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THE Q-PROCESS IN A MULTITYPE BRANCHING PROCESSES

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Abstract: The importance of the Q-process in the study of Galton-Watson processes is well known. The Q-process for multitype branching processes has not yet been studied. This paper fills this gap. Moreover we compute explicitly some classical quantities when the generating function of the fertility law of the multitype branching process is a fractional linear functions with the same denominator for each type.

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1. Introduction

The aim of this paper is to extend the study of Q-processes to the case of multitype branching processes (for an introduction see Athreya and Ney [1], p. 56). In the spirit of Joffe and Letac [2] we compute explicitly some classical quantities when the fertility law of the multitype branching process is a fractional linear function with the same denominator for each component. This paper is a complement to the paper of Joffe and Letac [2]; therefore we assume that the

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reader is familiar with its contents and we use the same notations.

Let us also recall that for two vectors $\mathbf{t}, \mathbf{s}$ in $\mathbb{R}^k$ we denote by $\mathbf{ts}$ the vector whose components are the product of the components of $\mathbf{t}$ and $\mathbf{s}$ and by $\mathbf{s^t}$ the vector whose i component is $s_i^{t_i}$. Let us denote by $\mathbf{I}_k$ the unit matrix of order k and by $\mathbf{e_i}$ the i^{th} unit vector of $\mathbb{R}^k$.

In this paper we consider $\mathbf{Z} = (\mathbf{Z}_n)_{n\geq 0}$, a k-type multitype branching process with generating function $\mathbf{f} := \mathbb{E}(\mathbf{s}^{\mathbf{Z}_1}|\mathbf{Z}_0 = \mathbf{e}_i)$, it is well known that the generating function $\mathbf{f}_n := \mathbb{E}(\mathbf{s}^{\mathbf{Z}_n}|\mathbf{Z_0} = \mathbf{e_i})$ is the n^{th} iterate of $\mathbf{f}$.

Let $\mathbf{M}$ be the classical mean matrix of the birth law. Let ρ denote the eigenvalue of $\mathbf{M}$ of largest module, with left eigenvector $\boldsymbol{\nu}$ and right eigenvector $\boldsymbol{\mu}$, normalized by $\langle \boldsymbol{\nu}, \mathbf{1}\rangle = 1$ and $\langle \boldsymbol{\nu}, \boldsymbol{\mu}\rangle = 1$.

We assume that $\mathbf{M}$ is strictly positive and that the proces $\mathbf{Z_n}$ is positive regular and not singular (Athreya and Ney [1], p. 184).

When $\rho \neq 1$ we can extend to the multitype case results of the single type. For a critical process we are only able to deal with the fractional linear case. Section 2 deals with the general case and Section 3 with the fractional linear case.

2. The Q-Process

To construct the Q- process we proceed as for the one-type case in Athreya and Ney [1], p. 58: Let T be the extinction time of the process.

$$\begin{aligned}
&\mathbb{P}(\mathbf{Z}_{n_1} = \mathbf{j}_1, \mathbf{Z}_{n_2} = \mathbf{j}_2, \ldots \mathbf{Z}_{n_l} = \mathbf{j}_l | \mathbf{Z}_0 = \mathbf{i}, n_k + k < T < \infty) \\
&\qquad = \mathbb{P}(\mathbf{Z}_{n_1} = \mathbf{j}_1, \mathbf{Z}_{n_2} = \mathbf{j}_2, \ldots \mathbf{Z}_{n_l} = \mathbf{j}_l) \frac{\sum_{\mathbf{i}\neq \mathbf{0}} \mathbb{P}_k(\mathbf{j}, \mathbf{i})\mathbf{q^i}}{\sum_{\mathbf{i}\neq \mathbf{0}} \mathbb{P}_{n_l+k}(\mathbf{j}, \mathbf{i})\mathbf{q^i}} \\
&\qquad = \mathbb{P}(\mathbf{Z}_{n_1} = \mathbf{j}_1, \mathbf{Z}_{n_2} = \mathbf{j}_2, \ldots \mathbf{Z}_{n_l} = \mathbf{j}_l) \frac{\mathbf{f^j}_k(\mathbf{q}) - \mathbf{f^j}_k(0)}{\mathbf{f^i}_{n_l+k}(\mathbf{q}) - \mathbf{f^i}_{n_l+k}(0)} \,. \quad (1)
\end{aligned}$$

To study the behaviour of the last expression as $k \to \infty$ we have to distinguish two cases:

Case $\rho < 1$. In this case $\mathbf{q} = \mathbf{1}$, assuming that $\mathbb{E}\log \mathbf{Z} < \infty$, Theorems 2 and 4 of Joffe and Spitzer [3] state that $\lim_{n\to\infty} \frac{\mathbf{1} - f_n(\mathbf{s})}{\rho^n} = Q(\mathbf{s})\boldsymbol{\mu}$ with $Q(\mathbf{s}) > \mathbf{0}$.

