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ON k-TREES WITH GIVEN LEAFAGES

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Abstract: The leafage $l(G)$ of a chordal graph G is the minimum number of leaves of a tree whose subtrees compose an intersection representation of G. In [3] I.-J Lin, T.A.McKee and D.West give upper and lower bounds of $l(G)$ and algorithms for computing the leafages of non-clique k-trees. We show that for an integer t, there exists a k-tree whose leafage is t. And we consider some properties on k-trees.

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1. Introduction

In this paper, we consider finite undirected simple graphs. For a vertex v in G, the *neighborhood* of v is the set of vertices which are adjacent to v, and denoted by $N_G(v)$. For a subset $S \subset V(G)$, $\langle S\rangle_G$ is the induced subgraph of S. A *clique* in the graph G is the vertex set of a complete subgraph. For a clique C, C is a k-clique if $|C| = k$. In some cases we consider that a clique is a complete subgraph. A vertex v is called a *simplicial vertex* if $N_G(v)$ is a clique. The neighborhood of a simplicial vertex is called a *simplicial neighborhood*. For a graph G, the *derived graph* G^- is the induced subgraph obtained by deleting all simplicial vertices of G.

A graph G is a *chordal graph* if and only if every cycle of length greater than or equal to four has a chord, that is, G does not contain induced subgraphs isomorphic to C_n ($n \geq 4$). Many researches deal with properties of chordal graphs (see [1], [2], [4], [5]). We already know the following results on chordal

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graphs.

Theorem 1. (see Buneman [1], Gavril [2] and Walter [5]) *For a graph G, G is a chordal graph if and only if G is the intersection graph of a family of subtrees of a tree.*

For a chordal graph G, a tree of Theorem 1 is called a *host tree* of G and a family of subtrees of Theorem 1 is called a *subtree representation* of G. The *leafage* $l(G)$ of a chordal graph G is the minimum number of leaves of a host tree in a subtree representation of G.

In [3] Lin, McKee and West also deal with k-trees. A *k-tree* is a chordal graph that can be constructed from a complete graph K_k by a sequence of vertex additions in which the neighborhood of each new vertex is a k-clique of the current graph. K_k and K_{k+1} are k-trees, and trees are 1-trees. k-trees other than complete graphs are called *non-clique k-trees.*

Proposition 2. (see Lin, McKee and West [3]) *Let T be a tree other than a star. Then $l(T)$ is the number of leaves in the derived graph T^-.*

In [3] Lin, McKee and West report some properties of k-trees as follows.

Proposition 3. (see Rose [4]) *Every minimal cutset of a k-tree induces a k-clique.*

Proposition 4. (see Lin, McKee and West [3]) *Let G be a non-clique k-tree. Then simplicial vertices of G are degree k vertices. And the set of simplicial vertices forms an independent set of G.*

Fact 5. (see [3]) *(1) Non-clique k-trees are the connected graphs in which the largest clique has $k+1$ vertices and every minimal cutset induces a k-clique.*
(2) If we delete a simplicial vertex of a k-tree, we obtain a smaller k-tree. And the construction procedure defining a k-tree can begin from any k-tree.

By Fact 5, we obtain the next result.

Proposition 6. *Let G be a non-clique k-tree. Then G^- is also k-tree.*

Proof. Let v be a simplicial vertex of G. Since v is not adjacent to other simplicial vertices of G, simplicial vertices of G other than v are also simplicial vertices of $G - v$. Thus G^- is also k-tree by Fact 5 (2). □

We already know the following properties on k-trees. N_2 is a graph with two vertices and whose edge set is an empty set.

Fact 7. *(1) A clique k-tree is K_k or K_{k+1}.*

(2) The minimal non-clique k-tree is $N_2 + K_k$ and $|V(G)| \geq k + 2$ for a non-clique k-tree G.

(3) For a non-clique k-tree G, a degree of each vertex of G is greater than or equal to k and G^- is a connected graph.

For a non-clique k-tree G, $r(G)$ is the number of distinct simplicial neighborhoods of G that are not cutsets in G^-. In [3] Lin, McKee and West give the following result on leafages of k-trees.

Theorem 8. (see Lin, McKee and West [3]) *Let G be a non-clique k-tree. Then $l(G) = \max\{2, r(G)\}$.*

This theorem gives a method of computing for leafages of non-clique k-trees. In the next section, we consider a k-tree with leafage t, where t is any integer.

2. On Leafages of k-Trees

Based on properties of k-trees in Section 1, we have the following result.

Proposition 9. *Every vertex of a non-clique k-tree G is in a maximal clique K_{k+1} of G.*

Proof. We use induction on the number of vertices of G. First we consider the case $|V(G)| = k + 2$. Then G is a minimal non-clique k-tree and $N_2 + K_k$. So every vertex of G is in a maximal clique K_{k+1}.

Next we consider the case $|V(G)| \geq k + 3$. For a simplicial vertex v, $G - v$ is a non-clique k-tree. By the induction hypothesis, each vertex of $G - v$ is in a maximal clique K_{k+1}. Since v is a simplicial vertex of G, $\{v\} \cup N_G(v)$ induces a clique K_{k+1}. So v and each vertex of $N_G(v)$ are in a maximal clique K_{k+1} of G. □

In [3] Lin, McKee and West show the following property on non-clique k-trees.

