THE COMPACT AND NONCOMPACT STRUCTURES FOR
GENERALIZED CAMASSA-HOLM-KP AND
BOUSSINESQ EQUATIONS

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Abstract: The aim of this paper is to investigate two generalized nonlinear Boussinesq equations in (1+1) dimensional space and two modified nonlinear Camassa-Holm-KP equations in (2+1) dimensional space. The compactons, solitons, solitary patterns and periodic solutions for each of the equations are expressed analytically under various circumstances. The qualitative change in the physical structures of the solutions is also analyzed.

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1. Introduction

Nonlinear partial differential equations which describe the nonlinear waves existing in nature world have been investigated by many mathematical approaches, such as the Darboux transformation, the inverse scattering method.
the Bäcklund transformation, the Painlevé analysis, the tri-Hamiltonian operators, the finite difference method, the Adomian decomposition method, the tanh method, the sin−cos method, and so on [3], [4], [5], [2].

Fan and Zhang [3, 4, 5] derived the extended tanh method and homogeneous balance method and used them to study the generalized mKdV equation and the generalized ZK equation. The method presented in [3, 4, 5] was shown as a powerful tool to seek exact solutions of nonlinear equations. Many researchers have used the extended tanh method to find travelling wave solutions for various forms of partial differential equations.

Boyd [1] pointed that the Camassa-Holm equation is a model for small amplitude shallow water waves and it gives peaked periodic solutions which have discontinuous first derivative at each peak. New peaked solitary wave solutions for a modified Camassa-Holm equation were obtained by Tian and Song [9] in which many features of the solutions were discussed.


\[
\begin{align*}
(u_t + 2ku_x - u_{xx} - au^n u_x)_x + uy_y &= 0, \quad a > 0, \\
(u_t + 2ku_x - u_{xx} + au^n (u^n)_x)_x + uy_y &= 0, \quad a > 0,
\end{align*}
\]

and the modified Boussinesq equations in [11] are written by

\[
\begin{align*}
&u_{tt} - a(u^{n+1})_{xx} - b[u(u^n)_{xx}]_{xx} = 0, \\
&u_{tt} - a(u^{2n})_{xx} - b[u(u^n)_{xx}]_{xx} = 0.
\end{align*}
\]

The aim of this work is to develop a mathematical method, which is different from those in previous work [3, 4, 5, 2, 1, 9, 10, 11], to further study equations (1), (2), (3) and (4). In some sense, the method used here may be said to depend on the technique of reducing order for solving differential equations. The compactons, solitary patterns, solitons and periodic solutions for each of the four equations are expressed analytically. Moreover, assuming that the wave variable takes the form of \( \xi = \mu(x - ct) \) or \( \xi = \mu(x + y - ct) \), where \( \mu \) is an arbitrary nonzero number, we find that the travelling wave solutions obtained by our method are independent of wave number \( \mu \), while Wazwaz [10, 11] chose \( \mu \) as a fixed number.
2. The Generalized Camassa-Holm-Kp Equation (1)

We seek the travelling wave solutions for equation (1) in the form \( u = u(\xi) \) with wave variable \( \xi = \mu(x + y - ct) \), where constants \( \mu \neq 0, c \neq 0 \). The wave variable \( \xi \) turns equation (1) into the following ordinary differential equation (ODE)

\[
\mu[-\mu u_{\xi} + 2k\mu u + \mu^3 c u_{\xi\xi\xi} - au^n u_{\xi}] + \mu^2 u_{\xi\xi} = 0, \quad n \neq -1, \quad n \neq -2.
\]

Integrating equation (5) twice and ignoring the integral constants yield the second order ODE

\[
(-c + 2k + 1)u - \frac{au^{n+1}}{n+1} + \mu^2 cu_{\xi\xi} = 0.
\]

Using transformation \( \frac{du}{d\xi} = Z \), we have \( u_{\xi\xi} = Z \frac{dZ}{du} \) which changes equation (6) into the first order ODE

\[
\mu^2 c Z \frac{dZ}{du} = (c - 2k - 1)u + \frac{au^{n+1}}{n+1}.
\]

