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APPROXIMATION OF EXTREMAL SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: We discuss boundary value problems for first-order functional differential equations and formulate sufficient conditions under which there exist monotone sequences approximating extremal solutions of such problems. To obtain the results we apply the monotone iterative method.

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1. Introduction

Let us consider the problem

$$\begin{cases} x'(t) = f(t, x(t), [p(x)](t)), & t \in J = [0, T],\ T > 0\,, \\ x(0) = rx(T) + \lambda, \ \ r \in (0, 1], \end{cases} \tag{1}$$

where $f\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$, $C(J,\mathbb{R})$ is the space of continuous functions in J with the maximum norm.

In this paper we extend some result of Nieto and Rodríguez-López [5] where it was considered the case of functional differential equations with periodic boundary conditions and f satisfied a one sided Lipschitz condition with respect to the last two variables with corresponding constants.

Plan of this paper is as follows. In Section 2 we prove comparison theorem that will be important in following sections. In the next section we prove the existence of unique solution of linear problem associated to (1). In the last

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section we prove the existence of monotone sequences approximating extremal solutions of (1).

2. Comparison Theorem

Theorem 1. *Assume that* $x \in C^1(J,\mathbb{R})$, $M,N \in C(J,\mathbb{R})$, $M(t) > 0$, $N(t) \geq 0$ *and*

$$\begin{cases} x'(t) + M(t)x(t) + N(t)[p(\max\{x,\Theta\})](t) \geq 0, & t \in J\,, \\ x(0) \geq rx(T), \;\; r \in (0,1], \end{cases}$$

where $\Theta(t) = 0$, $t \in J$ *and* $p\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ *satisfies the condition*

$$p(\Theta) \leq 0 \quad on \quad J. \tag{2}$$

Moreover assume that

$$\int_a^b N(t)[p(\max\{v,\Theta\})]e^{\int_0^t M(s)ds}dt \leq \max_{t\in[0,b]}\{v(t)e^{\int_0^t M(s)ds}\} \tag{3}$$

if $0 \leq a < b \leq T$ *and* $v \in C(J,\mathbb{R})$ *with* $\max_{t\in[0,b]}\{v(t)e^{\int_0^t M(s)ds}\} \geq 0$.

Then $x(t) \geq 0, t \in J$.

Proof. Assume that it is not true that $x \geq 0$ on J. Then there exists $t_0 \in J$ such that $x(t_0) < 0$.

If $x \leq 0$ on J and $x \not\equiv 0$ then, in view of (2), we have

$$\begin{aligned} x'(t) &\geq -M(t)x(t) - N(t)[p(\max\{x,\Theta\})](t) \\ &= -M(t)x(t) - N(t)[p(\Theta)](t) \geq 0 \end{aligned}$$

for $t \in J$, so x is nondecreasing. Since $x(0) \geq rx(T) \geq x(T)$, then x is a constant function. Put $x(t) = k$, $k < 0$. Then, using (2), we obtain

$$0 \leq M(t)k + N(t)[p(\Theta)](t) \leq M(t)k, \;\; t \in J$$

which is a contradiction since $k < 0$. So there exists $t_* \in J$ such that $x(t_*) > 0$.

Let us consider a function

$$z(t) = x(t)e^{\int_0^t M(s)ds}, \;\; t \in J\,,$$

whose sign coincides with the sign of x. Then we have

$$x'(t)e^{\int_0^t M(s)ds} + M(t)x(t)e^{\int_0^t M(s)ds} \geq -N(t)[p(\max\{x,\Theta\})](t)e^{\int_0^t M(s)ds}, \quad t \in J\,,$$

that is

$$z'(t) \geq -N(t)[p(\max\{x,\Theta\})](t)e^{\int_0^t M(s)ds}, \quad t \in J. \tag{4}$$

It is easy to see that $z(0) = x(0)$, $z(t_*) > 0$ and $z(t_0) < 0$.

Two cases are possible.