Proposition 10. (see Lin, McKee and West [3]) *Let G be a non-clique k-tree with distinct simplicial neighborhoods. Then G has distinct simplicial neighborhoods that are not cutsets of G^-.*

For a non-clique k-tree G, if G has exactly one simplicial neighborhood, then G is $N_s + K_k$, because the set of simplicial vertices is the independent set. Thus we obtain the following result.

Proposition 11. *Let G be a non-clique k-tree and s be an integer greater than or equal to 2. If G is not isomorphic to $N_s + K_k$, then G has distinct simplicial neighborhoods that are not cutsets of G^-.*

Using these properties on simplicial neighborhoods that are not cutsets of the derived graph, we obtain the following results on k-trees with given number of leafages. A *caterpillar* T is a tree for which there exists a path P such that each vertex of T is in P or adjacent to a vertex of P. By Proposition 2, for a tree T other than stars, $l(T) = 2$ if and only if T^- has 2 leaves, that is, T^- is a path, and T is a caterpillar. So we obtain the following result.

Proposition 12. *Let T be a tree other than a star, and $|V(T)| \geq 4$. Then $l(T) = 2$ if and only if T is a caterpillar.*

In the case $l(T) \geq 3$, we have the following fact.

Fact 13. *Let T be a tree other than a star, and $|V(T)| \geq 4$. Then for an integer $l \geq 3$, $l(T) = l$ if T^- is homeomorphic to $K_{1,l}$.*

Next we deal with k-trees other than 1-trees. For an integer $k \geq 2$, $l(N_2 + K_k) = 2$. We also show a constructive method on k-trees.

Proposition 14. *Let G be a non-clique k-tree and u be a simplicial vertex of G. H is obtained from G adding a new vertex v, where v is adjacent to u and some $k-1$ vertices of $N_G(u)$. Then H is a non-clique k-tree and v is a simplicial vertex of H.*

Proof. Since u is a simplicial vertex of G, $\{u\} \cup N_G(u)$ is a $k+1$-clique. Since $N_H(v) \subseteq \{u\} \cup N_G(u)$, $N_H(v)$ is a k-clique. Thus H is a k-tree and v is a simplicial vertex of H. And there exists a vertex of $N_G(u)$ which is not adjacent to v. Thus H is not clique, and H is a non-clique k-tree. □

Proposition 15. *Let G be a non-clique k-tree, and u be a simplicial vertex of G whose simplicial neighborhood is X. H is obtained from G adding a new vertex v, where v is adjacent to u and some $k-1$ vertices of X. Then $H^- - N_H(v)$ is connected.*

Proof. Let w be a vertex of X which is not adjacent to v in H. Since $H^- - u = G^-$ and $N_H(v) = \{u\} \cup (X - \{w\})$, $H^- - N_H(v) = H^- - (\{u\} \cup (X - \{w\}) = G^- - (X - \{w\})$. By Propositions 3 and 6, G^- is a k-tree and a minimal cutset of G^- is a k-clique. Since $|N_G(u)| = |X| = k$ and $|X - \{w\}| = k-1$, $X - \{w\}$ is not a cutset of G^-. Therefore $G^- - (X - \{w\}) = H^- - N_H(v)$ is connected. □

Using these propositions, we obtain the following result.

Theorem 16. *For an integer $l \geq 2$, there exists a non-clique k-tree with $l(G) = l$, where $k \geq 2$.*

Proof. We use induction on the number of $l(G)$. In the case $l(G) = 2$, $l(N_2 + K_k) = 2$ for $k \geq 2$. Let G be a non-clique k-tree with $l(G) = l \geq 3$. Then there exists a simplicial neighborhood X which is not a cutset of G^- by Proposition 11.

The graph I is obtained from G adding new vertices u and v which are adjacent to all vertices of X. Then u and v are simplicial vertices of I, I is a non-clique k-tree and $r(I) = r(G) = l$.

Next we construct a new graph H from I adding new vertices u' and v', where u' is adjacent to u and some $k-1$ vertices of $N_I(u) = X$ and v' is adjacent to v and some $k-1$ vertices of $N_I(v) = X$. Then u' and v' are simplicial vertices of H and H is a non-clique k-tree.

Since $u \in N_H(u'), u \notin N_H(v'), v \in N_H(v')$ and $v \notin N_H(u')$, $N_H(u') \neq N_H(v')$. So simplicial neighborhoods of u' and v' are distinct. Let w be a simplicial vertex of G whose simplicial neighborhood is X. Since u, v and w are simplicial vertices of I, $u, v \notin N_H(w)$ and simplicial neighborhoods of u', v', w are distinct. By Proposition 15, $H^- - N_H(u')$ and $H^- - N_H(v')$ are connected. Thus simplicial neighborhoods $N_H(u')$ and $N_H(v')$ are not cutsets of H^-. On the other hand, since $N_{H^-}(u) = N_{H^-}(v) = X$, $N_H(w) = X$ is a cutset of H^-. Thus $r(H) = r(G) - 1 + 2 = r(G) + 1$ and $l(H) = l(G) + 1 = l + 1$ by Theorem 8. □

Propositon 2 deals with properties of $l(G)$. For $l \geq 2$ and $k \geq 2$, Propositon 12 and Fact 13 mean that there exists a 1-tree T with $l(T) = l$, and Theorem 16 means that there exists a k-tree G with $l(G) = l$.

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