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L^1-STABILITY OF CONSTANTS IN MULTI-DIMENSIONAL CONSERVATION LAWS WITH VISCOSITY

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Abstract: We study a class of multi-dimensional scalar viscous conservation laws. The solution of the Cauchy problem satisfies comparison principles, L^1-contraction property and preserves the total mass. After proving L^p-contraction ($p > 1$) of the solution and an L^2-decay estimate for the gradient, we state the L^1-stability of constant states, in the case where the initial disturbance of the initial value with respect to the fixed constant belongs to L^1 and has zero mass. Similarly to in one-dimensional problems, the L^1-stability of constants implies the L^1-stability of shock waves.

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1. Introduction

In the present paper we are interested in studying the following class of viscous conservation laws:

$$\partial_t u + \nabla_x \cdot (f(u)) = \nabla_x \cdot (B(u)\nabla_x u), \tag{1}$$

where $x \in \mathbf{R}^d$, $t > 0$; the unknown function u is defined in $\mathbf{R}^d \times (0, \infty)$ with values in $\mathbf{R}$. The non-linear convection f, is a given C^1-function $f : \mathbf{R} \longrightarrow \mathbf{R}^d$, which satisfies the assumption $f(0) = 0$; the dissipation term in the conservation law (1) is represented by the smooth function $B : \mathbf{R} \longrightarrow \mathbf{R}$. We assume that

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the second order operator $\nabla_x \cdot (B(u)\nabla_x u)$, is strictly elliptic, i.e. there exists a positive real constant δ in such a way that $B(\cdot) \geq \delta$.

We shall consider the Cauchy problem for the equation (1), with initial values in $L^1(\mathbf{R}^d)$. Since the functions f and B are assumed to be regular, the initial value problem with initial datum in $L^1(\mathbf{R}^d)$ is well-posed. In addition, it holds true that the solution of problem (1) satisfies comparison principles, contraction property in L^1-norm, and preserves the total mass (see [8]).

In this paper, we shall study the stability of the constant zero with respect to the L^1-distance, under the assumption that the initial disturbance has zero-mass and belongs to $L^1(\mathbf{R}^d)$.

Let c be a real constant and let us consider the solution to the Cauchy problem (1), with initial datum u_0. The constant state c is said to be L^1-stable if $\lim_{t\to+\infty} \|u(\cdot,t) - c\|_{L^1(\mathbf{R}^d)} = 0$. Since the mass is preserved, a necessary condition for stability is a zero-mass initial disturbance, i.e. $\int_{\mathbf{R}^d}(u_0(x)-c)dx = 0$.

Thanks to the properties satisfied by the semi-group of solutions of (1), we shall carry out the study of the L^1-stability of constants for multi-dimensional conservation laws, being the problem already worked out in [2], in the case where $d = 1$.

Concerning one-dimensional conservation laws with a linear diffusion operator, it has been proved by H. Freistühler and D. Serre in [3] and by D. Serre in [5], [7] and [8] that constants are L^1-stable: the main result of [3] tells us that in the case where $d = 1$, and $B = 1$, given a real constant c and an initial value $u_0 \in c + L^1(\mathbf{R}) \bigcap L^\infty(\mathbf{R})$, with $\int_{\mathbf{R}^d}(u_0(x) - c)dx = 0$, the solution u satisfies $\lim_{t\to+\infty} \|u(\cdot,t) - c\|_{L^1(\mathbf{R})} = 0$.

The interest in previous problem relies on a general argument proved in [3], which establishes that the L^1-stability of constants allows to give an answer to the question of the L^1-stability of shock waves. A planar shock is a bounded travelling wave solution of the form $u(x,t) = U(x\cdot\nu - st)$, having finite limit on each side $U(\pm\infty) = U_\pm$. The real constant s represents the velocity, the unit vector ν the direction of propagation. In [3] and [8] as well as in [7] for a model of radiating gases, it has been proved that, given a standing shock U and an initial value $u_0 \in U + L^1(\mathbf{R})$, with $\int_{\mathbf{R}}(u_0(x) - U)dx = 0$, the L^1-stability of constant states implies the L^1-stability of the travelling wave solution U.

