and J-proximal mapping of ϕ is a retraction. Suppose that there exists a constant $\rho>0$ such that for each $x,y\in E,\ x^*\in E^*$*

$$\|J_\rho^{\partial\phi(.,x)}(x^*)-J_\rho^{\partial\phi(.,y)}(x^*)\|\le\mu\|x-y\| \tag{3.3}$$

and the following conditions are satisfied

$$\lambda_j^2+(\lambda_g+\lambda_m)^2+\frac{\alpha^2\mu^2}{2}>\frac{\alpha^2(1-P^2)}{4},$$

$$0<\rho<\sqrt{\frac{4(\lambda_j^2+(\lambda_g+\lambda_m)^2)+2\alpha^2\mu^2-\alpha^2(1-P^2)}{4(\lambda_j\lambda_P\lambda_m+\lambda_j\lambda_f\lambda_s+\lambda_j\lambda_G\lambda_T)^2}}, \tag{3.4}$$

for

$$P=(1-2k+64C\lambda_g^2)^{1/2}+\lambda_m.$$

Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to $x,\ u,\ v$ and w, respectively and (x,u,v,w) is a solution of (GMCQVIP).

Proof. From Algorithm 3.1, we have

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&=\|x_n-g(x_n)+m(x_n)+J_\rho^{\partial\phi(.,x_n)}\{J(g(x_n)-m(x_n))-\rho(J(p(u_n)-(f(v_n)\\
&\qquad-h(w_n)))\}-[x_{n-1}-g(x_{n-1})+m(x_{n-1})+J_\rho^{\partial\phi(.,x_{n-1})}\{J(g(x_{n-1})\\
&\qquad\quad-m(x_{n-1}))-\rho(J(p(u_{n-1})-(f(v_{n-1})-h(w_{n-1})))]\}\|\\
&\qquad\le\|x_n-x_{n-1}-(g(x_n)-g(x_{n-1}))\|+\|m(x_n)-m(x_{n-1})\|\\
&\qquad+\|J_\rho^{\partial\phi(.,x_n)}\{J(g(x_n)-m(x_n))-\rho(J(p(u_n)-(f(v_n)-h(w_n)))\}
\end{aligned}$$

$$- J_\rho^{\partial\phi(.,x_{n-1})}\{J(g(x_{n-1}) - m(x_{n-1}))\| - \rho(J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))]\}\| . \quad (3.5)$$

By Proposition 2.1, we have (see, for example, the proof of [2, Theorem 3])

$$\|x_n - x_{n-1} - (g(x_n) - g(x_{n-1}))\|^2 \le (1 - 2k + 64C\lambda_g^2)\|x_n - x_{n-1}\|^2 . \quad (3.6)$$

By the Lipschitz continuity of m, we have

$$\|m(x_n) - m(x_{n-1})\| \le \lambda_m \|x_n - x_{n-1}\| . \quad (3.7)$$

Since $\|x + y\|^2 \le 2(\|x\|^2 + \|y\|^2)$, we have

$$\begin{aligned}
&\frac{1}{2}\|J_\rho^{\partial\phi(.,x_n)}\{J(g(x_n) - m(x_n)) - \rho(J(p(u_n) - (f(v_n) - h(w_n)))\} \\
&\quad - J_\rho^{\partial\phi(.,x_{n-1})}\{J(g(x_{n-1}) - m(x_{n-1})) - \rho(J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))\}\|^2 \\
&\qquad \le \|J_\rho^{\partial\phi(.,x_n)}\{J(g(x_n) - m(x_n)) - \rho(J(p(u_n) - (f(v_n) - h(w_n)))\} \\
&\quad - J_\rho^{\partial\phi(.,x_n)}\{J(g(x_{n-1}) - m(x_{n-1})) - \rho(J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))\}\|^2 \\
&\quad + \| J_\rho^{\partial\phi(.,x_n)}\{J(g(x_{n-1}) - m(x_{n-1})) - \rho(J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))\} \\
&\quad - J_\rho^{\partial\phi(.,x_{n-1})}\{J(g(x_{n-1}) - m(x_{n-1})) - \rho(J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))\}\|^2 \\
&\le \frac{1}{\alpha^2}\|J(g(x_n) - m(x_n)) - \rho(J(p(u_n) - (f(v_n) - h(w_n))) - \{J(g(x_{n-1}) - m(x_{n-1})) \\
&\qquad - \rho(J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))\}\|^2 + \mu\|x_n - x_{n-1}\|^2 \\
&\quad \le \frac{2}{\alpha^2}\|J(g(x_n) - m(x_n)) - J(g(x_{n-1}) - m(x_{n-1}))\|^2 + \frac{2\rho^2}{\alpha^2}\|J(p(u_n) - (f(v_n) \\
&\qquad - h(w_n))) - J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))\|^2 + \mu\|x_n - x_{n-1}\|^2 . \quad (3.8)
\end{aligned}$$

By the Lipschitz continuity of J, g and m, we have

$$\begin{aligned}
&\|J(g(x_n) - m(x_n)) - J(g(x_{n-1}) - m(x_{n-1}))\| \\
&\qquad \le \lambda_j(\|g(x_n) - g(x_{n-1})\| + \|m(x_n) - m(x_{n-1})\|) \\
&\quad \le \lambda_j(\lambda_g\|x_n - x_{n-1}\| + \lambda_m\|x_n - x_{n-1}\| = \lambda_j(\lambda_g + \lambda_m)\|x_n - x_{n-1}\| . \quad (3.9)
\end{aligned}$$

