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VULNERABILITY OF GENERALIZED PETERSEN GRAPHS VIA INVARIANTS

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Abstract: Effectiveness of a network decreases if some nodes or links of it break down any way. Thus, communication network must be constructed to be stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. The vulnerability value of a communications network shows the resistance of the network after the disruption of some nodes or connection links until the communication breakdown. In a network, as the number of nodes belonging to sub networks changes, the vulnerability of the network also changes and requires less vulnerability or greater degrees of stability. If a graph is considered as a modelling network, many graph parameters have been used to describe the vulnerability of communications network, including connectivity, integrity and tenacity. Several of these deal with two fundamental questions about the resulting graph. How many vertices can still communicate? How difficult is it to reconnect the graph? Vulnerability numbers of a graph measure its durability respect to break down. We consider the integrity and tenacity of generalized Petersen graphs and the relation with its invariant.

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1. Introduction

A network can be modelled by a graph whose vertices represent the nodes and whose edges represent the lines of communication. Its efficiency decreases when some vertices or edges are destroyed with any way. In graph theory, many graph parameters have been used to describe the vulnerability of communication networks (graph), like connectivity, integrity, tenacity. The connectivity of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph, it is denoted by k(G). It is shown that the connectivity of a graph is related to the number of disjoint paths joining two vertices. We consider two graphs that have the same connectivity. Which one is less vulnerable than other? At this time we assume the integrity and tenacity as vulnerability measures: The integrity of a graph G is defined as

$$\min_{S\subseteq V}\{|S| + m(G - S)\},$$

where V(G) is the vertices set of the graph G and m(G-S) is the maximum number of vertices in a component of G-S (see [4], [5], [1], [15], [13], [11], [9]). Integrity of a graph has the following properties: If G is a graph of order p, then $1 \leq I(G) \leq p$, and if H is a subgraph of G then $I(H) \leq I(G)$.

If $G = \bigcup_{i=1..n} = G_i$ then $\max_i I(G_i) \leq I(G) \leq \sum_{k=1}^{n} I(G_i) - n + 1$

For any graphs G and H, $I(G+H) = \min\{I(G)+|H|, I(H)+|G|\}$ Similarly, edge-integrity is worked (see [3], [2]). The *tenacity* of a graph G, another vulnerability measure, incorporates ideas of both toughness and integrity. It is defined by Cozzens (see [7]) as $T(G) = \min_{S\subset V}\{\frac{|S|+m(G-S)}{w(G-S)}\}$, where S is a cut-set of G, m(G-S) is the maximum order of a component of G-S and w(G-S) is the number of components of G-S A set S is called T-set of G if it give the tenacity of G. There are many examples of graphs which suggest that T is a suitable measure of vulnerability in that it is able to distinguish between graphs that intuitively should have different levels of stability. The tenacity for several classes of graphs is studied. It is shown that: If H is a spanning subgraph of G, then $T(H) \leq T(G)$. For any graph G, $T(G) \geq (k(G)+1)/\beta(G)$, where $\beta(G)$ is the independence number of G and $k(G)$ is the connectivity of G.

If G is not complete, then $T(G) \leq (n - \beta(G) + 1)/\beta(G)$, where n is the order of graph G.

For any graph G,$T(G) \geq (t(G)+1)/\beta(G)$, where t is the toughness of graph G (see [7], [10], [14]). For example consider the graphs with 6 vertices in Figure 1. Two graphs have the same connectivity which is 1. There, $m(G_1 - v) = 2$

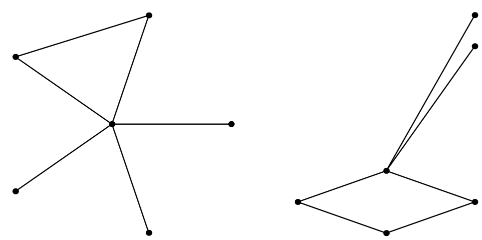

Figure 1: Stability comparisons of two graph that have same connectivity

and $I(G_1) = 3$, $T(G_1) = 3/4$, $m(G_2 - v) = 3$ and $I(G_2) = 4$, $T(G_2) = 4/3$. Consequently, $I(G_1) < I(G_2)$ and $T(G_1) < T(G_2)$.