Taking into account the results achieved for one-dimensional viscous conservation laws, we shall prove that the constants are L^1-stable also in the multi-dimensional models (1). Under the assumption $u_0 \in L^1(\ R)$, we prove L^p-contraction properties of the solution u, for every $p \geq 1$ and L^2-decay estimate for the gradient of u. Next, by means of Sobolev's inequality, we prove the main result on the L^1-stability of constant states. As a consequence of this result, similarly to in the one-dimensional case, we state the L^1-stability of standing shock waves in the x_1-direction.

2. L^1-Stability

The present section is concerned with the study of the L^1-stability of constant states. After proving L^p-contraction properties of the solution u, with $p \geq 1$, and L^2-decay estimate for the gradient of u, we shall discuss the L^1-stability of constant states. As explained in Introduction, the Cauchy problem for the viscous model (1) with initial value $u_0 \in L^1(\ R)$, is well-posed. The associated semi-group enjoys the following properties: conservation of mass, L^1-contraction and maximum principles.

As far as the proof of the L^1-contraction property for the solution of (1) is concerned, let us multiply by $\operatorname{sgn} u$ both sides of the equation (1) and integrate over $\mathbf{R}^d$

$$\frac{d}{dt}\int_{\mathbf{R}^d} |u|dx + \int_{\mathbf{R}^d} \nabla_x \cdot (f(u)\operatorname{sgn} u)dx = \int_{\mathbf{R}^d} \nabla_x \cdot (B(u)\nabla_x u)\operatorname{sgn} u dx.$$

If we approximate sgn$u$ by means of a sequence of monotone non-decreasing C^1-functions, we obtain $\frac{d}{dt}\int_{\mathbf{R}^d} |u|dx \leq 0$.

Moreover, since $B(\cdot) \geq \delta$, the second-order operator $\nabla_x \cdot (B(u)\nabla_x u)$, turns out to be elliptic and according to the results of [6], the equation (1) satisfies maximum principles.

et us prove now the following proposition.

Proposition 2.1. *If u is the solution of the Cauchy problem (1) with initial value $u_0 \in L^1(\mathbf{R}^d)\bigcap L^\infty(\mathbf{R}^d)$, then for every $p > 1$ the following results hold true:*

(i) the functions $t \longrightarrow \|u(\cdot,t)\|_{L^p(\mathbf{R}^d)}$, are monotone non-increasing;

(ii) the functions $t \longrightarrow \displaystyle\int_{\mathbf{R}^d} B(u)|\nabla_x u|^2 \frac{d}{du}(|u|^{p-1}\operatorname{sgn} u)dx$, are integrable on the interval $(0,\infty)$.

Proof. Let us fix $p > 1$ and denote by g the function $g : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ defined as follows

$$g(u,z) = -|u|^{p-1}\operatorname{sgn} u B(u)z + f(u)|u|^{p-1}\operatorname{sgn} u - \int_0^u \frac{d}{ds}(|s|^{p-1}\operatorname{sgn} s)f(s)ds. \quad (2)$$

We derive the following identity

$$\partial_t \frac{|u|^p}{p} + \nabla_x \cdot (g(u, \nabla_x u)) = -B(u)\nabla_x u \nabla_x (|u|^{p-1}\operatorname{sgn} u). \tag{3}$$

By integrating on $\mathbf{R}^d$, we get

$$\frac{d}{dt}\int_{\mathbf{R}^d} \frac{|u|^p}{p} dx = -\int_{\mathbf{R}^d} B(u)|\nabla_x u|^2 \frac{d}{du}(|u|^{p-1}\operatorname{sgn} u) dx. \tag{4}$$

