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THE SUFFICIENT CONDITION FOR A CONVEX BODY TO ENCLOSE ANOTHER IN A REAL SPACE FORM

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Abstract: We follow Hadwiger and Ren's ideas (see [5], [6], and [7]) to estimate the kinematic measure of a convex body D_1 with C^2-boundary ∂D_1 moving inside another convex body D_0 with the same kind of boundary ∂D_0 under the isometry group G in a real space form $\tilde{M}^4(c)$ of the constant curvature c. By using Chern and Yen's (see [4]) kinematic fundamental formula and C.-S. Chen's (see [2]) kinematic formula for the total square mean curvature $\int_{\partial D_0 \cup_g \partial D_1} H^2 dv$ we obtain a sufficient condition to guarantee that one convex body can enclose another in $\tilde{M}^4(c)$.

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1. Introduction

Many mathematicians have been interested in getting sufficient conditions to insure that a given domain D_1 of surface area F_1, bounded by boundary ∂D_1, of volume V_1 can be contained in a second domain D_0 of surface area F_0, bounded by boundary ∂D_0, of volume V_0 in an n-dimensional Euclidean space R^n. Hadwiger (see [5], [6]) was first to use the method of integral geometry to obtain some sufficient conditions for one domain to contain another in the Euclidean plane R^2. Ren [7] derived another condtion in R^2. It is natural to try to generalize Hadwiger's Theorem to higher-dimensional Euclidean space R^n ($n \geq 3$) ([8], [9] and [10]). Even restricted results, with some strong conditions placed on the two domains involved (for example, convexity and some topological con-

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ditions), were not available. Hadwiger and Ren used Blaschke's and Poincaré's formulas to estimate the kinematic measure of one domain moving into another under rigid motions in R^2. However, there is no general Poincaré formula in R^n ($n \geq 3$) which can be used in our discussion. There is general (or generalized) Blaschke's formula in R^n, i.e., Chern and Yen's kinematic fundamental formula. Our strategy is to find another formula to substitute for Poincaré's formula, upon which Hadwiger and Ren's results in R^2 are based. There are some different extensions of Poincaré's formula, but none of them are applicable for the situation of R^n.

In this paper we restrict our discussion to a real space form $\tilde{M}^4(c)$ of constant curvature c. There the situation is very concrete, and there are some well-known results we can use for the 2-dimensional intersection manifold $\partial D_0 \cap_g \partial D_1$ of two boundaries ∂D_0 and ∂D_1. We use Chern and Yen's kinematic fundamental formula and C.-S. Chen's gereral formulas [1] to substitute for Blaschke and Poincaré's formulas and obtain a sufficient condition for $\tilde{M}^4(c)$. Our work is a direct application for the ambient space of Chern and Yen's formula.

2. Preliminaries

Let M^p, M^q be two submanifolds in a homogeneours space G/H and I an invariant of the intersection submanifold $M^p \cap_g M^q$. Let dg be suitably normalized kinematic density of G. Evaluating the integral

$$\int_{\{g \in G: M^p \cap_g M^q \neq \emptyset\}} I(M^p \cap_g M^q) dg \tag{1}$$

in terms of invariants of M^p and M^q we obtain the so-called *kinematic formula*.

Let the boundary ∂D of a convex body D be a hypersurface of class C^2. It is known that at each point of a hypersurface Σ in $\tilde{M}^n(c)$ there are $n-1$ principal directions and $n-1$ principal curvatures $\kappa_1, ..., \kappa_{n-1}$. If dv denotes the area element of Σ, then the *r-th integral of mean curvature* is defined by

$$M_r(\Sigma) = \begin{pmatrix} n-1 \\ r \end{pmatrix}^{-1} \int_\Sigma \{\kappa_{i_1}, ..., \kappa_{i_r}\} dv, \tag{2}$$

where $\{\kappa_{i_1}, ..., \kappa_{i_r}\}$ denotes the *r-th elementary symmetric function of the principal curvatures.* In particular, M_0 is the area, M_{n-1} is a numerical multiple of the degree of mapping of Σ into the unit hypersphere defined by the normal field and M_1 is the total mean curvature of Σ. Let D_0 and D_1 be two convex bodies in $\tilde{M}^n(c)$ bounded by the hypersurfaces ∂D_0 and ∂D_1, which we assumed to be of class C^2. $(M^0)_i$, $(M^1)_i$ are the i-th integrals of mean curvature of ∂D_0 and ∂D_1, respectively. For simplicity we will denote these by M_i^0 and M_i^1. Chern and Yen's kinematic fundamental formula is

$$\int_{\{g: D_0 \cap_g D_1 \neq \emptyset\}} dg = O_{n-2} \cdots O_1 \left[O_{n-1}(V_0 + V_1) + \frac{1}{n} \sum_{h+1}^{n-1} \binom{n}{h+1} M_h^0 M_{n-2-h}^1 \right]. \quad (3)$$

