International Journal of Pure and Applied Mathematics
Volume 37 No. 4 2007, 507-511

THE HENSTOCK INTEGRAL OF SET-VALUED FUNCTIONS

She Xiang Hai[1 §], Fang Di Kong[2], Jin Shu Chen[3]
[1,2,3]School of Science
Lanzhou University of Technology
Lanzhou, Gansu, 730050, P.R. CHINA
[1]e-mail: haishexiang@lut.cn

Abstract: In this paper, the Henstock integral of set-valued functions is defined and discussed. Furthermore, the Henstock integral of set-valued functions is characterized by the real functions.

AMS Subject Classification: 28B20, 26E25, 54C60, 54C65, 90A14
Key Words: set-valued functions, the Henstock integral of set-valued functions, support functions $\sigma(x, F(t))$ of set

1. Introduction

Set-valued functions have been used repeatedly in economics. Furthermore, integrals of set-valued functions have been studied in connection with statistical problems. In this paper the Henstock integral of set-valued functions is defined and discussed. A necessary and sufficient condition of (HI) integrability for set-valued functions is given by means of real functions.

2. Preliminary

In this paper the symbol $T = [a, b]$ denotes a compact interval on R.

The set of intervals-point $\{(\xi_1, T_1), (\xi_2, T_2), \cdots, (\xi_k, T_k)\}$ is called a division of T that is $\xi_1, \xi_2, \cdots, \xi_k \in T$, intervals $T_1, T_2, \cdots, T_k$ are non-intersect and $\bigcup_{i=1}^{k} T_i = T$. Marking the division of T as $\Pi = \{(\xi_1, T_1), (\xi_2, T_2), \cdots, (\xi_k, T_k)\}$, shortening as $\Pi = \{\xi, [u, v]\}$, see [5].

Received: March 13, 2007
§Correspondence author

Definition 2.1. (see [5], [4]) Let Π be a division of T, if there exists function $\delta > 0$ on T, such that $\xi_i \in T_i \subset \big(\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)\big)$ $(i = 1, 2, \cdots, k)$, Π is said to be a δ-fine division of T.

Definition 2.2. (see [5]) Let $F : T \to R^n$, $F(t)$ is said to be Henstock integrable on T if there exists $a \in R^n$, for every $\varepsilon > 0$, there is a function $\delta(t) > 0$, such that for any δ-fine division $\Pi = \{\xi, [u, v]\}$ on T, we have

$$\|\sum_{(\Pi)} F(\xi)(v-u) - a\| < \varepsilon\,.$$

We write $(HR)\int_T F(t)dt = a$, here $\|\cdot\|$ stands for the norm of R^n.

The symbol $P_k(R^n)$ denotes the family of all nonempty compact convex subsets of R^n, define the addition and scalar multiplication in $P_k(R^n)$ as following, for $A, B \in P_k(R^n)$, $a \in R$,

$$A + B = \big\{x + y \mid x \in A, y \in B\big\}, \qquad aA = \big\{ax \mid x \in A\big\}\,.$$

For every $A, B \in P_k(R^n)$, define the Hausdorff metric of A and B by the equation, see [7],

$$d(A, B) = \max\big\{\sup_{a\in A}\inf_{b\in B}\|a - b\|, \sup_{b\in B}\inf_{a\in A}\|b - a\|\big\}\,.$$

Definition 2.3. For $A \in P_k(R^n), x \in S^{n-1}$, define the support function of A as $\sigma(x, A) = \sup_{y\in A}\langle y, x\rangle$, where S^{n-1} is the unit sphere of R^n, i.e., $S^{n-1} = \{x \in R^n : \|x\| = 1\}$, $\langle\cdot,\cdot\rangle$ is the inner product in R^n.

Remark 2.1. It is clear, for $A, B \in P_k(R^n), x \in S^{n-1}$: (1) if $k \geq 0$, $\sigma(x, kA) = k\sigma(x, A)$; (2) $\sigma(x, A + B) = \sigma(x, A) + \sigma(x, B)$.