Since $u_0 \in L^1(\mathbf{R}^d) \bigcap L^\infty(\mathbf{R}^d)$, we deduce the following identity

$$\int_{\mathbf{R}^d} \frac{|u|^p}{p} dx = \int_{\mathbf{R}^d} \frac{|u_0|^p}{p} dx - \int_0^t \int_{\mathbf{R}^d} B(u)|\nabla_x u|^2 \frac{d}{du}(|u|^{p-1}\operatorname{sgn} u) dx d\tau. \tag{5}$$

Thanks to the conditions fulfilled by the function B, we derive the properties (i) and (ii). ☐

Let p be an even natural number. Hence the identity (4) becomes

$$\frac{d}{dt}\int_{\mathbf{R}^d} \frac{|u|^p}{p} dx = -\int_{\mathbf{R}^d} B(u)|\nabla_x u|^2 (p-1) u^{p-2} dx. \tag{6}$$

In the case where p is odd, we obtain

$$\frac{d}{dt}\int_{\mathbf{R}^d} \frac{|u|^p}{p} dx = -\int_{\mathbf{R}^d} B(u)|\nabla_x u|^2 (p-1) u^{p-2} \operatorname{sgn} u dx. \tag{7}$$

As explained above, the conservation of mass yields a necessary condition for stability in L^1-norm. Thus we suppose that the mass of the initial disturbance with respect to the constant state zero is null: $\int_{\mathbf{R}^d} u_0(x)dx = 0$.

By taking into account the previous assumptions, we prove now the main result in the case where $d \geq 2$. The one-dimensional problem has been studied in [2].

Theorem 2.1. *Consider the solution u of the solution of the Cauchy problem (1) with initial value u_0. If $u_0 \in L^1(\mathbf{R}^d)$ and $\int_{\mathbf{R}^d} u_0(x)dx = 0$, then the constant state zero turns out to be stable with respect to the L^1-distance*

$$\lim_{t \to +\infty} \|u(\cdot, t)\|_{L^1(\mathbf{R}^d)} = 0. \tag{8}$$

Proof. In a first instance, we suppose that $u_0 \in L^1(\mathbf{R}^d) \bigcap L^\infty(\mathbf{R}^d)$.

Let us consider the identity (6) proved in the previous proposition, in the case where $p=2$. Since the function $t \to \int_{\mathbf{R}^d} B(u)|\nabla_x u|^2 dx$ is integrable on $(0,\infty)$, there exists a sequence $(t_j)_{j\in\mathbf{N}}$, such that

$$\lim_{t_j\to+\infty}\int_{\mathbf{R}^d} B(u(x,t_j))|\nabla_x u(x,t_j)|^2 dx = 0. \tag{9}$$

Moreover, due to the comparison properties,

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^d)} \le \|u_0(\cdot)\|_{L^\infty(\mathbf{R}^d)},$$

for every $t>0$. Thus

$$\int_{\mathbf{R}^d} B(u(x,t_j))|\nabla_x u(x,t_j)|^2|u(x,t_j)|dx \\ \le \|u_0\|_{L^\infty(\mathbf{R}^d)}\int_{\mathbf{R}^d} B(u(x,t_j))|\nabla_x u(x,t_j)|^2 dx. \tag{10}$$

Let us focus on the sequence $\nabla_x u(\cdot,t_j)$: if $\lim_{t_j\to\infty}\|\nabla_x u(\cdot,t_j)\|_{L^\infty(\mathbf{R}^d)}$, is not zero, then there exists a positive real constant ρ and a subsequence $(t_{j_n})_{n\in\mathbf{N}}$, in such a way that

$$\delta\rho^2\int_{\mathbf{R}^d}|u(x,t_{j_n})|dx \le \int_{\mathbf{R}^d} B(u(x,t_{j_n}))|\nabla_x u(x,t_{j_n})|^2|u(x,t_{j_n})|dx. \tag{11}$$

In view of (10) and (11), we get $\lim_{t_{j_n}\to+\infty}\|u(\cdot,t_{j_n})\|_{L^1(\mathbf{R}^d)} = 0$. Hence the L^1-contraction property yields the limit relation (8).