By the Lipschitz continuity of J, P, f, h, M, S and T, we have

$$\begin{aligned}
&\|J(p(u_n) - (f(v_n) - h(w_n))) - J(p(u_{n-1}) - (f(v_{n-1}) - h(w_{n-1})))\| \\
&\quad \le \lambda_j(\|P(u_n) - P(u_{n-1})\| + \|f(v_n) - f(v_{n-1})\| + \|h(w_n) - h(w_{n-1})\|) \\
&\qquad \le \lambda_j(\lambda_p\|u_n - u_{n-1}\| + \lambda_f\|v_n - v_{n-1}\| + \lambda_h\|w_n - w_{n-1}\|) \\
&\quad \le \lambda_j\lambda_p(1 + n^{-1})D(M(x_n, M(x_{n-1})) + \lambda_j\lambda_f(1 + n^{-1})D(S(x_n, S(x_{n-1}))
\end{aligned}$$

$$+\lambda_j\lambda_h(1+n^{-1})D(T(x_n,T(x_{n-1}))\le\lambda_j\lambda_p\lambda_M(1+n^{-1})\|x_n-x_{n-1}\|$$
$$+\lambda_j\lambda_f\lambda_S(1+n^{-1})\|x_n-x_{n-1}\|+\lambda_j\lambda_h\lambda_T(1+n^{-1})\|x_n-x_{n-1}\|$$
$$=[\lambda_j\lambda_p\lambda_M(1+n^{-1})+\lambda_j\lambda_f\lambda_S(1+n^{-1})$$
$$+\lambda_j\lambda_h\lambda_T(1+n^{-1})]\|x_n-x_{n-1}\|. \quad (3.10)$$

Combining (3.8) to (3.10), we have

$$\|J_\rho^{\partial\phi(.,x_n)}\{J(g(x_n)-m(x_n))-\rho(J(p(u_n)-(f(v_n)-h(w_n)))\}$$
$$-J_\rho^{\partial\phi(.,x_{n-1})}\{J(g(x_{n-1})-m(x_{n-1}))-\rho(J(p(u_{n-1})-(f(v_{n-1})-h(w_{n-1})))\}\|^2$$
$$\le[\frac{4}{\alpha^2}(\lambda_j^2(\lambda_g+\lambda_m)^2)+\frac{4\rho^2}{\alpha^2}(\lambda_j\lambda_p\lambda_M(1+n^{-1})+\lambda_j\lambda_f\lambda_S(1+n^{-1})$$
$$+\lambda_j\lambda_h\lambda_T(1+n^{-1})^2+2\mu^2]\|x_n-x_{n-1}\|^2. \quad (3.11)$$

From (3.5), (3.6), (3.7) and (3.11), it follows that

$$\|x_{n+1}-x_n\|\le(1-2k+64C\lambda_g^2)^{1/2}+\lambda_m+[\frac{4}{\alpha^2}(\lambda_j^2(\lambda_g+\lambda_m)^2)$$
$$+\frac{4\rho^2}{\alpha^2}\{\lambda_j\lambda_p\lambda_M(1+n^{-1})+\lambda_j\lambda_f\lambda_S(1+n^{-1})$$
$$+\lambda_j\lambda_h\lambda_T(1+n^{-1})\}^{1/2}+2\mu^2]^{1/2}\|x_n-x_{n-1}\|=\theta\|x_n-x_{n-1}\|\,, \quad (3.12)$$

where

$$\theta=(1-2k+64C\lambda_g^2)^{1/2}+\lambda_m+[\frac{4}{\alpha^2}(\lambda_j^2(\lambda_g+\lambda_m)^2)$$
$$+\frac{4\rho^2}{\alpha^2}(\lambda_j\lambda_p\lambda_M+\lambda_j\lambda_f\lambda_S+\lambda_j\lambda_h\lambda_T)^{1/2}+2\mu^2]^{1/2}.$$

From (3.4), we have $\theta<1$, and consequently $\{x_n\}$ is a cauchy sequence in E. Since E is a Banach space. There exists $x\in E$ such that $x_n\to x$ as $n\to\infty$. Since M, S and T are Lipschitz continuous mappings, it follows from (3.2) that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also cauchy sequences, we can assume that $u_n\to u$, $v_n\to v$ and $w_n\to w$ respectively. By Lipschitz continuity of $g,\ m,\ J,\ P,\ f,\ h,$ the condition (3.3) and Theorem 2.2, it follows that

$$g(x)=m(x)+J_\rho^{\partial\phi(.,x)}\{J(g(x)-m(x))-\rho(J(P(u)-(f(v)-h(w))))\}.$$

Now we will prove that $u\in M(x)$, $v\in S(x)$ and $w\in T(x)$. Infact, since $u_n\in M(x_n)$ and

$$d(u_n,M(x))\quad\le\quad\max\Big\{d(u_n,M(x)),\sup_{y\in M(x)}d(M(x_n),y)\Big\}$$

$$\begin{aligned} &\leq \max\left\{\sup_{z\in M(x)} d(z, M(x)), \sup_{y\in M(x)} d(M(x_n), y)\right\} \\ &= D(M(x_n), M(x)). \end{aligned}$$

We have

$$\begin{aligned} d(u_n, M(x)) &\leq \{\|u - u_n\| + d(u_n, M(x)) \leq \|u - u_n\| + D(M(x_n), M(x))\} \\ &\leq \|u - u_n\| + \lambda_M \|x_n - x\| \to 0, \text{ as } n \to \infty, \end{aligned}$$

which implies that $d(u, M(x)) = 0$. Since $M(x) \in CB(E)$, it follows that $u \in M(x)$. Similarly, we can prove that $v \in S(x)$ and $w \in T(x)$. By Theorem 3.1, (x, u, v, w) is a solution of (GMCQVIP). This completes the proof. □

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