Let M be an 2-dimensional closed surface in an 4-dimensional real space form $\tilde{M}^4(c)$, H the mean curvature of M. Then we have B.-Y. Chen's formula (see [2], [3], p. 207, Corollary 3.2, and p. 247, Theorem 7.1)

$$\int_M H^2 dv \geq 4\pi - c\mathrm{Vol}\,(M). \quad (4)$$

3. Main Theorem

Theorem 1. *Let $D_i (i = 0, 1)$ be convex bodies in 4-dimensional real space form $\tilde{M}^4(c)$, with C^2 boundaries ∂D_i, and let $V_i, F_i, M_r^i, \tilde{H}_i, \tilde{R}_i^c$ be the volume of D_i, the surface area of D_i, the rth integral of mean curvature of ∂D_i, the total square mean curvature, and integral of scalar curvature of ∂D_i, respcetively. H_i and R_i represent, respectively, the mean curvature and scalar curvature of ∂D_i. Then a sufficient condition for D_0 to contain D_i or for D_1 to contain D_0 is*

$$8\pi^2[2\pi^2(V_0 + V_1) + F_0 M_2^1 + F_1 M_2^0 + \frac{3}{2} M_1^0 M_1^1] - \frac{64\pi^3}{15(4\pi - c\mathrm{Vol}\,(M))}[(18\tilde{H}_0 - \tilde{R}_0^c)F_1 + (18\tilde{H}_1 - \tilde{R}_1^c)F_0] > 0. \quad (5)$$

Moreover:

(1) *if $V_1 \geq V_0$, then D_1 can contain D_0;*

(2) *if $V_1 \leq V_0$, then D_1 can be contained in D_0.*

This formula comes from estimating the kinematic measure of the set of rigid motions which move one convex body inside another in $\tilde{M}^4(c)$, i.e.,

$$m\{g \in G : gD_1 \subseteq D_0 \mathrm{or} gD_0 \subseteq D_1\} = \int_{\{g: D_0 \cap_g D_1 \neq \emptyset\}} dg - \int_{\{g: \partial D_0 \cap_g \partial D_1 \neq \emptyset\}} dg$$
$$\geq 8\pi^2[2\pi^2(V_0 + V_1) + F_0 M_2^1 + F_1 M_2^0 + \frac{3}{2} M_1^0 M_1^1] - \frac{64\pi^3}{15(4\pi - c\mathrm{Vol}\,(M))}[(18\tilde{H}_0 - \tilde{R}_0^c)F_1 + (18\tilde{H}_1 - \tilde{R}_1^c)F_0]. \quad (6)$$

4. The Proof of the Main Result

First, we estimate the integral

$$\Phi = \int_{\{g:\partial D_0 \cap_g \partial D_1 \neq \emptyset\}} dg. \tag{7}$$

For each $g \in G$, the intersection $\partial D_0 \cap_g \partial D_1$ may be composed of several connected components, i.e., $\partial D_0 \cap_g \partial D_1 = \sum_{i=1}^{N_g} M_i$, where each M_i is a connected closed surface and N_g is always finite and only depends on g. To our generic 2-dimensional submanifold $\partial D_0 \cap_g \partial D_1$ we have B.-Y. Chen's formula (4)

$$\int_M H^2 dv \geq 4\pi - c\mathrm{Vol}\,(M). \tag{4}$$

By (4) we have

$$\begin{aligned}(4\pi - c\mathrm{Vol}\,(M)) \int_{\{g:\partial D_0 \cap_g \partial D_1 \neq \emptyset\}} dg &\leq \int_{\{g:\partial D_0 \cap_g \partial D_1 \neq \emptyset\}} (\int_{\partial D_0 \cap_g \partial D_1} H^2 dv) dg \\ &= \frac{64\pi^3}{15}[(18\tilde{H}_0 - \tilde{R}_0^c)F_1 + (18\tilde{H}_1 - \tilde{R}_1^c)F_0] > 0\,, \end{aligned} \tag{8}$$

i.e.,

$$\begin{aligned}\Phi &= \int_{\{g:\partial D_0 \cap_g \partial D_1 \neq \emptyset\}} dg \\ &\leq \frac{64\pi^3}{15(4\pi - c\mathrm{Vol}\,(M))}[(18\tilde{H}_0 - \tilde{R}_0^c)F_1 + (18\tilde{H}_1 - \tilde{R}_1^c)F_0]. \end{aligned} \tag{9}$$