This implies that $f_{n,i}^{j_l}(\mathbf{s}) = 1 - j_1 Q(\mathbf{s})\mu_i \rho^n + o(\rho^n)$; this result and simple algebraic manipulations applied to (1) yield that:

$$\lim_{k\to\infty} \mathbb{P}(\mathbf{Z}_{n_1} = \mathbf{j}_1, \mathbf{Z}_{n_2} = \mathbf{j}_2, \ldots \mathbf{Z}_{n_l} = \mathbf{j}_l | \mathbf{Z}_0 = \mathbf{i}, n_k + k < T < \infty)$$

$$= \frac{\mathbb{P}(\mathbf{Z}_{n_1} = \mathbf{j}_1, \mathbf{Z}_{n_2} = \mathbf{j}_2, \dots \mathbf{Z}_{n_l} = \mathbf{j}_l)}{\rho^l} \frac{< \mathbf{j}, \boldsymbol{\mu} >}{< \mathbf{i}, \boldsymbol{\mu} >} =: Q(\mathbf{j}_1, \mathbf{j}_2, \dots \mathbf{j}_l).$$

Since $\mathbf{Z}_n$ is a Markov process it follows that Q defines a Markov process, the multitype Q-process with probability transition matrix given by

$$Q_{\mathbf{i},\mathbf{j}} = \frac{1}{\rho} P_{\mathbf{i},\mathbf{j}} \frac{< \mathbf{j}, \boldsymbol{\mu} >}{< \mathbf{i}, \boldsymbol{\mu} >}. \tag{2}$$

Theorem 1. *The Q- process associated with a subcritical multitype branching process satisfying $\mathbb{E} \log Z < \infty$ is positive recurent with stationary measure $\mathbf{m}$ given by:*

$$m_{\mathbf{i}} = Q(0)\ b_{\mathbf{i}} < \mathbf{i}, \boldsymbol{\mu} >, \quad b_{\mathbf{i}} = \lim_{n \to \infty} \mathbb{P}(\mathbf{Z}_n = \mathbf{i} | \mathbf{Z}_n \neq \mathbf{0}).$$

Proof. Since formula (2) yields $Q_{\mathbf{i},\mathbf{j}} = \frac{1}{\rho} P_{\mathbf{i},\mathbf{j}} \frac{<\mathbf{j},\boldsymbol{\mu}>}{<\mathbf{i},\boldsymbol{\mu}>}$, one can estimate $P^n_{\mathbf{i},\mathbf{j}}$ from Yaglom Theorem (Theorem 2 of Athreya and Ney [1], p.):

$$\lim_{n \to \infty} \mathbb{P}(\mathbf{Z}_n = \mathbf{i} | \mathbf{Z}_n \neq \mathbf{0}) = b(\mathbf{i}), \quad \sum b(\mathbf{i}) = 1,$$

and from Theorem 1 (Section 4) of Athreya and Ney [1], p. 186:

$$\lim_{n \to \infty} \mathbb{P}(\mathbf{Z}_n \neq \mathbf{0} | \mathbf{Z}_0 = \mathbf{i}) = Q(\mathbf{0}) < \mathbf{i}, \boldsymbol{\nu} > .$$

Those two results yield the theorem. □

Case $\rho > 1$. Assuming $\mathbf{q} \neq \mathbf{0}$ we reduce this case to the preceeding one using the transformation:

$$g_{1,i}(\mathbf{s}) := \frac{1}{q_i} f_{1,i}(\mathbf{qs}), \quad g_{n,i}(s) = \frac{1}{q_i} f_{n,i}(\mathbf{qs}), \quad i = 1 \dots k.$$

It follows easily that g is the generating function of a positive regular and non-singular subcritical process whose mean matrix is given by $M^*_{i,j} := \frac{q_j}{q_i} J_{i,j}(\mathbf{q})$, where J is the Jacobien of $\mathbf{f}$. Since all the partial derivatives of $\mathbf{g}$ at $\mathbf{1}$ are finite we can apply Theorem 1 to $\mathbf{g}$. A simple verification shows that the Q-process associated with $\mathbf{f}$ is the same as the one associated with $\mathbf{g}$. From Theorem 1 we obtain:

Theorem 2. *The Q- process associated with a supercritical multitype branching process with generating function $f(\mathbf{s})$ has the same law as the subcritical multitype branching process with generating function $g_i(\mathbf{s}) = \frac{1}{q_i} f_i(\mathbf{qs})$ and is positive recurent with stationary measure $\overline{\mathbf{m}}$ given by:*

$$\overline{m}_{\mathbf{i}} = \overline{Q}(0) \overline{b}(\mathbf{i}) < \mathbf{i}, \overline{\boldsymbol{\mu}} >,$$

where $\overline{Q}(0), \overline{b}(\mathbf{i})$ and $\overline{\boldsymbol{\mu}}$ are the quantities defined in Theorem 1 for the generating function $g(\mathbf{s})$.

