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INVESTIGATION OF A MATHEMATICAL MODEL OF THE MESSAGE SWITCHING SYSTEM

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Abstract: The paper is designated to the analysis of queueing systems, arising in the network and communications theory. The purpose of this research in the queueing theory is the theorem on the law of the iterated logarithm (LIL) in multiphase queueing systems and its application to the mathematical model of the message switching system. First we investigate the extremal values of the cumulative idle time of a customer (maximum and minimum of the cumulative idle time of a customer).

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1. Introduction

The modern queueing theory is one of the powerful tools for a quantitative and qualitative analysis of communication systems, computer networks, transportation systems, and many other technical systems. The paper is designated to the analysis of queueing systems, arising in the network and communications theory (called multiphase queueing systems, tandem queues or series of queueing systems). We note that multiphase queueing systems can be representative for modelling practical multi-stage service systems in a variety of disciplines, most notable manufacturing (asembly lines), computer networking (packet switch

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structures), and telecommunications (e.g. cellular mobile networks), etc.

Interest in the field of multiphase queueing systems has been stimulated by theoretical values of the results as well as by their possible applications in information and computing systems, communication networks, and automated technological processes (see, for example, Saati et al [15]). The methods of investigation of single-phase queueing systems are considered in Borovkov [2, 3], etc. The asymptotic analysis of models of queueing systems in heavy traffic is of special interest (see, for example, Kingman [8, 9], Iglehart et al [4, 5], etc.). The papers of Kobyashi [10], Reiman [14] and others described the beginning of the investigation of diffusion approximation to queueing networks. Intermediate models – multiphase queueing systems – are considered rarer due to serious technical difficulties (see, for example, book of Karpelevich et al [7]). At first, the LIL is considered by investigating the cumulative idle time of a customer in multiphase queueing systems (maximum and minimum of the cumulative idle time of a customer).

Let the cumulative idle time of a customer be unrestricted in the phases of a queueing sys-tem the principle of service being "first come, first served" (FCFS).

We present some definitions in the theory of metric spaces (see, for example, Billingsley [1]). Let C be a metric space consisting of real continuous functions in [0, 1] with a uniform metric

$$\rho(x,y) = \sup_{0\le t\le 1} |x(t) - y(t)|,\ x,y \in C.$$

Let D be a space of all real-valued right-continuous functions in [0,1] having left limits and endowed with the Skorokhod topology induced by the metric d (under which D is complete and separable). Also, note that $d(x,y) \le \rho(x,y)$ for $x,y \in D$.

In this paper, we will constantly use an analog of the theorem on converging together (see, for example, Billingsley [1]).

Theorem 1.1. *Let* $\varepsilon > 0$ *and* $\boldsymbol{X}_n, \boldsymbol{Y}_n, \boldsymbol{X} \in D$. *If* $\boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\, d(\boldsymbol{X}_n, \boldsymbol{X}) > \varepsilon\right)$ $= 0$, *and* $\boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\, d(\boldsymbol{X}_n, \boldsymbol{Y}_n) > \varepsilon\right) = 0$, *then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\, d(\boldsymbol{Y}_n, \boldsymbol{X}) > \varepsilon\right) = 0. \tag{1}$$

We investigate here a k-phase queue (i.e., after a customer has been served in the j-th phase of the queue, he is routed to the $(j+1)$-st phase of the queue, and, after the service in the k-th phase of the queue, he leaves the queue). Let us denote by t_n the time of arrival of the n-th customer; by $S_n^{(j)}$– the service time of the n-th customer in the j-th phase, $z_n = t_{n+1} - t_n$; and by $\tau_{j,n+j}$ - departure of the n-th customer from the j-th phase of the queue, $j = 1, 2, \cdots, k$.

Let interarrival times (z_n) at the multiphase queueing system and service times $(S_n^{(j)})$ in each phase of the queue for $j = 1, 2, \cdots, k$ be mutually independent identically distributed random variables.

Next, denote by $BI_{j,n}$ the idle time of the n-th customer in the j-th phase of the multiphase queue; $\hat{F}_{j,n} = \sum_{l=1}^{n} BI_{j,l}$ stands for a cummulative idle time of the n-th customer in the j-th phase of the multiphase queue, $j = 1, 2, \ldots, k$.