Using Chern and Yen's formula (3) and (9) we obtain (6):

$$\begin{aligned} m\{g \in G : gD_1 \subseteq D_0 \mathrm{or} gD_0 \subseteq D_1\} &= \int_{\{g:D_0 \cap_g D_1 \neq \emptyset\}} dg - \int_{\{g:\partial D_0 \cap_g \partial D_1 \neq \emptyset\}} dg \\ &\geq 8\pi^2[2\pi^2(V_0 + V_1) + F_0 M_2^1 + F_1 M_2^0 + \frac{3}{2} M_1^0 M_1^1] \\ &\quad - \frac{64\pi^3}{15(4\pi - c\mathrm{Vol}\,(M))}[(18\tilde{H}_0 - \tilde{R}_0^c)F_1 + (18\tilde{H}_1 - \tilde{R}_1^c)F_0]. \end{aligned}$$

5. Remarks

Let D_0 and D_1 be two domains in $\tilde{M}^4(c)$ bounded by the hypersurfaces ∂D_0 and ∂D_1, which we assume to be of class C^2. Denote by $\chi(\cdot)$ the *Euler-Poincaré* characteristic. We have Chern and Yen's kinematic formula

$$\int_{\{g:D_0 \cap_g D_1 \neq \emptyset\}} \chi(D_0 \cap_g D_1) dg = O_{n-2} \cdots O_1$$

$$\left[O_{n-1}(\chi(D_0)V_1 + \chi(D_1)V_0) + \frac{1}{n}\sum_{h+1}^{n-1}\left(\begin{array}{c} n \\ h+1 \end{array}\right) M_h^0 M_{n-2-h}^1\right]. \quad (10)$$

Then we have:

Theorem 2. *Let $D_i (i = 0, 1)$ be two domains in 4-dimensional real space form $\tilde{M}^4(c)$, bounded by the C^2-somooth boundaries ∂D_i. Suppose that $V_i, F_i, M_r^i, \tilde{H}_i, \tilde{R}_i^c$ are as in Theorem 1. Denote by $\chi(\cdot)$ the Euler-Poincaré characteristic. Moreover, assume that for all rigid mortion $g \in G$, in $\tilde{M}(c)$, $\chi(D_0 \cap_g D_1) \le n_0$, a finite integer. Then a sufficient condition for D_1 to enclose, or to be enclosed in, D_0* is

$$2\pi^2(\chi(D_0)V_1 + \chi(D_1)V_0) + F_0M_2^1 + F_1M_2^0 + \frac{3}{2}M_1^0M_1^1 - \frac{2n_0}{15}[(18\tilde{H}_0 - \tilde{R}_0^c)F_1 + (18\tilde{H}_1 - \tilde{R}_1^c)F_0] > 0. \quad (11)$$

Moreover:

(1) *if $V_1 \ge V_0$, then D_1 can contain D_0;*

(2) *if $V_1 \le V_0$, then D_1 can be contained in D_0.*

(11) comes from estimating the kinematic measure of one domain moving into another under the rigid motions in $\tilde{M}(c)$, i.e.,

$$\begin{aligned} m\{g \in G : gD_1 \subseteq D_0 \text{or} gD_0 \subseteq D_1\} &= \int_{\{g: D_0 \cap_g D_1 \neq \emptyset\}} dg - \int_{\{g: \partial D_0 \cap_g \partial D_1 \neq \emptyset\}} dg \\ &\ge \frac{8\pi^2}{n_0}[2\pi^2(V_0 + V_1) + F_0M_2^1 + F_1M_2^0 + \frac{3}{2}M_1^0M_1^1] \\ &\quad - \frac{16\pi^2}{15}[(18\tilde{H}_0 - \tilde{R}_0^c)F_1 + (18\tilde{H}_1 - \tilde{R}_1^c)F_0]. \quad (12) \end{aligned}$$

If D_0 and D_1 are convex bodies, we have $\chi(D_0) = \chi(D_1) = \chi(D_0 \cap_g D_1) = 1$, $n_0 = \frac{4\pi}{4\pi - c\mathrm{Vol}(M)}$ and Theorem 2 becomes Theorem 1.

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