Integrating equation (7) with respect to \( u \) and ignoring the constant of integration, we have

\[
\left( \frac{du}{d\xi} \right)^2 = Z^2 = \frac{1}{\mu^2} \left( \frac{u^{n+2}}{c} - \frac{nu^n}{c(n+2)(n+1)} \right).
\]

Setting \( u^n = W \), we get

\[
u = W^{\frac{1}{n}}, \quad du = \frac{1}{n} W^{\frac{1}{n}-1} dW,
\]

which makes equation (8) become

\[
W \sqrt{\frac{(c - 2k - 1)}{c} + \frac{2aW}{c(n+2)(n+1)}} = \pm \frac{n}{\mu} d\xi.
\]

Case 2.1. \( \frac{(c-2k-1)}{c} < 0 \) and \( n > 0 \). Solving equation (10), we have

\[
W = \left( \frac{2k + 1 - c}{2a} \right) \frac{(n+1)(n+2)}{2n} \sec^2 \left[ \frac{n\xi}{2\mu} \sqrt{\frac{2k + 1 - c}{c}} \right]
\]

or

\[
W = \left( \frac{2k + 1 - c}{2a} \right) \frac{(n+1)(n+2)}{2n} \csc^2 \left[ \frac{n\xi}{2\mu} \sqrt{\frac{2k + 1 - c}{c}} \right].
\]

Using \( \xi = x + y - ct \), we get that the periodic solutions of equation (1) are expressed in the form

\[
u = \left\{ \frac{(2k + 1 - c)(n+1)(n+2)}{2a} \sec^2 \left[ \frac{n\xi}{2} \sqrt{\frac{2k + 1 - c}{c}} (x + y - ct) \right] \right\}^{\frac{1}{n}}
\]
and
\[ u = \left\{ \frac{(2k + 1 - c)(n + 1)(n + 2)}{2a} \csc^2 \left[ \frac{n}{2} \sqrt{\frac{2k + 1 - c}{c}(x + y - ct)} \right] \right\}^{\frac{1}{n}}. \] (14)

Case 2.2. \( \frac{(c - 2k - 1)}{c} > 0 \) and \( n > 0 \). The solitons of equation (1) take the form
\[ u = \left\{ \frac{(2k + 1 - c)(n + 1)(n + 2)}{2a} \sech^2 \left[ \frac{n}{2} \sqrt{\frac{c - 2k - 1}{c}(x + y - ct)} \right] \right\}^{\frac{1}{n}}. \] (15)

and
\[ u = \left\{ -\frac{(2k + 1 - c)(n + 1)(n + 2)}{2a} \csch^2 \left[ \frac{n}{2} \sqrt{\frac{c - 2k - 1}{c}(x + y - ct)} \right] \right\}^{\frac{1}{n}}. \] (16)

Case 2.3. \( \frac{(c - 2k - 1)}{c} < 0, n < 0, n \neq -1 \) and \( n \neq -2 \). The compacton solutions are given by
\[
\begin{cases}
  u = \left\{ \frac{2a}{(2k + 1 - c)(n + 1)(n + 2)} \cos^2 \left[ \frac{n}{2} \sqrt{\frac{2k + 1 - c}{c}(x + y - ct)} \right] \right\}^{\frac{1}{n}}, \\
  \quad | \frac{n}{2} \sqrt{\frac{2k + 1 - c}{c}(x + y - ct)} | < \frac{\pi}{2}, \\
  u = 0, \quad \text{otherwise},
\end{cases}
\] (17)

and
\[
\begin{cases}
  u = \left\{ \frac{2a}{(2k + 1 - c)(n + 1)(n + 2)} \sin^2 \left[ \frac{n}{2} \sqrt{\frac{2k + 1 - c}{c}(x + y - ct)} \right] \right\}^{\frac{1}{n}}, \\
  \quad | \frac{n}{2} \sqrt{\frac{2k + 1 - c}{c}(x + y - ct)} | < \pi, \\
  u = 0, \quad \text{otherwise}.
\end{cases}
\] (18)

