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APPROXIMATIONS FOR DEPARTURE PROCESSES IN MULTIPHASE QUEUEING SYSTEMS

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Abstract: The modern queueing theory is one of the powerful tools for a quantitative and qualitative analysis of communication systems, computer networks, transportation systems, and many other technical systems. In this paper, we deal with approximations of multiphase queueing systems (tandem queues, queues in series). Multiphase queueing systems are of special interest both in theory and in practical applications (message switching systems, processes of conveyor production, retransmission of video images, etc.). In this paper, the theorems have been proved on the law of the iterated logarithm (LIL) for departure flows of customers served under various heavy traffic conditions in multiphase queueing systems.

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1. Introduction

The modern queueing theory is one of the powerful tools for a quantitative and qualitative analysis of communication systems, computer networks, transportation systems, and many other technical systems. Thus, the paper is designated to the analysis of queueing systems, arising in the networks theory and communications theory (called multiphase queueing systems, tandem queues or series

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of queueing systems). The multiphase queueing system is a special case of the open Jackson network (see, for example, the book of Karpelevich et al [17]).

So, in this paper, we investigated the LIL for the departure flows of customers served under various heavy traffic conditions in multiphase queueing systems. Limit theorems (diffusion approximations) and the LIL for the queueing system under the conditions of heavy traffic are closely connected (they belong to the same field of research, i.e., investigations on the theory of queueing systems in heavy traffic). Therefore, first we shall try to trace the development of research on the general theory of queueing systems in heavy traffic.

One of the main trends of research in the queueing theory is related with the asymptotic analysis of explicit formulas or equations that decribe the distribution of one or other characteristics of a queueing system. To make an analysis of this kind we, certainly, assume the existence itself of such explicit formulas or equations, and, in addition, an unrestricted approximation of a queueing system to some limit. It was namely in this field that in 1962 J. Kingman obtained the first results on the behaviour of single-server queueing systems in heavy traffic (see Kingman [19, 20]). The single-server queue case, where intervals between the arrival time of customers to a system are independent nonnegative identically distributed random variables and there is only one server that works independently of external flow under the conditions of heavy traffic, has been studied in detail (see Kendall [18], Prohorov [28], etc). Later on, there appeared many of works designated to the various aspects of diffusion approximations of the models of queueing theory (see survey papers by Whitt [31] and books by Borovkov [5, 6]), Karpelevich et al [17]). The authors of works (see Harrison [12]), Harrison et al [13], Kobyashi [21], Reiman [29] and others) laid the basis for investigations on the diffusion approximation of queueing networks.

Due to technical difficulties, intermediate models – multiphase queues – are considered infrequently. Let us consider the works on the diffusion approximation of multiphase queues more in detail. In Harrison [11], it has been proved that a stationary distribution of waiting time in a two-phase queueing system is approximated by a limit distribution of two-dimensional diffusion processes with reflection. In Grigelionis et al [10], it has been proved that the limit processes for waiting time of a customer in a heavy traffic queueing system can also have discontinuous trajectories. The book of Karpelevich et al [17] mainly deals with a multiphase queueing system with identical service in phases of the system and also states the general theory of diffusion processes with reflection. In Minkevičius [22], functional limit theorems in a multiphase queueing system for important probability characteristics (waiting time of a customer and queue length of customers) have been proved.

The works on departure time or the number of departures for the queues in heavy traffic are sparse. One of the first papers of this kind by Iglehart et al [16] proved functional limit theorems for the number of departures and the departure time of customers in single-server queues. Whitt [33, 34], Glynn et al [8] investigated departures in multiclass and multiserver queues. Greenberg et al [9] presented the study of simulations for a departure process in multiphase queueing systems. Whitt [32], Albin et al [1], Glynn et al [7] investigated the limiting behaviour of the departure time of customers in multiphase queueing systems. Prabhahar et al [27] investigated the convergence of departures on tandem queues. Minkevičius [25] proved functional limit theorems for the number of departures in multiphase queueing systems for various conditions of heavy traffic.

The works on the LIL of queueing systems in heavy traffic are also not so numerous. In Iglehart [15], the LIL for multiple channel queues is studied. In Minkevičius [24], the LIL for the queue length of customers and a virtual waiting time of customers in multiphase queues are proved. Sakalauskas et al [26] also present the proof of a theorem on the LIL for a virtual waiting time of the open Jackson network in heavy traffic.

Note that various variants on the LIL in various domains of applications can be found in the survey of Bingham [4].

We submit some definitions from the theory of metric spaces (see, for example, Billingsley [3]).

Let $\mathbf{C}$ be a metric space consisting of real continuous functions in $[0,1]$ with a uniform metric $\rho(x,y) = \sup\limits_{0\le t\le 1} |x(t)-y(t)|$, $x, y \in \mathbf{C}$.

