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BORDERLINE BEHAVIOR FOR 2×2 ITERATIVE SYSTEMS

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Abstract: The study of 2×2 linear iterative systems leads naturally to an analysis of the eigenvalues and eigenvectors of the corresponding system matrix. The phase portraits for such systems have been previously examined and outlined; however the outline lacks the analysis of the many borderline cases in the trace-determinant plane. In this paper we fill in some of these details and look at the general solutions for the most interesting cases in terms of eigenvectors. In particular, we find generalized eigenvectors when required.

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1. Systems of Iterative Equations

Linear iterative systems take the form:

$$\begin{cases} x_{n+1} = ax_n + by_n\,, \\ y_{n+1} = cx_n + dy_n\,, \end{cases} \qquad \text{or in vector form} \quad \vec{Y}_{n+1} = A\vec{Y}_n,$$

where $\vec{Y}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From our knowledge of the solutions to

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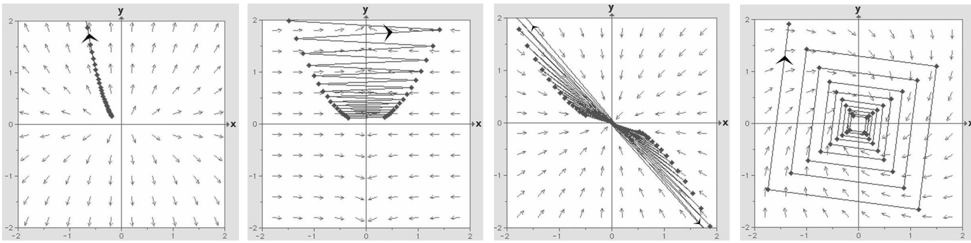
the second-order iterative equations, one seeks solutions to the system of the form $\vec{Y}_n = \lambda^n \vec{v}$, where λ is an eigenvalue for matrix A and $\vec{v}$ is a corresponding eigenvector. In the case where matrix A has two distinct eigenvalues λ_1 and λ_2 (real or complex), it is well-known that the general solution $\vec{Y}_n$ takes the form: $\vec{Y}_n = c_1\lambda_1^n\vec{v_1} + c_2\lambda_2^n\vec{v_2}$, where $\vec{v_1}, \vec{v_2}$ are corresponding eigenvectors for λ_1 and λ_2 respectively (see [1]). The case of repeated eigenvalues will be discussed separately in Section 3.

One is interested in the long-term behavior of solutions to initial value problems, and the study of the phase portraits of such systems illuminates this behavior. In the case of two real eigenvalues λ_1 and λ_2 where $|\lambda_1| > 1, |\lambda_2| > 1$, solutions to initial value problems approach the origin, which in this case is called a *source* (see Figure 1(a)); if on the other hand $|\lambda_1| < 1$ and $|\lambda_2| < 1$, the origin is a *Sink*; and if $|\lambda_1| < 1, |\lambda_2| > 1$, the origin is a *Saddle*. An important aspect therefore that distinguishes linear systems of iterative equations from linear systems of differential equations is that the *size* of the eigenvalues constitutes a major factor in the classification of the origin. But also if the eigenvalues have opposite signs then the origin becomes a *flip source* (see Figure 1(b)), *flip sink*, or *flip saddle* respectively. Furthermore, if both eigenvalues are negative then the origin is a *double flip source* (see Figure 1(c)), *double flip sink*, or *double flip saddle* respectively. If the eigenvalues are complex, the corresponding eigenvectors have complex entries and the qualitative behavior of the solution is better understood by noticing that the real and imaginary parts of any one solution are themselves solutions to the system. More precisely, the sine and cosine terms that appear in the real and imaginary parts of the general solution to the system produce a spiralling effect on that solution. Thus, if the magnitude of λ is bigger than one ($||\lambda|| > 1$), the solution will spiral outwards, away from the origin, and the origin is called a *spiral source* (see Figure 1(d)); if on the other hand $||\lambda|| < 1$, the solution spiral inwards to the origin, and the origin is a *spiral sink*; and if $||\lambda|| = 1$ the origin is a *Center*. Figure 1 shows the various types of sources.