Case 1. Let $x(T) < 0$. Then $z(T) < 0$. Take $\xi \in [0,T)$ such that $z(\xi) = \max_{t\in[0,T]} z(t) > 0$. Integrating (4) on $[\xi,T]$ and using (3), we obtain

$$\begin{aligned}- z(\xi) > z(T) - z(\xi) &\geq -\int_\xi^T N(t)[p(\max\{x,\Theta\})](t)e^{\int_0^t M(s)ds}dt \\ &\geq -\max_{t\in[0,T]}\{x(t)e^{\int_0^t M(s)ds}\} = -\max_{t\in[0,T]} z(t) = -z(\xi)\end{aligned}$$

which is a contradiction.

Case 2. Let $x(T) \geq 0$. Then $z(T) \geq 0$ and $z(0) \geq 0$ since $x(0) \geq rx(T) \geq 0$. Put $t_1 \in (0,T)$ such that $z(t_1) = \min_{t\in[0,T]} z(t) < 0$. Let $\xi \in [0,t_1)$ such that $z(\xi) = \max_{t\in[0,t_1]} z(t) \geq 0$. Integrating (4) on $[\xi,t_1]$ and using (3), we obtain

$$\begin{aligned}-z(\xi) &> z(t_1) - z(\xi) \geq -\int_\xi^{t_1} N(t)[p(\max\{x,\Theta\})](t)e^{\int_0^t M(s)ds}dt \\ &\geq -\max_{t\in[0,t_1]}\{x(t)e^{\int_0^t M(s)ds}\} = -\max_{t\in[0,t_1]} z(t) = -z(\xi).\end{aligned}$$

It is also a contradition.

Thus $x \geq 0$ on J. That ends the proof. □

3. Linear Problem

Now consider the problem

$$\begin{cases} x'(t) + M(t)x(t) + N(t)[p(x)](t) = \sigma(t), & t \in J\,, \\ x(0) = rx(T) + \lambda, & r \in (0,1]\,, \end{cases} \tag{5}$$

where $M, N \in C(J,\mathbb{R})$, $M(t) > 0$, $N(t) \geq 0$, $t \in J$, $p\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ and $\sigma \in C(J,\mathbb{R})$.

Let $y_0, z_0 \in C(J,\mathbb{R})$, $y_0 \le z_0$ on J. Define function $q\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as

$$[q(x)](t) = \max\{y_0(t), \min\{x(t), z_0(t)\}\} = \begin{cases} y_0(t) & \text{if } x(t) < y_0(t)\,, \\ x(t) & \text{if } y_0(t) \le x(t) \le z_0(t)\,, \\ z_0(t) & \text{if } x(t) > z_0(t)\,, \end{cases}$$

for $x \in C(J,\mathbb{R})$, $t \in J$.

Theorem 2. *Assume that:*

1. $M, N \in C(J,\mathbb{R})$, $M(t) > 0$, $N(t) \ge 0$, $t \in J$, $p\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ *and* $\sigma \in C(J,\mathbb{R})$.

2. $y_0, z_0 \in C^1(J,\mathbb{R})$, $y_0 \le z_0$ *are, respectively, lower and upper solutions of* (5), *i.e.*

$$\begin{cases} y_0'(t) + M(t)y_0(t) + N(t)[p(y_0)](t) \le \sigma(t), & t \in J\,, \\ y_0(0) \le r y_0(T) + \lambda, \end{cases}$$

and

$$\begin{cases} z_0'(t) + M(t)z_0(t) + N(t)[p(z_0)](t) \ge \sigma(t), & t \in J\,, \\ z_0(0) \ge r z_0(T) + \lambda\,. \end{cases}$$

3. $p\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ *satisfies conditions* (2) *and* (3).

4. p *is continuous and* $p([y_0, z_0])$ *is bounded in* $C(J,\mathbb{R})$.