Lemma 2.1. (see [3]) *If $A \in P_k(R^n), x \in S^{n-1}$, then $A = \big\{y \in R^n | \langle y, x\rangle \leq \sigma(x, A), x \in S^{n-1}\big\}$.*

Lemma 2.2. (see [3]) *If $A, B \in P_k(R^n)$, then $d(A, B) = \sup_{x\in S^{n-1}}\big|\sigma(x, A) - \sigma(x, B)\big|$.*

3. The Main Results

Definition 3.1. The set-valued function $F : T \to P_k(R^n)$ is said to be Henstock integrable on T, if there exists $I \in P_k(R^n)$, for every $\varepsilon > 0$, there is a function $\delta(t) > 0$, such that for any δ-fine division $\Pi = \{\xi, [u, v]\}$ on T, we have

$$d\big(\sum_{(\Pi)} F(\xi)(v-u), I\big) < \varepsilon\,.$$

Write $(HI)\int_T F(t)dt = I$.

Remark 3.1. If the set-valued function $F : T \to P_k(R^n)$ is (HI) integrable on T, then the integral value is unique.

Theorem 3.1. *The set-valued function $F : T \to P_k(R^n)$ is (HI) integrable on T if and only if $\sigma\big(x, F(t)\big)$ is (HR) integrable on T uniformly for $x \in S^{n-1}$, and for any $x \in S^{n-1}$, we have*

$$\sigma\big(x, (HI)\int_T F(t)dt\big) = (HR)\int_T \sigma\big(x, F(t)\big)dt\,.$$

Proof. Since set-valued function $F : T \to P_k(R^n)$ is (HI) integrable on T, then there exists $I \in P_k(R^n)$, given $\varepsilon > 0$, there exists $\delta(t) > 0$, for any $\delta-$ fine division $\Pi = \{\xi, [u, v]\}$ on T, we have

$$d\big(\sum_{(\Pi)} F(\xi)(v-u), I\big) < \varepsilon\,.$$

By Remark 2.1 and Lemma 2.2,

$$\begin{aligned}\sup_{x\in S^{n-1}} &\big|\sum_{(\Pi)} \sigma\big(x, F(\xi)\big)(v-u) - \sigma(x, I)\big| \\ &= \sup_{x\in S^{n-1}} \big|\sigma\big(x, \sum_{(\Pi)} F(\xi)(v-u)\big) - \sigma(x, I)\big| = d\big(\sum_{(\Pi)} F(\xi)(v-u), I\big) < \varepsilon\,.\end{aligned}$$

Since $\sigma\big(x, F(t)\big)$ is (HR) integrable on T uniformly for $x \in S^{n-1}$, then there exists $A_x \in R$, given $\varepsilon > 0$, there is $\delta(t) > 0$, for any $\delta-$ fine division $\Pi = \{\xi, [u, v]\}$ on T and for any $x \in S^{n-1}$,

$$\big|\sum_{(\Pi)} \sigma(x, F(\xi))(v-u) - A_x\big| < \varepsilon\,.$$

We can prove that $\big\{y \in R^n | \langle y, x\rangle \le A_x, x \in S^{n-1}\big\} \in P_k(R^n)$. It is clear that $\big\{y \in R^n | \langle y, x\rangle \le A_x, x \in S^{n-1}\big\}$ is nonempty set. Let $a, b \in \big\{y \in R^n | \langle y, x\rangle \le A_x, x \in S^{n-1}\big\}$, then for any $x \in S^{n-1}$,

$$\langle a, x\rangle \le A_x, \qquad \langle b, x\rangle \le A_x,$$

so for any $s \in [0,1]$, $\langle sa+(1-s)b, x\rangle = s\langle a,x\rangle + (1-s)\langle b,x\rangle \le A_x$, such that $\{y \in R^n | \langle y,x\rangle \le A_x, x \in S^{n-1}\}$ is the convex subsets of R^n. Let $\{a_m\}_{m=1}^{\infty} \subset \{y \in R^n | \langle y,x\rangle \le A_x, x \in S^{n-1}\}$, then for each $x \in S^{n-1}$, $\langle a_m, x\rangle \le A_x = (HR)\int_T \sigma(x, F(t))dt$, i.e. $\langle a_m, x\rangle$ is bounded. So for any $x \in S^{n-1}, \langle a_m, x\rangle$ there exist convergent subsequence $\langle a_{m_k}, x\rangle$. Let $\lim\limits_{k\to\infty}\langle a_{m_k}, x\rangle = b_x$, then $b_x \le A_x$, apparently,