Case 2.4. \( \frac{(c - 2k - 1)}{c} > 0, n < 0, n \neq -1 \) and \( n \neq -2 \). We obtain the solitary pattern solutions as follows
\[ u = \left\{ \frac{2a}{(2k + 1 - c)(n + 1)(n + 2)} \cosh^2 \left[ \frac{n}{2} \sqrt{\frac{c - 2k - 1}{c}(x + y - ct)} \right] \right\}^{\frac{1}{n}}. \] (19)

and
\[ u = \left\{ \frac{2a}{(2k + 1 - c)(n + 1)(n + 2)} \sinh^2 \left[ \frac{n}{2} \sqrt{\frac{c - 2k - 1}{c}(x + y - ct)} \right] \right\}^{\frac{1}{n}}. \] (20)

The exact travelling wave solutions from (13) to (20) are independent of wave number \( \mu \) while Wazwaz [10] found \( \mu \) as a concrete number.
3. The Generalized Camassa-Holm-Kp Equation (2)

The wave variable $\xi = \mu(x + y - ct)$ with $\mu \neq 0$ and $c \neq 0$ carries equation (2) into the ODE

$$\mu[-\mu cu_x + 2k\mu u_x + \mu^3 cu_x - \mu au^n(u^n)_{\xi}]_{\xi} + \mu^2 u_{\xi\xi} = 0, \quad n \neq \pm \frac{1}{2}. \quad (21)$$

Integrating equation (21) twice and letting the constant of integration to be zero yield

$$(-c + 2k + 1)u + \frac{a}{2} u^{2n} + \mu^2 cu_{\xi\xi} = 0. \quad (22)$$

Setting $u_{\xi} = Z$, we have $u_{\xi\xi} = Z \frac{dZ}{du}$ and

$$(-c + 2k + 1)u + \frac{a}{2} u^{2n} + \mu^2 cZ \frac{dZ}{du} = 0, \quad (23)$$

which results in

$$\left(\frac{du}{d\xi}\right)^2 = \frac{1}{\mu^2} \left( -c - 2k - 1 \right) u^2 - \frac{a u^{2n+1}}{c(2n+1)}. \quad (24)$$

Letting $W = u^{2n-1}$, we have

$$u = W^{\frac{1}{2n-1}}, \quad du = \frac{1}{2n-1} W^{-\frac{1}{2n-1}} dW. \quad (25)$$

It follows from (25) that equation (24) becomes

$$dW \sqrt{\frac{c-2k-1}{e} - \frac{aW}{c(2n+1)}} = \pm \frac{2n-1}{\mu} d\xi. \quad (26)$$

Case 3.1. $\frac{c-2k-1}{c} < 0$ and $n > \frac{1}{2}$. It follows from equation (26) that the periodic solutions of equation (2) take the form

$$u = \left( \frac{(c - 2k - 1)(2n + 1)}{a} \right) \sec^2 \left[ \frac{2n - 1}{2} \sqrt{\frac{2k + 1 - c}{c}} (x + y - ct) \right]^{\frac{1}{2n-1}} \quad (27)$$

and

$$u = \left( \frac{(c - 2k - 1)(2n + 1)}{a} \right) \csc^2 \left[ \frac{2n - 1}{2} \sqrt{\frac{2k + 1 - c}{c}} (x + y - ct) \right]^{\frac{1}{2n-1}}. \quad (28)$$

Case 3.2. $\frac{c-2k-1}{c} > 0$ and $n > \frac{1}{2}$. We obtain the following solitons for equation (2)
\[ u = \left\{ \frac{(c - 2k - 1)(2n + 1)}{a} \right\} \text{sech}^2\left( \frac{2n - 1}{2} \sqrt{\frac{c - 2k - 1}{c}} (x + y - ct) \right) \left( \frac{1}{2n-1} \right) \]  

and

\[ u = \left\{ -\frac{(c - 2k - 1)(2n + 1)}{a} \right\} \times \text{csch}^2\left( \frac{2n - 1}{2} \sqrt{\frac{c - 2k - 1}{c}} (x + y - ct) \right) \left( \frac{1}{2n-1} \right). \]  