In this paper, theorems on the LIL for departure processes of the customers served in multiphase queues in heavy traffic are proved. The main tool for the analysis of multiphase queues in heavy traffic is a LIL to the renewal processes (the proof can be found in Iglehart [14]).

2. Laws of the Iterated Logarithm

We are investigating here the k-phase queueing systems (see, for example, Karpelevich et al [17]). A queueing system consisting of k consecutive service units is said to be a k-phase queueing system. The service process at the i-th service unit is called the i-th phase of service. Upon completing its service at the i-th service unit ($i = 1, \ldots, k-1$), a customer enters immediately the $(i+1)$-st phase. Upon completing its service at the k-th service unit, a customer leaves a system. Queues of unbounded length are allowed at any service unit: FCFS service principle is applied. Let us denote by t_n the time of arrival of the n-th customer; by $S_n^{(j)}$ the service time of the multiphase queueing systems; $z_n = t_{n+1} - t_n$. Let us introduce mutually independent renewal processes $x_j(t) = \{\max_k \sum_{i=1}^{k} S_i^{(j)} \leq t\}$ (such a total number of customers can be served in the j-th phase of multiphase queueing systems until time t if devices are working without time wasted), $e(t) = \{\max_k \sum_{i=1}^{k} z_i \leq t\}$ (total number of customers which arrive at multiphase queueing systems until time t). Next, denote by $\tau_j(t)$ the total number of customers after service departure from the j-th phase of multiphase queueing systems until time t; by $Q_j(t)$ the queue length of customers in the j-th phase of multiphase queueing systems at a time moment t; $v_j(t) = \sum_{i=1}^{j} Q_i(t)$ stands for the summary queue length of customers in the j-th phase of multiphase queueing systems at the time moment t, $j = 1, 2, \ldots, k$ and $t > 0$.

Suppose that the random variables are defined on one common probability space (Ω, F, P).

Let interarrival times (z_n) at the multiphase queueing system and service times $(S_n^{(j)})$ in every phase of the multiphase queueing system for ($j = 1, 2, \ldots, k$) be mutually independent identically distributed random variables.

Let us define $\mu_j = (ES_1^{(j)})^{-1}$, $\mu_0 = (Ez_1)^{-1}$, $\sigma_j^2 = DS_1^{(j)} \cdot (ES_1^{(j)})^{-3} > 0$, $\sigma_0^2 = Dz_1 \cdot (Ez_1)^{-3} > 0$, $\hat{x}_j(t) = e(t) - x_j(t)$, $\tilde{x}_j(t) = x_{j-1}(t) - x_j(t)$, $j = 1, 2, \ldots, k$ and $t > 0$.

In Minkevičius [23], the relations

$$Q_j(t) = \tau_{j-1}(t) - \tau_j(t), \tag{1}$$

$$Q_j(t) = f_t(\tau_{j-1}(\cdot) - x_j(\cdot)), \tag{2}$$

are obtained for ($j = 1, 2, \ldots, k$) and $f_t(x(\cdot) = x(t) - \inf_{0 \leq s \leq t} x(s)$.

Next, using (1)-(2), we obtain that

$$\tau_j(t) = \tau_{j-1}(t) - Q_j(t) = x_j(t) + \inf_{0 \leq s \leq t} (\tau_{j-1}(s) - x_j(s)), \tag{3}$$

for $j = 1, 2, \ldots, k$ and $\tau_0(t) = e(t)$.

Also, note that

$$v_j(t) = e(t) - \tau_j(t),\ j = 1, 2, \ldots, k. \tag{4}$$

At first assume the following condition to be fulfilled

$$\mu_0 > \mu_1 > \cdots > \mu_k > 0. \tag{5}$$

One of the main results of the paper is a theorem on the LIL for the departure flows of customers served in multiphase queueing systems.

Theorem 2.1. *If conditions (5) are fulfilled, then*

$$P\left(\varliminf_{n\to\infty} \frac{\tau_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)} = -1\right) = P\left(\varlimsup_{n\to\infty} \frac{\tau_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)} = 1\right) = 1,$$

$j = 1, 2, \ldots, k,\ 0 \le t \le 1$ *and* $a(n) = \sqrt{2n \ln\ln n}$.

Proof. First, denote

$$x_j^n(t) = \frac{x_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)},$$

$$\tau_j^n(t) = \frac{\tau_j(nt) - \mu_j \cdot n \cdot t}{\sigma_j \cdot a(n)}, \quad j = 1, 2, \ldots, k,\ 0 \le t \le 1.$$

Note that (see (3))

$$x_j(t) - \tau_j(t) = \sup_{0\le s\le t} (x_j(s) - \tau_{j-1}(s)),\ j = 1, 2, \ldots, k \text{ and } t > 0.$$

Thus,

$$x_j(nt) - \tau_j(nt) = \sup_{0\le s\le nt} (x_j(s) - \tau_{j-1}(s)) = \sup_{0\le s\le t} (x_j(ns) - \tau_{j-1}(ns)),$$

$j = 1, 2, \ldots, k$ and $t > 0$.