$$\langle \lim_{k\to\infty} a_{m_k}, x\rangle = \lim_{k\to\infty}\langle a_{m_k}, x\rangle = b_x \le A_x,$$

i.e. $\lim\limits_{k\to\infty} a_{m_k} \in \{y \in R^n | \langle y,x\rangle \le A_x, x \in S^{n-1}\}$. So, $\{y \in R^n | \langle y,x\rangle \le A_x, x \in S^{n-1}\}$ is a compact set on R^n.

Let $\{y \in R^n | \langle y,x\rangle \le A_x, x \in S^{n-1}\} = A \in P_k(R^n)$. By Lemma 2.1, $\sigma(x,A) = A_x$. So, by Lemma 2.2 and the Remark 2.1, we have,

$$\begin{aligned} d\Big(\sum_{(\Pi)} F(\xi)(v-u), A\Big) &= \sup_{x\in S^{n-1}} \big|\sigma(x, \sum_{(\Pi)} F(\xi)(v-u)) - \sigma(x,A)\big| \\ &= \sup_{x\in S^{n-1}} \big|\sum_{(\Pi)} \sigma(x, F(\xi))(v-u) - A_x\big| \le \varepsilon\,. \quad \square \end{aligned}$$

Theorem 3.2. *Let $F, G : T \to P_k(R^n)$ is the set-valued function on T, then:*

(1) If F, G are all (HI) integral on T, $\alpha, \beta \in R$, such that $\alpha F + \beta G$ is (HI) integral on T, and

$$(HI)\int_T \big(\alpha F(t) + \beta G(t)\big)dt = \alpha(HI)\int_T F(t)dt + \beta(HI)\int_T G(t)dt\,.$$

(2) If F is (HI) integral on T, then F is (HI) integral on any subinterval of T, and if exist mutually exclusive intervals $T_1, T_2, \cdots, T_m$, so as to $T = \bigcup\limits_{i=1}^{m} T_i$, then

$$(HI)\int_T F(t)dt = \sum_{i=1}^{m}(HI)\int_{T_i} F(t)dt\,.$$

(3) If F is (HI) integral on T, $F = G$ a.e. on T, then G is (HI) integral on T, and

$$(HI)\int_T F(t)dt = (HI)\int_T G(t)dt\,.$$

(4) If F is the simple set-valued function on T, then F is (HI) integral on T.

By Theorem 3.1 and the property of (HR) integral of real functions, the above conclusions are clear.

References

[1] Robert J. Aumann, Integrals of set-valued functions, *Journal of Mathematical Analysis and Applications*, **12** (1965), 1-12.

[2] R.A. Gordon, The McShane integral of Banach-valued functions, *Illinois Journal of Mathematics*, **34** (1990), 557-567.

[3] Shexiang Hai, Zengtai Gong, The McShane integral of set-valued function, *Fuzzy Systems and Mathematics*, To Appear, In Chinese.

[4] P.Y. Lee, *Lanzhou Lectures on Henstock Integration*, Singapore, World Scientific (1989).

[5] P.Y. Lee, R. Vyborny, *The Integral: An Easy Approach after Kurzweil and Henstock*, Cambridge University Press (2000).

[6] Congxin Wu, Zengtai Gong, On Henstock integrals of interval-valued functions and fuzzy-valued functions, *Fuzzy Sets and Systems*, **115** (2000), 377-391.

[7] Congxin Wu, Ma Ming, Jinxuan Fang, *Structure Theory of Fuzzy Analysis*, Guizhou Scientific Publication (1994), In Chinese.