**Case 3.3.** \( c - 2k - 1 < 0, \ n < \frac{1}{2} \) and \( n \neq -\frac{1}{2} \). The compacton solutions are expressed in the form

\[
\begin{cases}
  u = \left\{ -\frac{a}{(c-2k-1)(2n+1)} \cos^2\left( \frac{2n-1}{2} \sqrt{\frac{2k+1-c}{c}} (x + y - ct) \right) \left( \frac{1}{2n-1} \right), \\
  \quad \left| \frac{2n-1}{2} \sqrt{\frac{2k+1-c}{c}} (x + y - ct) \right| < \frac{\pi}{2}, \\
  u = 0, \quad \text{otherwise},
\end{cases}
\]

and

\[
\begin{cases}
  u = \left\{ \frac{a}{(c-2k-1)(2n+1)} \sin^2\left( \frac{2n-1}{2} \sqrt{\frac{2k+1-c}{c}} (x + y - ct) \right) \left( \frac{1}{2n-1} \right), \\
  \quad \left| \frac{2n-1}{2} \sqrt{\frac{2k+1-c}{c}} (x + y - ct) \right| < \pi, \\
  u = 0, \quad \text{otherwise}.
\end{cases}
\]

**Case 3.4.** \( c - 2k - 1 > 0, \ n < \frac{1}{2} \) and \( n \neq -\frac{1}{2} \). This case admits equation (2) to have the solitary pattern solutions given by

\[
\begin{cases}
  u = \left\{ \frac{a}{(c-2k-1)(2n+1)} \cosh^2\left( \frac{2n-1}{2} \sqrt{\frac{c+2k-1}{c}} (x + y - ct) \right) \left( \frac{1}{2n-1} \right), \\
  \quad \left| \frac{2n-1}{2} \sqrt{\frac{2k+1-c}{c}} (x + y - ct) \right| < \frac{\pi}{2}, \\
  u = 0, \quad \text{otherwise},
\end{cases}
\]

and

\[
\begin{cases}
  u = \left\{ -\frac{a}{(c-2k-1)(2n+1)} \sinh^2\left( \frac{2n-1}{2} \sqrt{\frac{c+2k-1}{c}} (x + y - ct) \right) \left( \frac{1}{2n-1} \right), \\
  \quad \left| \frac{2n-1}{2} \sqrt{\frac{2k+1-c}{c}} (x + y - ct) \right| < \pi, \\
  u = 0, \quad \text{otherwise}.
\end{cases}
\]

**4. The Modified Boussinesq Equation (3)**

The wave variable \( \xi = \mu(x - ct) \) with \( \mu \neq 0 \) and \( c \neq 0 \) makes the modified Boussinesq equation (3) become the following ordinary differential equation (ODE)

\[
\mu^2 c^2 u_{\xi\xi} - a \mu^2 (u^{n+1})_{\xi\xi} - \mu^4 b[u^n]_{\xi\xi} = 0. \]  

(35)
Integrating (35) twice and letting the integral constants to be zero, we find
\[ c^2 - au^n - b\mu^2(u^n)_{\xi\xi} = 0. \] (36)
The transform \( Z = \frac{du^n}{d\xi} \) and the identity \( du^n = nu^{n-1}du \) turn (36) into
\[ (c^2 - au^n)nu^{n-1}du - b\mu^2 Z dZ = 0. \] (37)
Integrating (37) and ignoring the integral constant, we get
\[ b\mu^2 \left( \frac{du^n}{d\xi} \right)^2 = 2c^2 u^n - au^{2n}. \] (38)
Setting \( u^n = W^2 \) yields
\[ u = W^2, \quad du = \frac{2}{n} W^{\frac{2}{n}-1} dW. \] (39)
Substituting equation (39) into equation (38), we obtain
\[ \left( \frac{dW}{d\xi} \right)^2 = \frac{1}{\mu^2} \left[ c^2 - \frac{1}{2b} a \right] W^2. \] (40)