When $j = 1, 2, \ldots, k$, let

$$\delta_{j,n} = \begin{cases} S_{n-(j-1)}^{(j)} - z_n, & \text{if } n \geq k\,, \\ 0, & \text{if } n < k. \end{cases}$$

Let us denote $S_{j,n} = \sum_{l=1}^{n-1} \delta_{j,l}$, $S_{0,n} \equiv 0$, $\hat{S}_{j,n} = S_{j-1,n} - S_{j,n}$, $x_{j,n} = \tau_{j,n} - t_n$, $x_{0,n} \equiv 0$, $\hat{x}_{j,n+1} = x_{j,n} - \delta_{j,n+1}$, $\hat{x}_{0,n} \equiv 0$, $z_{j,n} = \hat{x}_{j,n} - S_{j,n}$, $\alpha_j = M\delta_{j,n}$, $\alpha_0 \equiv 0$, $Dz_n = \sigma_0^2$, $DS_n^{(j)} = \sigma_j^2$, $\tilde{\sigma}_j^2 = \sigma_0^2 + \sigma_j^2$, $S_n^{(0)} = z_n$, $j = 1, 2, \ldots, k$, $[x]$ as the integer part of number x.

We assume that the following conditions are fulfilled:

There exists a constant $\gamma > 0$ such that

$$\sup_{n \geq 1} M|S_n^{(j)}|^{4+\gamma} < \infty, \; j = 0, 1, 2, \ldots, k \tag{2}$$

and

$$\alpha_k < \alpha_{k-1} < \cdots < \alpha_1 < 0. \tag{3}$$

In this paper, we mostly use the equations presented in Minkevičius [11]:

$$\hat{x}_{j,n} = \max_{0 \leq l \leq n} (\hat{x}_{j-1,l} - S_{j,l}) + S_{j,n}, \quad \hat{x}_{0,n} \equiv 0, \; n \geq k, \; j = 1, 2, \ldots, k. \tag{4}$$

2. Main Results

One of the results of the paper is a theorem on LIL for the maximum of the comulative idle time of a customer in multiphase queues.

We prove such a theorem.

Theorem 2.1. *If conditions (2) and (3) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}\hat{F}_{j,l}-(-\alpha_j)\cdot n}{\tilde{\sigma}\cdot a(n)}=1\right)$$
$$=\boldsymbol{P}\left(\underline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}\hat{F}_{j,l}-(-\alpha_j)\cdot n}{\tilde{\sigma}\cdot a(n)}=-1\right)=1,\quad j=1,2,\dots,k$$

and $a(n)=\sqrt{2n\ln\ln n}$.

Proof. Denote a random function in D as follows

$$\tilde{F}_j^n(t)=\frac{\max\limits_{0\le l\le [nt]}\hat{F}_{j,l}-(-\alpha_j)\cdot [nt]}{a(n)},\quad \tilde{Z}_j^n(t)=\frac{\max\limits_{0\le l\le [nt]}\hat{z}_{j,l]}-(-\alpha_j)\cdot [nt]}{a(n)},$$
$$\tilde{S}_j^n(t)=\frac{\max\limits_{0\le l\le [nt]}S_{j,l}-(-\alpha_j)\cdot n\cdot t}{a(n)},\quad \hat{F}_j^n(t)=\frac{\hat{F}_{j,[nt]}-(-\alpha_j)\cdot [nt]}{a(n)},$$
$$\hat{Z}_j^n(t)=\frac{\hat{z}_{j,[nt]}-(-\alpha_j)\cdot [nt]}{a(n)},\quad \hat{S}_j^n(t)=\frac{S_{j,[nt]}-(-\alpha_j)\cdot [nt]}{a(n)},$$
$$j=1,2,\dots,k\ \text{ and }\ 0\le t\le 1.$$