2. The Trace-Determinant Plane

For linear 2×2 iterative systems $\vec{Y}_{n+1} = A\vec{Y}_n$, the parabola $\det(A) = \frac{1}{4}(\mathrm{tr}(A))^2$ is the determining factor for the existence of two distinct eigenvalues, one repeated eigenvalue, or complex eigenvalues for the system matrix A. The eigenvalues are solutions to the characteristic equation of A given by: $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$. Thus, in case $(\mathrm{tr}(A))^2 - 4\det(A) < 0$, the iterated points will either



(a) Source (b) Flip source (c) Double Flip source (d) Spiral source

Figure 1: Various types of sources

spiral toward or away from the origin (or simply form closed loops if $||\lambda|| = 1$). On the other hand, in case $(\mathrm{tr}\,(A))^2 - 4\det(A) > 0$, the origin is in general either a saddle (or flip/double flip saddle), a source (or flip/double flip source), or a sink (or flip/double flip sink). The qualitative behavior for solutions of iterative systems that correspond to points in the source/sink/saddle/center regions of the trace-determinant plane has been previously extensively outlined in [1] and [2]. The literature on this subject however lacks a detailed description of the qualitative behavior on the boundaries of these regions, specifically the parabola. In what follows we fill in this important gap.

3. The Parabola $\det(A) = \frac{1}{4}(\mathrm{tr}\,(A))^2$

We look at the system $\vec{Y}_{n+1} = A\vec{Y}_n$ where $\det(A) = \frac{1}{4}(\mathrm{tr}\,(A))^2$. In this case, the characteristic equation has a double root. The general form of the solution requires the existence of two independent eigenvectors; however, this is not always guaranteed. The main contribution of this paper, summarized in the next theorem, describes the algebraic form of the solution in either case.

Theorem 1. *Consider the system $\vec{Y}_{n+1} = A\vec{Y}_n$ having one repeated eigenvalue $\lambda \neq 0$, and let $\vec{v}$ denote a corresponding eigenvector.*

1. If a second independent eigenvector exists for the repeated eigenvalue λ, then the general solution of the system takes the form $Y_n = \lambda^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, where x_0, y_0 are some initial conditions.

2. If another independent eigenvector does not exist, then the general solution of the system takes the form: $\vec{Y}_n = c_1\lambda^n\vec{v} + c_2\lambda^n(n\vec{v} + \vec{u})$, where $(A - \lambda I)\vec{u} = \lambda\vec{v}$.

Proof. 1. In the 2×2 case, the eigenspace belonging to a double eigenvalue can be two-dimensional only if the matrix is a multiple of the identity, i.e. the matrix has the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, and any vector can be an eigenvector. The iterative system is therefore the completely decoupled system $x_{n+1} = \lambda x_n$ and $y_{n+1} = \lambda y_n$. For some arbitrary initial conditions $x(0) = x_0, y(0) = y_0$, the general solution takes the form $x_n = \lambda^n x_0, y_n = \lambda^n y_0$, and part 1 follows.

2. Now, if no second independent eigenvector exists, then a solution of a different form must be found. As in the case of 2×2 linear systems of differential equations, one seeks to find a *generalized eigenvector*, which is not really an eigenvector for the matrix but it does the job for finding the general solution of the system. More precisely, one seeks a second solution of the form $\vec{Y}_n = n\lambda^n \vec{v} + \lambda^n \vec{u}$, where $\vec{u}$ is the generalized eigenvector. To find the form of $\vec{u}$ we note that $\vec{Y}_{n+1}$ must equal $A\vec{Y}_n$, and hence $(n+1)\lambda^{n+1}\vec{v} + \lambda^{n+1}\vec{u} = A\left(n\lambda^n \vec{v} + \lambda^n \vec{u}\right)$. Setting $n = 0$, we obtain $\lambda\vec{v} + \lambda\vec{u} = A\vec{u}$ or $(A - \lambda I)\,\vec{u} = \lambda\vec{v}$. We have thus proved part 2 of the theorem. □

We now proceed to discuss the phase portraits in this boundary case, that is along the parabola in the trace-determinant plane. As we mentioned earlier, in case there are two independent eigenvectors corresponding to the repeated eigenvalue λ, the general solution takes the form $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \lambda^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. Thus

1. If $|\lambda| < 1$, the iteration converges in a linear fashion to the origin and flips in the case where λ is negative. We shall call the origin a *linear* sink or a *linear* flip sink.

2. If $|\lambda| > 1$, then the iteration diverges in a linear fashion and flips in the case where λ is negative. The origin is therefore a *linear* source (flip source).

3. If $\lambda = 1$, then every point is an equilibrium point.

4. If $\lambda = -1$, then the iteration forms a *2-point cycle* that flips between (x_0, y_0) and $(-x_0, -y_0)$.

5. Finally if $\lambda = 0$, it is obvious that the iteration converges directly to the origin (a *one-iteration* sink).

Figure 2 summarizes all the possibilities.