Even though, generally it is an NP-Complete problem to determine these measures of a graph, it is possible to provide closed form formulate for measures of large classes of graphs. A large number of topologies have been proposed and studied for interconnection networks. Such a topology is usually modelled as an undirected graph where the set of vertices represents the processors and the set of edges represents the bi-directional communication links between the processors. Existing topologies include paths, cycles, meshes, complete binary trees, X-trees, pyramids, hypercubes, meshes of trees, cube connected-cycles, de Brujin graphs and so on (see [16]). Among these graphs, hypercube family has been popular because of such properties as symmetry, regularity, diameter, radius and Hamiltonian. For a connected graph G, we define the distance d(u, v) between two vertices u and v as the minimum of the lengths of the u-v paths of G. Under the distance function, the set V(G) is a metric space (see [8], [17]). The eccentricity e(v) of a vertex v of connected graph G is the number $\max\limits_{u\in V(G)}\{d(v,u)\}$. The radius rad G is defined as $\min\limits_{v\in V(G)}\{e(v)\}$ while the diameter diam G is $\max\limits_{v\in V(G)}\{e(v)\}$. It therefore follows that $\text{diam}(G) = \max\limits_{u\in V(G)}\{d(v,u)\}$. The n th power of the graph G has the same vertex set as G and vertices u and v are adjacent in G^n if their distance in G is at most n (see [18], [6]). Graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edges sets E_1 and E_2 respectively. The Cartesian product $G = G_1xG_2$ has $V = V_1xV_2$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E_1$. Generally, for n 5 relatively prime to $\lceil(n-1)/2\rceil$,the generalized Petersen graph of order 2n is constructed from two copies of C_n. Label the consecutive vertices of one cycle by u_0,u_1,u_2,,u_{n-1} and those of the other by $v_0, v_1, v_2, , v_{n-1}$. Then join each vertex u_i to $V_{i+\lceil (n-1)/2 \rceil \ modn}$. The generalized Petersen graph of order 2n is denoted by GP_n (see [6], [12]).

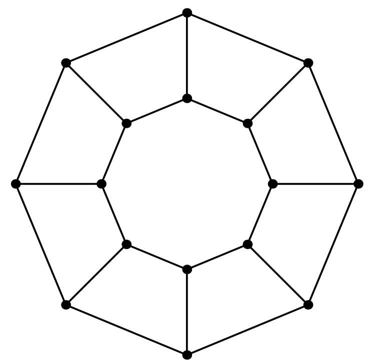

Figure 2: The graph GP_8. Generalized Petersen graph for $n = 8$

All generalized Petersen graphs are symmetric and 3-regular, that motivates us to design a new family of interconnection networks topology which uses the Petersen graph as a building block.

Petersen graph of 10 vertices has degree 3 and diameter 2 as compared to three-dimensional hypercube with diameter 3.

In this paper, we search the integrity and tenacity of generalized Petersen graphs networks and the relations between theirs invariant and integrity or tenacity. A subset S of V(G) such that every edge of G has at least one end in S is called a covering of G. The number of vertices in a minimum covering of G is the covering number of G and is denoted by $\alpha(G)$. A subset S of V(G) is independent if no two of its vertices are adjacent. The number of vertices in a maximum independent set of G is the independence number of graph G and is denoted by $\beta(G)$. In a graph order of n, $\alpha(G) + \beta(G) = n$. A vertex dominating set for a graph G is a set S of vertices such that every vertex of G belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in a graph G is called the vertex dominating number of G and is denoted by $\sigma(G)$. For every graph G, $\sigma(G) \leq \beta(G)$ (see [8], [17], [18], [6]). In this work, the first integer larger than x is denoted by $\lceil x \rceil$, the first integer small than x is denoted by $\lfloor x \rfloor$, and the absolute value of x is denoted by $|x|$.