5.

$$p(\max\{x - y_0, \Theta\}) \ge p(q(x)) - p(y_0) \quad \text{on} \quad J, x \in C(J,\mathbb{R}), \tag{6}$$

$$p(\max\{z_0 - x, \Theta\}) \ge p(z_0) - p(q(x)) \quad \text{on} \quad J, x \in C(J,\mathbb{R}), \tag{7}$$

$$p(\max\{f - g, \Theta\}) \ge p(f) - p(g) \quad \text{on} \quad J, f, g \in C(J,\mathbb{R}), f, g \in [y_0, z_0]. \tag{8}$$

Then there exists a unique solution of (5) *in the sector* $[y_0, z_0] = \{w \in C(J,\mathbb{R}) : y_0(t) \le w(t) \le z_0(t), t \in J\}$.

Proof. Consider the problem

$$\begin{cases} x'(t) + M(t)x(t) = \sigma(t) - N(t)[p(q(x))](t), & t \in J\,, \\ x(0) = r x(T) + \lambda, \quad r \in (0,1]\,. \end{cases} \tag{9}$$

Note that the map $q\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous. Define the operator $\Psi\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as $x \to \Psi(x)$, where

$$[\Psi(x)](t) =$$

$$e^{-\int_0^t M(s)ds}\left[\frac{\lambda + re^{-\int_0^T M(s)ds}\int_0^T(\sigma(\xi) - N(\xi)[p(q(x))](\xi))e^{\int_0^\xi M(s)ds}d\xi}{1-re^{-\int_0^T M(s)ds}}\right.$$
$$\left. + \int_0^t(\sigma(\xi) - N(\xi)[p(q(x))](\xi))e^{\int_0^\xi M(s)ds}d\xi\right].$$
$$= \int_0^T G(t,\xi)(\sigma(\xi) - N(\xi)[p(q(x))](\xi))d\xi + \frac{\lambda e^{-\int_0^t M(s)ds}}{1-re^{-\int_0^T M(s)ds}},$$

where

$$G(t,\xi) = \begin{cases} \dfrac{e^{\int_t^\xi M(s)ds}}{1-re^{-\int_0^T M(s)ds}}, & 0 \le \xi \le t \le T\,, \\ \dfrac{re^{\int_t^\xi M(s)\ ds}}{e^{\int_0^T M(s)ds} - r}, & 0 \le t < \xi \le T\,. \end{cases}$$

We show that Ψ is completely continuous. Ψ is continuous. If $S \subset C(J,\mathbb{R})$ is bounded set, then for every $v \in S, t \in J$

$$|[\Psi(v)](t)| \le \int_0^T |G(t,\xi)|\ d\xi(\|\sigma\| + N_1 R) + \frac{|\lambda|}{1-re^{-\int_0^T M(s)ds}} = \delta \tag{10}$$

because $\|N\| \le N_1$ and $\|p(q(v))\| \le R$, since $q(v) \in [y_0, z_0]$. Moreover

$$|[\Psi(v)]'(t)| = |-M(t)[\Psi(v)](t) + \sigma(t) - N(t)[p(q(v))](t)| \le M_1\delta + \|\sigma\| + N_1 R,$$

since $\|M(t)\| \le M_1$, $t \in J$. Thus $\Psi(S)$ is bounded and equicontinuous and therefore relatively compact. Hence Ψ is completely continuous.

If $u = \eta\Psi u$ for $\eta \in (0,1)$, then, by condition (10),

$$\|u\| = \|\eta\Psi u\| \le \|\Psi u\| = \max_{t\in[0,T]} |[\Psi(u)](t)|$$
$$\le KT(\|\sigma\| + N_1 R) + \frac{|\lambda|}{1-re^{-\int_0^T M(s)ds}},$$

since $|G(t,\xi)| \le K$. Thus Ψ has a fixed point $x \in C(J,\mathbb{R})$. We have

$$x(t) = \int_0^T G(t,\xi)\{\sigma(\xi) - N(\xi)[p(q(x))](\xi)\}\ ds + \frac{\lambda e^{-\int_0^t M(s)ds}}{1-re^{-\int_0^T M(s)ds}},\ \ t \in J.$$

Hence $x \in C^1(J,\mathbb{R})$ is a solution of (9).