3. The Fractional Linear Case

Let us recall from Letac Joffe [2] that the branching process $\mathbf{Z}$ governed by a regular fractional linear birth law can be parameterized by $\mathbf{t} \in T, T = \{\mathbf{t} = (t_1, \ldots, t_k) \in (0,, 1)^k : t_1 + \ldots t_k < 1\}$ and by the mean matrix $\mathbf{M}$. Let $\mathbf{w} = \frac{\mathbf{t}}{t_0}$, where $t_0 = 1 - (t_1 \ldots t_k)$.

Case $\rho < 1$. Since one has that $Q(0) = \lim_{n\to\infty} \frac{\boldsymbol{\nu}(\mathbf{1} - \mathbf{f}_n(0))}{\rho^n}$ we obtain from formulas (11) and (3) of Letac and Joffe [2] that $Q(0) = \frac{1}{1+\mathbf{w}(\mathbf{I}_k - \mathbf{M})\mathbf{1}}$. From Theorem 1 and Theorem 1 of Letac Joffe [2] we obtain:

Theorem 3. *Let $\mathbf{Z}$ be a branching process governed by a regular fractional linear birth law, parameterized by $\mathbf{t} \in T$ and by the mean $\mathbf{M}$. Denote by $\boldsymbol{\nu}$ the left eigenvector of $\mathbf{M}$ for the eigenvalue normalized by $< \boldsymbol{\nu}, \mathbf{1} >= 1$. Let us assume that $\rho < 1$.Let $\mathbf{w}^{(\infty)} = \mathbf{w}(\mathbf{I_k} - \mathbf{M})^{-1}$ and*

$$\mathbf{t}^{(\infty)} = \frac{\mathbf{w}^{\infty}}{1+ < \mathbf{w}^{(\infty)}, \mathbf{1} >}.$$

Then the Q-process associated with $\mathbf{Z}$ is positive regular with invariant measure $\mathbf{m}$ given by:

$$m_{\mathbf{i}} = \frac{1}{1+ < \mathbf{w}^{(\infty)}, \mathbf{1} >} \quad b_{\mathbf{i}} \ < \mathbf{i}, \boldsymbol{\mu} > .$$

Furthermore the $b_{\mathbf{i}}$ have a generating function given by:

$$\frac{< \boldsymbol{\nu} - \mathbf{t}^{(\infty)}, \mathbf{s} >}{1- < \mathbf{t}^{(\infty)}, \mathbf{s} >}.$$

Case $\rho = 1$. The same computation that leads to (1) in Section 2 gives: $Q_{\mathbf{i},\mathbf{j}} = P_{\mathbf{i},\mathbf{j}} \frac{<\mathbf{j},\boldsymbol{u}>}{<\mathbf{i},\boldsymbol{u}>}$ and $Q^n_{\mathbf{i},\mathbf{j}} = P^n_{\mathbf{i},\mathbf{j}} \frac{<\mathbf{j},\boldsymbol{u}>}{<\mathbf{i},\boldsymbol{u}>}$. Since the process $\mathbf{Z}$ is transient it follows that $\sum_n P^n_{\mathbf{i},\mathbf{j}} < \infty$, hence $\sum_n Q^n_{\mathbf{i},\mathbf{j}} < \infty$ and we obtain:

Theorem 4. *Let $\mathbf{Z}$ be a branching process governed by a regular fractional linear birth law with $\rho = 1$. Then the Q-process associated with $\mathbf{Z}$ is transient.*

Case $\rho > 1$. Let $\mathbf{Z}$ be a branching process governed by a regular fractional linear birth law, parameterized by $\mathbf{t} \in T$ and by the mean $\mathbf{M}$. Let us assume that $\rho > 1$ and let $f(\mathbf{s})$ denote the generating function which governs $\mathbf{Z}$. We apply Theorem 2 of Section 2 to reduce the study of the Q process associated with $\mathbf{Z}$ to the one associated with the subcritical process with generating function: $g_i(\mathbf{s}) := \frac{1}{q_i} f_i(\mathbf{qs})$. Observe that $\mathbf{g}(\mathbf{s})$ is also the generating function of a regular fractional linear birth law. Denote by $\overline{\mathbf{t}}$ and $\overline{\mathbf{M}}$ the parametrization of $\mathbf{g}(\mathbf{s})$.