2. Basic Results on Integrity and Tenacity of a Graph

In this section firstly, we define the basic network topology's (see [16]) and give the their integrity and tenacity. In a linear array network, each processor in this network (except the processor at the ends) has a direct communication link to two other processors. Such an interconnection network is called a path and it is denoted by P_n in graph theory where the number of vertices is n. A linear array with a wraparound connection is referred to as a ring. A ring is called a cycle and is denoted by C_n in graph theory. In a completely connected network, each processor has a direct communication link to every other processor in the network. A completely connected network is called complete graph and is denoted by K_n, which has n vertices in graph theory. In a star-connection network, one processor acts as the central processor. Every other processor has a communication link connecting it to this processor. A star-connected network is named star and is denoted by $K_{1,n}$ in graph theory. A two set completely connected network, has two sets of processors. Each processor in one of these sets has a direct communication link to every processor in the other set. A two set completely-connected network is called complete bipartite graph and is denoted by $K_{q,n}$ which has q+n vertices, in graph theory. A hypercube is a multi-dimensional mesh of processors with exactly two processors in each dimension. A d-dimensional hypercube consists of p=2d processors. A hypercube is denoted by Q_d in graph theory. The two-dimensional mesh is an extension of linear array to two dimensions. In a two-dimensional mesh, each processor has a direct communication link connecting it to four other processors. If both dimensions of mesh contain an equal number of processors, then it is called a square mesh; otherwise it is called a rectangular mesh. In graph theory, a two-dimensional square mesh is defined as P_nxP_n and a rectangular two-dimensional mesh is defined as P_qxP_n, where the Cartesian product of two graphs is denoted by x (see [11]). A tree network is one in which there is only path between any pair of processors. In a complete binary tree network, all internal processors have two children and all leaves are at the same depth. A complete binary tree is denoted by T_h in graph theory, where h is the height of the tree.

The integrity of a complete graph K_n is $I(K_n) = n$. The integrity of a star $K_{1,n}$ is $I(K_{1,n}) = 2$. The integrity of a path Pn is $I(P_n) = \lceil 2\sqrt{(n+1)}\rceil - 2$. The integrity of a cycle C_n is $I(C_n) = \lceil 2\sqrt{n}\rceil - 1$.

The integrity of a mesh $P_m * P_n$ for $n \geq 2$, if $n = 2r + k$ and $0 < k < 2 * r$

$$I(P_2xP_n) = \begin{cases} 2I(P_n) - 1\,, & 0 \leq k \leq r/2 \text{ or } r \leq k \leq 3r/2\,, \\ 2I(P_n)\,, & \text{otherwise}\,, \end{cases}$$

$$I(P_3xP_n) = \begin{cases} I(P_3xP_n) \leq 2(\lceil \sqrt{3n-1} \rceil - 1), & \text{if n is odd} , \\ I(P_3xP_n) \geq 2(\lceil \sqrt{3n-1} \rceil - 1), & \text{if n is even} . \end{cases}$$

The integrity of a complete binary tree T_h is

$$T_h = \begin{cases} 32^{\lfloor \frac{h}{2} \rfloor} - 1, & \text{h is odd}, \\ 2 * 2^{\frac{h}{2}} - 1, & \text{h is even}, \end{cases}$$

where the height of the tree is denoted by h.

The integrity of a hypercube Q_d is $I(Q_d) = 2^{d-1}+1$,where d is the dimension of hypercube.

The tenacity of a complete graph K_n is $T(K_n) = n$.

The tenacity of a star $K_{1,n}$ is $T(K_{1,n}) = 2/n$.

The tenacity of a path P_n is

$$T(P_n) = \begin{cases} 1 + \frac{2}{n}, & \text{n is even}, \\ 1, & \text{n is odd}. \end{cases}$$

The tenacity of a cycle C_n is

$$T(C_n) = \begin{cases} \frac{\lceil \frac{n}{2} \rceil + 1}{\lfloor \frac{n}{2} \rfloor}, & \text{for n odd}, \\ 1 + \frac{2}{n}, & \text{for n even}. \end{cases}$$

The tenacity of $K_{q,n}$ a complete bipartite graph and $q \leq n$.

Then, $T(K_{q,n}) = \frac{q+1}{n}$.

The tenacity of a hypercube Q_d is $T(Q_d) = 1 + 1/2^{d-1}$, where d is the dimension of hypercube.