Using a triangle inequality and Whitt [17] we see that, for each fixed $\varepsilon>0$,

$$\begin{aligned}
&\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\tilde{F}_j^n,\hat{S}_j^n)>\varepsilon\right)\le\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\tilde{F}_j^n,\tilde{Z}_j^n)>\frac{\varepsilon}{3}\right)\\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\tilde{Z}_j^n,\tilde{S}_j^n)>\frac{\varepsilon}{3}\right)\\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\tilde{S}_j^n,\hat{S}_j^n)>\frac{\varepsilon}{3}\right)\le\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\hat{F}_j^n,\hat{Z}_j^n)>\frac{\varepsilon}{3}\right)\\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\hat{Z}_j^n,\hat{S}_j^n)>\frac{\varepsilon}{3}\right)\\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\tilde{S}_j^n,\hat{S}_j^n)>\frac{\varepsilon}{3}\right)\le\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,\rho(\hat{F}_j^n,\hat{Z}_j^n)>\frac{\varepsilon}{3}\right)\\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,\rho(\hat{Z}_j^n,\hat{S}_j^n)>\frac{\varepsilon}{3}\right)\\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,\rho(\tilde{S}_j^n,\hat{S}_j^n)>\frac{\varepsilon}{3}\right)\le\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\sup\limits_{0\le t\le 1}|\hat{F}_{j,[nt]}-\hat{z}_{j,[nt]}|}{a(n)}>\frac{\varepsilon}{3}\right)\\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\sup\limits_{0\le t\le 1}|\hat{z}_{j,[nt]}-(-S_{j,[nt]})|}{a(n)}>\frac{\varepsilon}{3}\right)
\end{aligned}\tag{5}$$

$$+ \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\sup\limits_{0\le t\le 1} |\max\limits_{0\le l\le [nt]}(-S_{j,l}) - (-S_{j,[nt]}|}{a(n)} > \frac{\varepsilon}{3}\right)$$

$$\le \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le n} |\hat{F}_{j,l} - \hat{z}_{j,l})|}{a(n)} > \frac{\varepsilon}{3}\right)$$

$$+ \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le n} |\hat{z}_{j,l} - (-S_{j,l})|}{a(n)} > \frac{\varepsilon}{3}\right)$$

$$+ \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le n} |\max\limits_{0\le m\le l}(-S_{j,m}) - (-S_{j,l})|}{a(n)} > \frac{\varepsilon}{3}\right), \ j = 1, 2, \ldots, k.$$

Thus, we have for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} d(\tilde{F}_j^n, \hat{S}_j^n) > \varepsilon\right) \le \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le n} |\hat{F}_{j,l} - \hat{z}_{j,l}|}{a(n)} > \frac{\varepsilon}{3}\right)$$

$$+ \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le n} |\hat{z}_{j,l} - (-S_{j,l})|}{a(n)} > \frac{\varepsilon}{3}\right) \tag{6}$$

$$+ \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le m} |\max\limits_{0\le m\le l}(-S_{j,m}) - (-S_{j,l})|}{a(n)} > \frac{\varepsilon}{3}\right), \ j = 1, 2, \ldots, k.$$

As one can see from Minkevičius [12], the first and second term in inequality (6) converges to zero. So, we prove now that the third term in inequality (6) also converges to zero. Hence, we get that, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\frac{\max\limits_{0\le l\le m} |\max\limits_{0\le m\le l}(-S_{j,m}) - (-S_{j,l})|}{a(n)} > \varepsilon\right)$$

$$= \mathbf{P}\left(\frac{\max\limits_{0\le l\le m} (\max\limits_{0\le m\le l}(-S_{j,m}) - (-S_{j,l}))}{a(n)} > \varepsilon\right) = \mathbf{P}\left(\frac{\max\limits_{0\le l\le m} \max\limits_{0\le m\le l} S_{j,m}}{a(n)} > \varepsilon\right)$$

$$= \mathbf{P}\left(\frac{\max\limits_{0\le m\le l}(-S_{j,m}}{a(n)} > \varepsilon\right) = \mathbf{P}\left(\frac{\max\limits_{0\le m\le l}(-\sum\limits_{i=1}^{j} \hat{S}_{i,l})}{a(n)} > \varepsilon\right)$$

$$\leq \mathbf{P}\left(\frac{\sum_{i=1}^{j}\max_{0\leq m\leq l}(-\hat{S}_{i,l})}{a(n)} > \varepsilon\right) \leq \sum_{i=1}^{j}\mathbf{P}\left(\frac{\max_{0\leq m\leq l}(-\hat{S}_{i,l})}{a(n)} > \frac{\varepsilon}{j}\right)$$

$$\leq \sum_{i=1}^{k}\mathbf{P}\left(\frac{\max_{0\leq m\leq l}(-\hat{S}_{i,l})}{a(n)} > \frac{\varepsilon}{k}\right), \quad j = 1, 2, \ldots, k. \quad (7)$$