Now we prove that $y_0 \le x \le z_0$ on J, which means that x is a solution of (5) since $[g(x)](t) = x(t)$ if $y_0(t) \le x(t) \le z_0(t)$, $t \in J$. Put $m = x - y_0 \in C^1(J, \mathbb{R})$. In view of (6) we have

$$\begin{aligned} m'(t) &+ M(t)m(t) + N(t)[p(\max\{m,\Theta\})](t) \\ &= x'(t) + M(t)x(t) - y_0'(t) - M(t)y_0(t) + N(t)[p(\max\{x-y_0,\Theta\})](t) \\ &\ge \sigma(t) - N(t)[p(q(x))](t) - \sigma(t) + N(t)[p(y_0)](t) \\ &\qquad + N(t)[p(q(x))](t) - N(t)[p(y_0)](t) = 0, \quad t \in J\,, \end{aligned}$$

and

$$m(0) = x(0) - y_0(0) \ge r(x(T) - y_0(T)) = rm(T).$$

In view of Theorem 1 we have $m \ge 0$ on J. Thus $x \ge y_0$ on J.

Similarly let $m = z_0 - x \in C^1(J, \mathbb{R})$. Using (7) we obtain

$$\begin{aligned} m'(t) &+ M(t)m(t) + N(t)[p(\max\{m,\Theta\})](t) \\ &= z_0'(t) + M(t)z_0(t) - x'(t) - M(t)x(t) + N(t)[p(\max\{z_0-x,\Theta\})](t) \\ &\ge 0, \quad t \in J\,, \end{aligned}$$

and

$$m(0) = z_0(0) - x(0) \ge r(z_0(T) - x(T)) = rm(T).$$

Thus, by Theorem 1, we have $m \ge 0$ on J and $z_0 \ge x$ on J.

Finally, we prove uniqueness of solution. Let $x_1, x_2 \in [y_0, z_0]$ be solutions of (5). Put $u = x_1 - x_2$, $v = x_2 - x_1$. Using (8) we have

$$\begin{aligned} u'(t) &+ M(t)u(t) + N(t)[p(\max\{u,\Theta\})](t) \\ &= x_1'(t) + M(t)x_1(t) - x_2'(t) - M(t)x_2(t) + N(t)[p(\max\{x_1-x_2,\Theta\})](t) \\ &\ge \sigma(t) - N(t)[p(x_1)](t) - \sigma(t) + N(t)[p(x_2)](t) + N(t)[p(x_1)](t) - N(t)[p(x_2)](t) \\ &= 0, \quad t \in J\,, \end{aligned}$$

and analogically

$$\begin{aligned} v'(t) &+ M(t)v(t) + N(t)[p(\max\{v,\Theta\})](t) \\ &= x_2'(t) + M(t)x_2(t) - x_1'(t) - M(t)x_1(t) + N(t)[p(\max\{x_2-x_1,\Theta\})](t) \\ &\ge 0, \quad t \in J. \end{aligned}$$

Moreover $u(0) = ru(T)$ and $v(0) = rv(T)$. In view of Theorem 1 we have $x_1 = x_2$ on J. □

4. Monotone Iterative Method

Theorem 2. *Assume that:*

1. $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, *there exist functions* $M, N \in C(J, \mathbb{R})$, $M(t) > 0, N(t) \geq 0$ *such that*

$$f(t, u(t), [p(u)](t)) - f(t, v(t), [p(v)](t)) \geq -M(t)(u(t) - v(t)) - N(t)([p(u)](t) - [p(v)](t)), \quad (11)$$

for $t \in J$, $u, v \in C(J, \mathbb{R})$, $y_0 \leq v \leq u \leq z_0$ *on* J.