**Case 4.1.** \( \frac{a}{b} > 0 \) and \( n > 0 \). Solving ordinary differential equation (40) and noting (39), we find the compacton solutions of equation (3) take the form
\[ \begin{cases} 
  u = \left\{ \frac{2c^2}{a} \cos^2 \left[ \frac{1}{2} \sqrt{\frac{a}{b}}(x - ct) \right] \right\}^{\frac{1}{n}}, & \text{if } \frac{1}{2} \sqrt{\frac{a}{b}}(x - ct) < \frac{\pi}{2}, \\
  u = 0, & \text{otherwise},
\end{cases} \] (41) and
\[ \begin{cases} 
  u = \left\{ \frac{2c^2}{a} \sin^2 \left[ \frac{1}{2} \sqrt{\frac{a}{b}}(x - ct) \right] \right\}^{\frac{1}{n}}, & \text{if } \frac{1}{2} \sqrt{\frac{a}{b}}(x - ct) < \pi, \\
  u = 0, & \text{otherwise}.
\end{cases} \] (42)

**Case 4.2.** \( \frac{a}{b} < 0 \) and \( n > 0 \). The solitary pattern solutions are expressed in the form
\[ u = \left\{ \frac{2c^2}{a} \cosh^2 \left[ \frac{1}{2} \sqrt{-\frac{a}{b}}(x - ct) \right] \right\}^{-\frac{1}{n}} \] (43) and
\[ u = \left\{ -\frac{2c^2}{a} \sinh^2 \left[ \frac{1}{2} \sqrt{-\frac{a}{b}}(x - ct) \right] \right\}^{-\frac{1}{n}}. \] (44)

**Case 4.3.** \( \frac{a}{b} > 0 \) and \( n < 0 \). The periodic solutions of the modified Boussinesq equation (3) are given by
\[ u = \left\{ \frac{a}{2c^2} \sec^2 \left[ \frac{1}{2} \sqrt{\frac{a}{b}}(x - ct) \right] \right\}^{-\frac{1}{n}} \] (45) and
\[ u = \left\{ \frac{a}{2c^2} \csc^2 \left[ \frac{1}{2} \sqrt{\frac{a}{b}}(x - ct) \right] \right\}^{-\frac{1}{n}}. \] (46)
Case 4.4. \( \frac{a}{b} < 0 \) and \( n < 0 \). This case permits equation (3) to have the soliton solutions given in the form
\[
 u = \left\{ \frac{a}{2c^2} \text{sech}^2 \left[ \frac{1}{2} \sqrt{-\frac{a}{b}} (x - ct) \right] \right\}^{-\frac{1}{n}} \tag{47}
\]
and
\[
 u = \left\{ -\frac{a}{2c^2} \text{csch}^2 \left[ \frac{1}{2} \sqrt{-\frac{a}{b}} (x - ct) \right] \right\}^{-\frac{1}{n}}. \tag{48}
\]

5. The Modified Boussinesq Equation (4)

The wave variable \( \xi = \mu(x - ct) \) with \( \mu \neq 0 \) and \( c \neq 0 \) carries equation (4) into the following ODE
\[
 \mu^2 c^2 u_{\xi\xi} - a\mu^2 (u^{2n})_{\xi\xi} - b\mu^4 [u^n (u^n)]_{\xi\xi} = 0. \tag{49}
\]
Integrating (49) twice and ignoring the constants of integration give rise to
\[
 c^2 u - au^{2n} - b\mu^2 u^n (u^n)_{\xi\xi} = 0. \tag{50}
\]
Transforming \( \frac{du^n}{d\xi} = Z \) makes above equation to become
\[
 c^2 u - au^{2n} - b\mu^2 u^n Z \frac{dZ}{nu^{n-1} du} = 0. \tag{51}
\]
Integrating (51) with respect to \( u \) and letting the constant of integration to be zero, we have
\[
 \left( \frac{du^n}{d\xi} \right)^2 = \frac{n}{b\mu^2} \left( 2c^2 u - \frac{au^{2n}}{n} \right). \tag{52}
\]
Setting \( u^{2n-1} = W^2 \) and carrying out some calculations, we have
\[
 \left( \frac{dW}{d\xi} \right)^2 = \left( \frac{2n-1}{2n} \right)^2 \frac{n}{b\mu^2} \left( 2c^2 - \frac{aW^2}{n} \right). \tag{53}
\]