Thus, we obtain (see (7)) that, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\frac{\max_{0\leq l\leq m}|\max_{0\leq m\leq l}(-S_{j,m}) - (-S_{j,l})|}{a(n)} > \varepsilon\right)$$

$$\leq \sum_{i=1}^{k}\mathbf{P}\left(\frac{\max_{0\leq m\leq l}(-\hat{S}_{i,l})}{a(n)} > \frac{\varepsilon}{k}\right), \quad j = 1, 2, \ldots, k. \quad (8)$$

But we have proved (see Minkevičius [12]) that, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\lim_{n\to\infty}\frac{\max_{0\leq l\leq n}(-\hat{S}_{j,l})}{a(n)} > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \quad (9)$$

We get from this and (8) that, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max_{0\leq l\leq m}|\max_{0\leq m\leq l}(-S_{j,m}) - (-S_{j,l})|}{a(n)} > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \quad (10)$$

Consequently, the third term in inequality (6) also converges to zero (see (10)). Thus (see again (6)), we prove that, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{F}_j^n, \hat{S}_j^n) > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \quad (11)$$

Using the theorem on the LIL for random functions $\hat{S}_{j,n}(t)$, $j = 1, 2, \ldots, k$ (see, for example, Strassen (1964)) we achieve that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{(-S_{j,n}) - (-\alpha_j)\cdot n}{\tilde{\sigma}_j\cdot a(n)} = 1\right) = 1$$

and (12)

$$\mathbf{P}\left(\underline{\lim_{n\to\infty}}\frac{(-S_{j,n}) - (-\alpha_j)\cdot n}{\tilde{\sigma}_j\cdot a(n)} = -1\right) = 1, \quad j = 1, 2, \ldots, k.$$

Applying (1) and (12), we obtain that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}\hat{F}_{j,l}-(-\alpha_j)\cdot n}{\tilde{\sigma}\cdot a(n)}=1\right)=1\,, \tag{13}$$

and

$$\mathbf{P}\left(\underline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}\hat{F}_{j,l}-(-\alpha_j)\cdot n}{\tilde{\sigma}\cdot a(n)}=-1\right)=1,\quad j=1,2,\ldots,k.$$

The proof of Theorem 2.1 is complete. □

The next theorem, establishing the LIL for the minimum of the cumulative idle time of a customer in a multiphase queueing system, is proved analogously as Theorem 2.1.

Theorem 2.2. *If conditions (2) and (3) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\frac{\min\limits_{0\le l\le n}\hat{F}_{j,l}}{a(n)}=0\right)=1,\quad j=1,2,\ldots,k.$$

Proof. Denote a random function in D as follows

$$\bar{F}_j^n(t)=\frac{\min\limits_{0\le l\le [nt]}\hat{F}_{j,l}}{a(n)},\quad \bar{Z}_j^n(t)=\frac{\min\limits_{0\le l\le [nt]}\hat{z}_{j,l]}}{a(n)},\quad \bar{S}_j^n(t)=\frac{\min\limits_{0\le l\le [nt]}(-S_{j,l})}{a(n)},$$

$j=1,2,\ldots,k$ and $0\le t\le 1$.