From this we get for arbitrary $\varepsilon > 0$ that

$$\begin{aligned}
P\left(\rho(x_j^n, \tau_j^n) > \varepsilon\right) &= P\left(\sup_{0\le t\le 1} (x_j(nt) - \tau_j(nt)) > \varepsilon \cdot \sigma_j \cdot a(n)\right) \\
&\le P\left(\sup_{0\le t\le 1} (\sup_{0\le s\le t} (x_j(ns) - \tau_{j-1}(ns))) > \varepsilon \cdot \sigma_j \cdot a(n)\right) \\
&\le P\left(\sup_{0\le t\le 1} (x_j(nt) - \tau_{j-1}(nt)) > \varepsilon \cdot \sigma_j \cdot a(n)\right) \\
&\le P\left(\sup_{0\le t\le 1} (x_j(nt) - x_{j-1}(nt)) > \frac{\varepsilon}{2} \cdot \sigma_j \cdot a(n)\right) \\
&+ P\left(\sup_{0\le t\le 1} (x_{j-1}(nt) - \tau_{j-1}(nt)) > \frac{\varepsilon}{2} \cdot \sigma_j \cdot a(n)\right) \le \cdots
\end{aligned}$$

$$\le \sum_{i=1}^{j} P\left(\sup_{0\le t\le 1}(-\tilde{x}_i(nt)) > \frac{\varepsilon}{j}\cdot\sigma_j\cdot a(n)\right)$$
$$\le \sum_{i=1}^{k} P\left(\sup_{0\le t\le 1}(-\tilde{x}_i(nt)) > \frac{\varepsilon}{j}\cdot\sigma_j\cdot a(n)\right)$$
$$\le \sum_{i=1}^{k} P\left(\sup_{0\le t\le 1}(-\tilde{x}_i(nt)) > \frac{\varepsilon}{k}\cdot\sigma_j\cdot a(n)\right), \quad j=1,2,\dots,k. \tag{6}$$

We achieve that, if conditions (5) are fulfilled, then (see, for example, Iglehart [15])

$$\frac{\sup\limits_{0\le t\le 1}(-\tilde{x}_j(nt))}{a(n)} \Rightarrow 0,\ j=1,2,\dots,k. \tag{7}$$

But, applying the law of the iterated logarithm to the renewal processes (see Iglehart [14]), we obtain that

$$P\left(\varliminf_{n\to\infty} x_j^n(t) = -1\right) = P\left(\varlimsup_{n\to\infty} x_j^n(t) = 1\right) = 1,\ j=1,2,\dots,k \tag{8}$$

and $0\le t\le 1$. From this, (6) and (7) we get that

$$P\left(\varliminf_{n\to\infty} \tau_j^n(t) = -1\right) = P\left(\varlimsup_{n\to\infty} \tau_j^n(t) = 1\right) = 1,\ j=1,2,\dots,k\,,$$

and $0\le t\le 1$. The proof of the theorem is complete. □

Then we prove the following theorem about the LIL for departure flows of customers served in multiphase queues.

Assume the following condition to be fulfilled

$$0<\mu_0<\mu_1<\dots<\mu_k. \tag{9}$$

Theorem 2.2. *If conditions (5) are fulfilled, then*

$$P\left(\varliminf_{n\to\infty}\frac{\tau_j(nt)-\mu_0\cdot n\cdot t}{\sigma_0\cdot a(n)} = -1\right) = P\left(\varlimsup_{n\to\infty}\frac{\tau_j(nt)-\mu_0\cdot n\cdot t}{\sigma_0\cdot a(n)} = 1\right) = 1,$$

$j=1,2,\dots,k,\quad 0\le t\le 1$.

Proof. At first, denote families of random variables as follows:

$$e^n(t) = \frac{e(nt)-\mu_0\cdot n\cdot t}{\sigma_0\cdot a(n)},\quad \hat{\tau}_j^n(t) = \frac{\tau_j(nt)-\mu_0\cdot n\cdot t}{\sigma_0\cdot a(n)},$$
$$v_j^n(t) = \frac{v_j(nt)-\mu_0\cdot n\cdot t}{\sigma_0\cdot a(n)},$$
$$Q_j^n(t) = \frac{Q_j(nt)}{a(n)},\quad j=1,2,\dots,k \text{ and } 0\le t\le 1.$$