Let us discuss now the case where $\lim_{t_j\to\infty}\|\nabla_x u(\cdot,t_j)\|_{L^\infty(\mathbf{R}^d)} = 0$.

Thanks to the identity (6), we deduce $\lim_{t_j\to\infty}\|\nabla_x u(\cdot,t_j)\|_{L^2(\mathbf{R}^d)} = 0$, and $\|u(\cdot,t_j)\|_{L^2(\mathbf{R}^d)} \le \|u_0(\cdot)\|_{L^2(\mathbf{R}^d)}$.

We discuss first the case where $d>2$: due to Sobolev's inequality, we obtain

$$\|u(\cdot,t_j)\|_{L^{2^*}(\mathbf{R}^d)} \le C(d,2)\|\nabla_x u(\cdot,t_j)\|_{L^2(\mathbf{R}^d)}, \tag{12}$$

where $2^* = \frac{2d}{d-2}$. Therefore $\lim_{t_j\to\infty}\|u(\cdot,t_j)\|_{L^{2^*}(\mathbf{R}^d)} = 0$. Let us fix $r > \max(2^*,\, d)$ and let q be a natural number, such that $2^* < r < q$.

As a consequence of (6) and (7) with $p=q$, we deduce $u(\cdot,t)\in L^q(\mathbf{R}^d)$, for every $t>0$. By interpolation, we obtain $\lim_{t_j\to\infty}\|u(\cdot,t_j)\|_{L^r(\mathbf{R}^d)} = 0$.

Furthermore,

$$\int_{\mathbf{R}^d}|\nabla_x u(x,t_j)|^r dx \le \|\nabla_x u(x,t_j)\|^{r-2}_{L^\infty(\mathbf{R}^d)}\int_{\mathbf{R}^d}|\nabla_x u(x,t_j)|^2 dx.$$

Thus $\lim_{t_j\to\infty}\|u(\cdot,t_j)\|_{W^{1,r}(\mathbf{R}^d)}=0$. By means of Morrey's Theorem,

$$\lim_{t_j\to\infty}\|u(\cdot,t_j)\|_{L^\infty(\mathbf{R}^d)}=0\,. \tag{13}$$

Let us establish now the limit relation (11) in the case where $d=2$. In that case the following inequality holds true

$$\|u(\cdot,t_j)\|^{2\alpha}_{L^{2\alpha}(\mathbf{R}^d)}\le\alpha\|u(\cdot,t_j)\|^{2(\alpha-1)}_{L^{2(\alpha-1)}(\mathbf{R}^d)}\|\nabla_x u(\cdot,t_j)\|^2_{L^2(\mathbf{R}^d)}, \tag{14}$$

where $\alpha>1$. After setting $\alpha=2$, from previous inequality we obtain

$$\lim_{t_j\to\infty}\|u(\cdot,t_j)\|_{L^4(\mathbf{R}^d)}=0.$$

Thus we recover the limit relation (11) by means of Morrey's Theorem.

Similarly to in the one-dimensional case studied in [2], owing to the L^1-contraction property satisfied by the semi-group of the solutions of (1), we obtain the following relations

$$\begin{aligned}&\|u(x,t_j)\|_{L^1(\mathbf{R}^d)}\le\|u_0(x)\|_{L^1(\mathbf{R}^d)}\,,\\&\|u(x+y,t_j)-u(x,t_j)\|_{L^1(\mathbf{R}^d)}\le\|u_0(x+y)-u_0(x)\|_{L^1(\mathbf{R}^d)}\,,\end{aligned} \tag{15}$$

for every $y\in\mathbf{R}^d$.