2. $y_0, z_0 \in C^1(J, \mathbb{R})$, $y_0 \leq z_0$ *are, respectively, lower and upper solutions of* (1), *i.e.*

$$\begin{cases} y_0'(t) \leq f(t, y_0(t), [p(y_0)](t)), & t \in J\,, \\ y_0(0) \leq r y_0(T) + \lambda, \end{cases}$$

and

$$\begin{cases} z_0'(t) \geq f(t, z_0(t), [p(z_0)](t)), & t \in J\,, \\ z_0(0) \geq r z_0(T) + \lambda\,. \end{cases}$$

3. $p \colon C(J, \mathbb{R}) \to C(J, \mathbb{R})$ *satisfies conditions* (2), (3) *and* (6)-(8), p *is continuous and* $p([y_0, z_0])$ *is bounded in* $C(J, \mathbb{R})$.

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ *converging uniformly to the extremal solutions of* (1) *in* $[y_0, z_0]$.

Proof. For each $\eta \in [y_0, z_0]$, $t \in J$ consider the problem

$$\begin{cases} x'(t) + M(t)x(t) + N(t)[p(x)](t) = \sigma_\eta(t), \\ x(0) = r x(T) + \lambda, \end{cases} \quad (12)$$

where $\sigma_\eta(t) = f(t, \eta(t), [p(\eta)](t)) + M(t)\eta(t) + N(t)[p(\eta)](t)$. Note that y_0, z_0 are, respectively, lower and upper solutions of (12). Indeed, by (11), we have

$$\begin{aligned} y_0'(t) &+ M(t)y_0(t) + N(t)[p(y_0)](t) \\ &\leq f(t, y_0(t), [p(y_0)](t)) + M(t)y_0(t) + N(t)[p(y_0)](t) \\ &\leq f(t, \eta(t), [p(\eta)](t)) + M(t)(\eta(t) - y_0(t)) \\ &\quad + N(t)([p(\eta)](t) - [p(y_0)](t)) + M(t)y_0(t) + N(t)[p(y_0)](t) \\ &= f(t, \eta(t), [p(\eta)](t) + M(t)\eta(t) + N(t)[p(\eta)](t)\,, \end{aligned}$$

and similarly

$$\begin{aligned}
&z_0'(t) + M(t)z_0(t) + N(t)[p(z_0)](t) \\
&\qquad \geq f(t, z_0(t), [p(z_0)](t)) + M(t)z_0(t) + N(t)[p(z_0)](t) \\
&\qquad\qquad \geq f(t, \eta(t), [p(\eta)](t) + M(t)\eta(t) + N(t)[p(\eta)](t).
\end{aligned}$$

In view of Theorem 2, problem (12) has a unique solution $x_\eta \in [y_0, z_0]$.

Define operator $A\colon [y_0, z_0] \to [y_0, z_0]$ as $\eta \to A\eta = x_\eta$. First we prove that A is monotone. Let $y_0 \leq \eta_1 \leq \eta_2 \leq z_0$ on J and put $m = A\eta_2 - A\eta_1$. Then in view of (11) and (8) we have

$$\begin{aligned}
&m'(t) + M(t)m(t) + N(t)[p(\max\{m, \Theta\})](t) \\
&\qquad = (A\eta_2)'(t) + M(t)A\eta_2(t) - (A\eta_1)'(t) \\
&\qquad - M(t)A\eta_1(t) + N(t)[p(\max\{A\eta_2 - A\eta_1, \Theta\})](t) \\
&\qquad = f(t, \eta_2(t), [p(\eta_2)](t)) - f(t, \eta_1(t), [p(\eta_1)](t)) \\
&\qquad - N(t)[p(u_{\eta_2})](t) + M(t)\eta_2(t) + N(t)[p(\eta_2)](t) + N(t)[p(u_{\eta_1})](t) \\
&\qquad - M(t)\eta_1(t) - N(t)[p(\eta_1)](t) + N(t)[p(\max\{u_{\eta_2} - u_{\eta_1}, \Theta\})](t) \\
&\qquad \geq 0, \quad t \in J,
\end{aligned}$$

and

$$m(0) = u_{\eta_2}(0) - u_{\eta_1}(0) = r(u_{\eta_2}(T) - u_{\eta_1}(T)) = rm(T).$$