From (4) we obtain that, for $j = 1, 2, \ldots, k$,

$$P\left(\rho(e^n, \hat{\tau}_j^n) > \varepsilon\right) = P\left(\sup_{0\le t\le 1} (e(nt) - \tau_j(nt)) > \varepsilon \cdot a(n)\right)$$
$$= P\left(\sup_{0\le t\le 1} v_j(nt) > \varepsilon \cdot \sigma_1 \cdot a(n)\right) \le P\left(\sup_{0\le t\le 1} v_k(nt) > \varepsilon \cdot a(n)\right) \tag{10}$$
$$\le \sum_{i=1}^{k} P\left(\sup_{0\le t\le 1} Q_i(nt) > \frac{\varepsilon}{k} \cdot a(n)\right) = \sum_{i=1}^{k} P\left(\rho(Q_i^n, 0) > \frac{\varepsilon}{k}\right).$$

It has been proved (see Minkevičius [23]) that

$$Q_j(t) \le \sup_{0\le s\le t} \tilde{x}_j(s) - \tilde{x}_j(t),\ j = 1, 2, \ldots, k \text{ and } \ t > 0.$$

Thus,

$$Q_j(nt) \le \sup_{0\le s\le nt} \tilde{x}_j(s) - \tilde{x}_j(nt) = \sup_{0\le s\le t} \tilde{x}_j(ns) - \tilde{x}_j(nt), \tag{11}$$

$j = 1, 2, \ldots, k$ and $0 \le t \le 1$. Therefore, if $a(n) \ge \sqrt{n},\ n \ge e^e$, for arbitrary $\varepsilon > 0$

$$P\left(\rho(Q_i^n, 0) > \varepsilon\right) \le P\left(\sup_{0\le t\le 1} (\sup_{0\le s\le t} \tilde{x}_j(ns) - \tilde{x}_j(nt)) > \varepsilon \cdot a(n)\right)$$
$$\le P\left(\frac{\sup_{0\le t\le 1} (\sup_{0\le s\le t} \tilde{x}_j(ns) - \tilde{x}_j(nt))}{\sqrt{n}} > \varepsilon\right),\ 0 \le t \le 1. \tag{12}$$

Similarly as in Minkevičius [23], we see that, if conditions (9) are fulfilled, then

$$\frac{\sup_{0\le s\le t} \tilde{x}_j(ns) - \tilde{x}_j(nt)}{\sqrt{n}} \Rightarrow 0,\ j = 1, 2, \ldots, k \ \text{ and } \ 0 \le t \le 1. \tag{13}$$

Applying the continuous mapping theorem (see Billingsley [3], Theorem 4.1)) and (13), we get that

$$\frac{\sup_{0\le t\le 1} (\sup_{0\le s\le t} \tilde{x}_j(ns) - \tilde{x}_j(nt))}{\sqrt{n}} \Rightarrow 0,\ j = 1, 2, \ldots, k. \tag{14}$$

From this and (12), we obtain that, for arbitrary $\varepsilon > 0$,

$$P\left(\overline{\lim_{n\to\infty}}\, \rho(Q_j^n, 0) > \varepsilon\right) = 0,\ j = 1, 2, \ldots, k.$$

Thus, (see (10)), we get that, for arbitrary $\varepsilon > 0$,

$$P\left(\overline{\lim_{n\to\infty}}\, \rho(e^n, \hat{\tau}_j^n) > \varepsilon\right) = 0. \tag{15}$$

But similarly as in (8), we obtain that

$$P\left(\overline{\lim_{n\to\infty}}\, e^n(t) = 1\right) = P\left(\underline{\lim_{n\to\infty}}\, e^n(t) = -1\right) = 1,\ j = 1, 2, \ldots, k,\ 0 \le t \le 1.$$

From this and (15), we have that

$$P\left(\overline{\lim_{n\to\infty}}\, \hat{\tau}_j^n(t) = 1\right) = P\left(\underline{\lim_{n\to\infty}}\, \hat{\tau}_j^n(t) = -1\right) = 1,\ j = 1, 2, \ldots, k,\ \ 0 \le t \le 1.$$

The proof of the theorem is complete. □

3. Conclusions

In this section we present two corollaries of Theorem 1 and Theorem 2.

Corollary 3.1. *We see in this case that departure flow of customer in k-phase queueing system depends of service intensivity $\mu_k, j = 1, 2, \ldots, k$.*

Corollary 3.2. *If conditions (5) are fulfilled, then for fixed $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for every $n \ge n(\varepsilon)$,*

$$(1-\varepsilon)\cdot\sigma_j\cdot a(n) + \mu_j\cdot n\cdot t \le \tau_j(nt) \le (1+\varepsilon)\cdot\sigma_j\cdot a(n) + \mu_j\cdot n\cdot t,$$
$$j = 1, 2, \ldots, k, \quad 0 \le t \le 1\ ,$$

with probability one.

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