The tenacity of a mesh is P_qxP_n is $T(P_2xP_n) = 1 + 1/n$

$$T(P_3xP_n) = \begin{cases} 1, & \text{for n odd}, \\ 1 + \frac{1}{\frac{3n}{2}}, & \text{for n even}. \end{cases}$$

The tenacity of a complete binary tree T_h is

$$T(T_h) = \begin{cases} \frac{2+4^{k+1}}{2(4^k-1)}, & \text{if h is odd} \quad and\ k = \lfloor \frac{h}{2} \rfloor, \\ \frac{2+4^{k+1}}{2(4^{k+1}-1)}, & \text{if h is even} \quad and\ k = \frac{h}{2}. \end{cases}$$

where h denotes height of the tree.

3. Vulnerability of Generalized Petersen Graphs via Invariants

In this section we prove theorems on the integrity and tenacity of generalized Petersen graphs which related to some invariant of graphs.

Corollary 3.1. *For $n > 6$, cover number of generalized Petersen graph is equal to $\alpha(GPn) = 2\lceil \frac{n}{2} \rceil$.*

Theorem 1. $I(GPn) = 2\lceil \frac{n}{2} \rceil + 1$.

Proof. If we remove the minimum cover set from the graph GP_n, resulting graph is disconnected and of order its largest component is 1. Then $I(GPn) = \alpha(GPn) + 1 = 2\lceil \frac{n}{2} \rceil + 1$. □

Corollary 3.2. *The independence number $\beta(GP_n)$ of GP_n is equal to $2\lfloor \frac{n}{2} \rfloor$.*

Theorem 2. $T(GPn) = \frac{2\lceil \frac{n}{2} \rceil + 1}{2\lfloor \frac{n}{2} \rfloor}$.

Proof. If we remove the minimum cover set from the graph GP_n, resulting graph is disconnected and of order its largest component is 1 also the number of components is equal to the independence number of GP_n. Then $T(GPn) = \frac{2\lceil \frac{n}{2} \rceil + 1}{2\lfloor \frac{n}{2} \rfloor}$. □

Theorem 3. *$I(K_m + GP_n) = \alpha(GP_n) + m + 1$, where K_m denotes of order m complete graph.*

Proof. If we remove the minimum cover set from the graph GP_n, resulting graph is connected and the graph Km+mK1. Also, if we remove the vertices of the graph K_m from this graph resulting graph become disconnected and its largest component has one vertex. Then, $I(K_m + GP_n) = \alpha(GP_n) + m + 1$. □

Theorem 4. $T(K_m + GP_n) = \frac{\alpha(GP_n) + m + 1}{\beta(GP_n)}$.

Proof. The proof of theorem is similar to Theorem 2. □

Theorem 5. *Let $L(GP_n)$ be line graph of the graph GP_n. Then, $I(L(GP_n)) = n + 2I(C_n) - 1$.*

Proof. When we remove n vertices of graph L(GP_n) that become connecting edges two C_n of the graph (GP_n), resulting graphs are disconnect two C_n. Thus, $I(L(GP_n)) = n + 2T(C_n) - 1$. □

Theorem 6. *Let $L((GP_n)$ be line graph of the graph (GP_n). Then, $T(L(GP_n)) = \frac{n + 2T(C_n) - 1}{2}$.*

Proof. It is clear from the Theorem 5 and the definition of tenacity. □

Corollary 3.3. *The dominating number of generalized Petersen graph with 2n vertices is* $\lfloor \frac{2n}{4k} \rfloor$*, for* $2 \leq k \leq d$*, where d denotes the diameter of the graph.*

Theorem 7. $I((GP_n)^k) = \sigma((GP_n)^k) + I(C_{2n-\sigma((GP_n)^k)})$.

Proof. If we remove the minimum dominating set from the k-th power graph of (GP_n), resulting graph is the cycle $C_{2n-\sigma(GP_n)^k)}$. Its integrity is given in Section 2. □

4. Conclusion

If we want to design a new communication network as possible as less vulnerable. When its any node or link failure, it keeps up its function on subnetworks. In this time we can consider the connected graph as its model. Integrity and tenacity are new measures of the graph vulnerability. They guide us at network design. In this work, we search integrity and tenacity on specially graph classes that are named generalized Petersen graph.

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