Case 5.1. \( \frac{a}{b} > 0 \) and \( 2n - 1 > 0 \). Solving differential equation (53), we find the compacton solutions of equation (4) written by
\[
 \left\{ \begin{array}{ll}
 u = \left\{ \frac{2c^2}{a} \cos^2 \left[ \frac{2n-1}{2n} \sqrt{-\frac{a}{b}} (x - ct) \right] \right\}^{\frac{1}{2n-1}}, & | \frac{2n-1}{2n} \sqrt{-\frac{a}{b}} (x - ct) | < \frac{\pi}{2}, \\
 u = 0, & \text{otherwise},
\end{array} \right. \tag{54}
\]
and
\[
 \left\{ \begin{array}{ll}
 u = \left\{ \frac{2c^2}{a} \sin^2 \left[ \frac{2n-1}{2n} \sqrt{-\frac{a}{b}} (x - ct) \right] \right\}^{\frac{1}{2n-1}}, & | \frac{2n-1}{2n} \sqrt{-\frac{a}{b}} (x - ct) | < \pi, \\
 u = 0, & \text{otherwise}.
\end{array} \right. \tag{55}
\]

Case 5.2. \( \frac{a}{b} < 0 \) and \( 2n - 1 > 0 \). The solitary pattern solutions of equation (4) are given by
\[ u = \left\{ \frac{2c^2n}{a} \cosh^2 \left[ \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}(x - ct)} \right] \right\}^{\frac{1}{2n-1}} \] (56)

and

\[ u = \left\{ -\frac{2c^2n}{a} \sinh^2 \left[ \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}(x - ct)} \right] \right\}^{\frac{1}{2n-1}}. \] (57)

Case 5.3. \( \frac{a}{b} > 0, n \neq 0 \) and \( 2n - 1 < 0 \). This case allows equation (4) to have periodic solutions in the form

\[ u = \left\{ \frac{a}{2c^2n} \sec^2 \left[ \frac{2n - 1}{2n} \sqrt{\frac{a}{b}(x - ct)} \right] \right\}^{\frac{1}{2n}} \] (58)

and

\[ u = \left\{ -\frac{a}{2c^2n} \csc^2 \left[ \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}(x - ct)} \right] \right\}^{\frac{1}{2n}}. \] (59)

Case 5.3. \( \frac{a}{b} < 0, n \neq 0 \) and \( 2n - 1 < 0 \). This case makes equation (4) have the solutions given by

\[ u = \left\{ \frac{a}{2c^2n} \text{sech}^2 \left[ \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}(x - ct)} \right] \right\}^{\frac{1}{2n}} \] (60)

and

\[ u = \left\{ -\frac{a}{2c^2n} \text{csch}^2 \left[ \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}(x - ct)} \right] \right\}^{\frac{1}{2n}}. \] (61)

6. Conclusion

A mathematical method based on solving differential equations is developed to study the generalized Camassa-Holm-KP equations (1) and (2), and the modified Boussinesq equations (3) and (4). The compactons, solitary patterns, solitons and periodic solutions obtained in this paper for each of these four equations are in full agreement with those presented in Wazwaz [10, 11] in which the tanh method and sine-cosine technique are used. However, our method shows that the travelling wave solutions for equations (1) to (4) are independent of wave number \( \mu \) which is an arbitrary nonzero number.

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