In view of Theorem 1 $A\eta_1 \leq A\eta_2$ on J. Define sequences $\{y_n\}$, $\{z_n\}$ such that $y_{n+1} = Ay_n$, $z_{n+1} = Az_n$ for $n \geq 0$. We have $y_0 \leq y_1$ on J. Indeed, put $m = y_1 - y_0$. Then

$$m'(t) + M(t)m(t) + N(t)[p(\max\{m, \Theta\})](t) \geq 0$$

and $m(0) \geq rm(T)$, so $m \geq 0$ on J. Similarly $z_1 \leq z_0$ on J.

We show that y_1 is a lower solution of (1). We have

$$\begin{aligned}
&y_1'(t) + M(t)y_1(t) + N(t)[p(y_1)](t) \\
&= f(t, y_0(t), [p(y_0)](t)) + M(t)y_0(t) + N(t)[p(y_0)](t) \\
&\leq M(t)y_1(t) + N(t)[p(y_1)](t) + f(t, y_1(t), [p(y_1)](t)).
\end{aligned}$$

Thus

$$y_1'(t) \leq f(t, y_1(t), [p(y_1)](t)).$$

Analogically z_1 is an upper solution of (1). Using mathematical induction it can be proved that

$$y_0 \leq y_1 \leq \cdots \leq y_n \leq z_n \leq \cdots \leq z_1 \leq z_0 \quad \text{on} \quad J.$$

Sequences $\{y_n, z_n\}$ are uniformly bounded and equicontinuous on J. In view of Arzela-Ascoli theorem there exist subsequences $\{y_{n_k}, z_{n_k}\}$ of $\{y_n, z_n\}$ converging uniformly on J to some continuous functions y, z respectively. Since y_{n_k}, z_{n_k} satisfy integral equations

$$\begin{cases} y_{n_{k+1}}(t) = y_{n_k}(0) + \int\limits_0^t f(s, y_{n_k}(s), [p(y_{n_k})](s))\ ds \\ + \int\limits_0^t M(s)[y_{n_k}(s) - y_{n_{k+1}}(s)] + N(s)[[p(y_{n_k})](s) - [p(y_{n_{k+1}})](s)]\ ds\,, \\ y_{n_{k+1}}(0) = r y_{n_{k+1}}(T) + \lambda, \end{cases}$$

$$\begin{cases} z_{n_{k+1}}(t) = z_{n_k}(0) + \int\limits_0^t f(s, z_{n_k}(s), [p(z_{n_k})](s))\ ds \\ + \int\limits_0^t M(s)[z_{n_k}(s) - z_{n_{k+1}}(s)] + N(s)[[p(z_{n_k})](s) - [p(z_{n_{k+1}})](s)]\ ds\,, \\ z_{n_{k+1}}(0) = r z_{n_{k+1}}(T) + \lambda, \end{cases}$$

we have

$$\begin{cases} y(t) = y(0) + \int\limits_0^t f(s, y(s), [p(y)](s))\ ds, \\ y(0) = r y(T) + \lambda, \end{cases}$$

$$\begin{cases} z(t) = z(0) + \int\limits_0^t f(s, z(s), [p(z)](s))\ ds, \\ z(0) = r z(T) + \lambda. \end{cases}$$

Hence y, z are solutions of (1). If $u \in [y_0, z_0]$ is a solution of (1) then $y_n \leq u \leq z_n, n \in \mathbb{N}$. When $n \to \infty$ we get $y \leq u \leq z$ which proves that y, z are extremal solutions of (1) in $[y_0, z_0]$. □

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