Similarly as in (6) we get that, for each fixed $\varepsilon>0$,

$$\begin{aligned}&\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\bar{F}_j^n,0)>\varepsilon\right)\le\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}|\hat{F}_{j,l}-\hat{z}_{j,l})|}{a(n)}>\frac{\varepsilon}{3}\right)\\ &+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}|\hat{z}_{j,l}-(-S_{j,l})|}{a(n)}>\frac{\varepsilon}{3}\right)\\ &+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}|\min\limits_{0\le m\le l}(-S_{j,m})|}{a(n)}>\frac{\varepsilon}{3}\right),\ j=1,2,\ldots,k.\end{aligned} \tag{14}$$

As we can see in (6), the first and second terms in inequality (14) converge to zero. Now we will prove that the third term in inequality (14) also converges to zero. However, similarly as in Minkevičius [12] we prove that, for each fixed

$\varepsilon > 0$,

$$\mathbf{P}\left(\frac{\max_{0\le l\le n}|\min_{0\le m\le l}(-S_{j,m})|}{a(n)} > \varepsilon\right) \le \sum_{i=1}^{k}\mathbf{P}\left(\frac{|\max_{0\le l\le n}(-S_{i,l})|}{a(n)} > \frac{\varepsilon}{k}\right), \tag{15}$$

$j = 1, 2, \ldots, k$.

Using (15) and (9) we achieve that, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max_{0\le l\le n}|\min_{0\le m\le l}(-S_{j,m})|}{a(n)} > \varepsilon\right) = 0,\ j = 1, 2, \ldots, k. \tag{16}$$

So the third term in inequality (14) converges to zero (see (16)) as well. Finally (see again (16)), we prove that, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\bar{F}_j^n, 0) > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \tag{17}$$

From this and (1) we obtain the proof of the theorem.

The proof of Theorem 2.2 is complete. □

3. On the Mathematical Model of Switching Facility

In this part of the paper, we will present an application of the proved theorem – a mathematical model of the message switching system.

As noted in the introduction, multiphase queueing systems are of special interest both in theory and in practical applications. Such systems consist of several service nodes, and each arriving customer is served at each of the consecutively located node (frequently called a phase). A typical example is provided by queueing systems with identical service. Such systems are very important in applications, especially to message switching systems. In fact, in many comunication systems the transmission time of customers does not vary in the delivery process.

So, we investigate a message switching system that consists of k phases and in which $S_n^j = S_n,\ \ j = 1,\ 2, \ldots, k$ (the service process is identical in the phases of the system).

Let $\delta_n = \begin{cases} S_{n-k} - z_n, & \text{if } n \ge k \\ 0, & \text{if } n < k. \end{cases}$

Also, note that $\alpha = M\delta_n$, $Dz_n = \sigma_0^2$, $DS_n = \sigma^2$, $\tilde{\sigma}^2 = \sigma_0^2 + \sigma^2$, $\hat{T}_{j,n} = \sum_{i=1}^{j}(W_n^{(i)} + S_n)$.

We assume that the following conditions are fulfilled:

There exists a constant $\gamma > 0$ such that

$$\sup_{n\geq 1} M|S_n|^{4+j} < \infty \tag{18}$$

and

$$\alpha > 0. \tag{19}$$

Similarly as in the proof of Theorem 2.1, we present the following theorem on the LIL for the maximum cumulative idle time of a data packet in message switching systems.

Theorem 3.1. *If conditions (18) and (19) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\leq l\leq n} \hat{F}_{j,l} - (-\alpha_j)\cdot n}{\tilde{\sigma}\cdot a(n)} = 1\right)$$
$$= \boldsymbol{P}\left(\underline{\lim_{n\to\infty}} \frac{\max\limits_{0\leq l\leq n} \hat{F}_{j,l} - (-\alpha_j)\cdot n}{\tilde{\sigma}\cdot a(n)} = -1\right) = 1, \quad j = 1, 2, \ldots, k$$

and $a(n) = \sqrt{2n\ln\ln n}$.

Corollary 3.2. *If conditions (18) and (19) are fulfilled, then for fixed* $\varepsilon > 0$ *there exists* $n(\varepsilon)$ *such that for every* $n \geq n(\varepsilon)$,

$$(1-\varepsilon)\cdot\tilde{\sigma}\cdot a(n) + \alpha\cdot n \leq \max_{0\leq l\leq n} \hat{F}_{j,l} \leq (1+\varepsilon)\cdot\tilde{\sigma}\cdot a(n) + \alpha\cdot n, \quad j = 1, 2, \ldots, k,$$

with probability one.

We see that the maximum cumulative idle time of the data packet in the first phase gives a major contribution to the cumulative idle time for the whole system (see, for example, Karpelevich et al [7], p. 71).

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