An easy computation shows that: $\overline{t_i} = \frac{t_i}{q_i}$. The computation of $\overline{\mathbf{M}}$ is more involved; one has:

$$\overline{M}_{i,j} = \frac{\partial g_i}{\partial s_j}((\mathbf{1}) = \frac{\mathbf{q_j}}{\mathbf{q_i}} \frac{\partial \mathbf{f_i}}{\partial \mathbf{s_j}}(\mathbf{q}).$$

From the explicit expression of $\mathbf{f}$ one obtains:

$$\frac{\partial f_i}{\partial s_j}(\mathbf{s}) = \frac{M_{ij}(1 + \langle \mathbf{w}, \mathbf{1}\rangle - \langle \mathbf{w}, \mathbf{s}\rangle) + w_j(\sum M_{ik}s_k - \sum M_{ik}}{(1 + \langle \mathbf{w}, \mathbf{1}\rangle - \langle \mathbf{w}, \mathbf{s}\rangle)^{\mathbf{2}}}$$

since from Theorem 3 of Letac and Joffe [2] one has: $\mathbf{q} = \mathbf{1} - \frac{\rho-1}{\langle \mathbf{w}, \boldsymbol{\mu}\rangle}\boldsymbol{\mu}$, it follows that:

$$\frac{\partial f_i}{\partial s_j}(\mathbf{q}) = \frac{\rho M_{ij} - \frac{\rho-1}{\langle \mathbf{w}, \boldsymbol{\mu}\rangle} w_j (\mathbf{M}\boldsymbol{\mu})_{\mathbf{i}}}{\rho^2} = \frac{1}{\rho}(M_{ij} - \frac{\rho - 1}{\langle \mathbf{w}, \boldsymbol{\mu}\rangle} w_j \mu_j),$$

where the last equality is obtained from $\mathbf{M}\boldsymbol{\mu} = \rho\boldsymbol{\mu}$. Finaly we have:

$$(\overline{\mathbf{M}})_{ij} = \frac{q_j}{q_i}\frac{1}{\rho}(M_{ij} - \frac{\rho - 1}{\langle \mathbf{w}, \boldsymbol{\mu}\rangle} w_j \mu_j). \tag{3}$$

Let us define $\overline{\boldsymbol{\mu}}$ by $\overline{(\boldsymbol{\mu})}_i := \frac{\mu_i}{q_i}$; then an easy verification shows that $\overline{\mathbf{M}}\boldsymbol{\mu} = \frac{1}{\rho}\boldsymbol{\mu}$; hence one has that $\frac{1}{\rho}$ is the largest eigenvalue of $\overline{\mathbf{M}}$ with the right eigenvector $\overline{\boldsymbol{\mu}}$, and let us denote by $\overline{\boldsymbol{\nu}}$ its left eigenvector normalised by $< \overline{\boldsymbol{\nu}}, \mathbf{1} >= \mathbf{1}$. From Theorems 2 and 3 we obtain:

Theorem 5. *Let* $\mathbf{Z}$ *be a branching process governed by a regular fractional linear birth law, parameterized by* $\mathbf{t}$ *and by the mean matrix* $\mathbf{M}$. *Let us assume that* $\rho > 1$.

Then the Q-process associated with $\mathbf{Z}$ is positive regular and has the same law that the Q-process associated with the process governed by the regular fractional linear birth law, parameterized by $\overline{(\mathbf{t})}_i = \frac{t_i}{q_i}$ and by the mean matrix $\overline{\mathbf{M}}$ given by (3).

Its invariant measure $\overline{\mathbf{m}}$ is given by:

$$\overline{m}_i = \frac{1}{1 + \langle \overline{\mathbf{w}}^{(\infty)}, \mathbf{1}\rangle} \quad b_{\mathbf{i}} \ \langle \mathbf{i}, \boldsymbol{\mu}\rangle.$$

where $\overline{\mathbf{w}}^{(\infty)} = \overline{\mathbf{w}}(\mathbf{I_k} - \overline{\mathbf{M}})^{-\mathbf{1}}$.

Furthermore the $b_{\mathbf{i}}$ have a generating function given by:

$$\frac{< \overline{\boldsymbol{\nu}} - \overline{\mathbf{t}}^{(\infty)}, \mathbf{s} >}{1 - \langle \overline{\mathbf{t}}^{(\infty)}, \mathbf{s}\rangle, \text{where} : \overline{\mathbf{t}}^{(\infty)} = } \frac{\overline{\mathbf{w}}^{(\infty)}}{1 + < \overline{\mathbf{w}}^{(\infty)}, \mathbf{1} >} \quad .$$

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