Thus thanks to the Riesz-Frchet-Kolmogorov Theorem, the sequence of functions $(u(\cdot,t_j))_{t_j}$ turns out to be relatively compact in $L^1(\mathbf{R}^d)$. Due to the result proved above, $\lim_{t_j\to\infty}\|u(\cdot,t_j)\|_{L^\infty(\mathbf{R}^d)}=0$. Therefore

$$\lim_{t_j\to\infty}\|u(\cdot,t_j)\|_{L^1(\mathbf{R}^d)}=0.$$

By means of the L^1-contraction property, we deduce $\lim_{t\to\infty}\|u(\cdot,t)\|_{L^1(\mathbf{R}^d)}=0$.

Through a density argument we intend to remove now the assumption $u_0\in L^\infty(\mathbf{R}^d)$.

Let us consider a sequence of functions $u_{0n}\in L^1(\mathbf{R}^d)\bigcap L^\infty(\mathbf{R}^d)$ such that $\lim_{n\to\infty}\|u_{0n}-u_0\|_{L^1(\mathbf{R}^d)}=0$, and $\int_{\mathbf{R}^d}u_{0n}(x)dx=0$. Let $u_n=u_n(x,t)$ be the solution to the Cauchy problem (1) with initial value u_{0n}.

Because of the L^1-contraction property, we get $\|u_n(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R}^d)}\le\|u_{0n}-u_0\|_{L^1(\mathbf{R}^d)}$. Thanks to the results proved above, it holds true that $\lim_{t\to\infty}\|u_n(\cdot,t)\|_{L^1(\mathbf{R}^d)}=0$. Hence, the proof of the Theorem is completed. □

Remark 2.1. If c is a real constant, then $\frac{d}{dt}\int_{\mathbf{R}^d}|u-c|dx\le0$. Besides, if the function $u_0-c\in L^1(\mathbf{R}^d)$ and $\int_{\mathbf{R}^d}(u_0(x)-c)dx=0$, then $\lim_{t\to\infty}\|u(\cdot,t)-c\|_{L^1(\mathbf{R}^d)}=0$. The result can be proved as in the case where $c=0$.

As well as in one-dimensional problems treated in [3] and [8], the L^1-stability of constants implies the multi-dimensional stability of shock waves.

Let us consider a travelling wave solution U to (1) in the x_1-direction. Up to the choice of a moving frame, assume that U is steady.

Thus $U(x,t) = U(x_1)$. Let U_- and U_+, with $U_- < U_+$, be the lower and upper bounds of the monotone function U. We fix a general initial value u_0, with $u_0 - U \in L^1(\mathbf{R}^d)$ and $\int_{\mathbf{R}^d}(u_0(x) - U(x_1))dx = 0$.

In accordance with [8], there exist two functions $u_{0\pm}$, such that $u_{0\pm} - U_\pm \in L^1(\mathbf{R}^d)$, $u_{0-} \le u_0 \le u_{0+}$, and $\int_{\mathbf{R}^d}(u_{0\pm}(x) - U_\pm)dx = 0$.

Let $u_\pm$ be the solutions of the Cauchy problems (1) with initial data $u_{0\pm}$. Due to the results proved above, $\lim_{t\to+\infty} \|u_\pm(\cdot,t) - U_\pm\|_{L^1(\mathbf{R}^d)} = 0$. Moreover, thanks to the maximum principle, $u_- \le u \le u_+$, and $\|u - U\|_{L^1(\mathbf{R}^d)} \le \|u_- - U_-\|_{L^1(\mathbf{R}^d)} + \|u_+ - U_+\|_{L^1(\mathbf{R}^d)}$. Therefore we obtain the following result.

Corollary 2.1. *Let U be a steady travelling wave solution of (1) in the x_1-direction, i.e. $U(x,t) = U(x_1)$. Let us consider the solution u of the Cauchy problem (1) with initial value u_0. If $u_0 - U \in L^1(\mathbf{R}^d)$ and $\int_{\mathbf{R}^d}(u_0(x) - U(x_1))dx = 0$, then $\lim_{t\to+\infty} \|u(\cdot,t) - U\|_{L^1(\mathbf{R}^d)} = 0$.*

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