In the event that we are unable to find a second independent eigenvector corresponding to the repeated eigenvalue, then it can be shown that there exists an invertible matrix P such that $P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where the first column of P is an eigenvector corresponding to the eigenvalue λ and the second column is

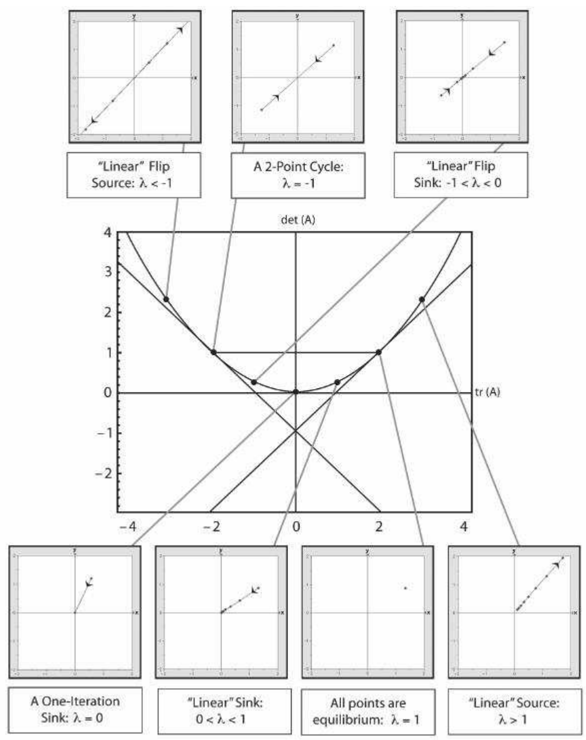


Figure 2: The parabola: Case of two independent eigenvectors

any linearly independent vector. In this canonical form, the unique eigenvector of the system is $\begin{pmatrix}1\\0\end{pmatrix}$, and the transition matrix P has the effect of rotating the phase plane of the original non-canonical system, more specifically rotating its eigenline. For further readings on this subject, you may refer to [3]. In this canonical form, one can easily show that the solution takes the form:

$$\begin{pmatrix}x_n\\y_n\end{pmatrix} = \lambda^n \begin{pmatrix}x_0\\y_0\end{pmatrix} + n\lambda^{n-1}\begin{pmatrix}y_0\\0\end{pmatrix}, \text{for any initial condition}(x_0, y_0),$$

a solution form that will help us understand the qualitative behavior. Thus:

1. If $|\lambda| > 1$, the iteration diverges and flips in the case when the eigenvalue is negative. The origin is called a *degenerate (flip) source.*

2. If $|\lambda| < 1$, the iteration converges to the origin, and flips in the case when the eigenvalue is negative. The origin is a *degenerate (flip) sink.*

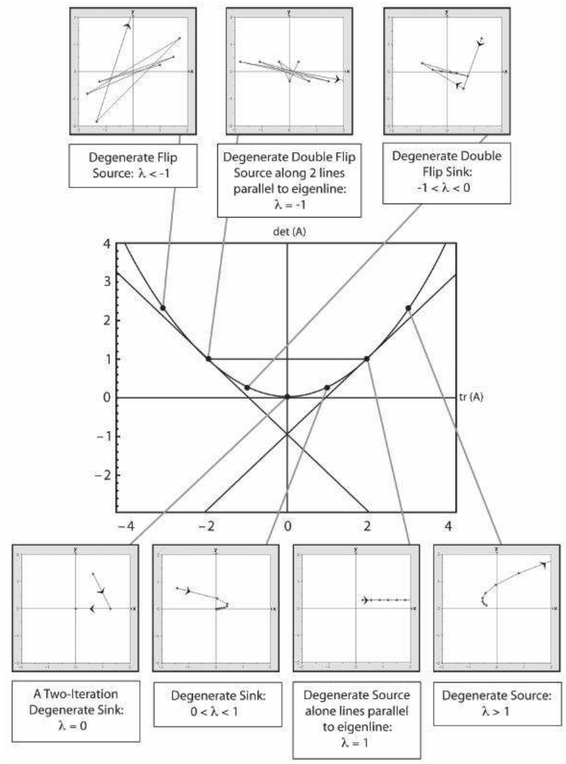


Figure 3: The parabola: Case of one eigenvector

3. If $\lambda = 1$, the general solution is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + n \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$, which diverges along a line parallel to the x-axis, the eigenline in the canonical case. In the non-canonical case, the divergence is along a line parallel to the corresponding eigenline.

4. If $\lambda = -1$, the general solution is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (-1)^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + n(-1)^{n-1} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$, which double flips and diverges along two lines parallel to the x-axis. In the non-canonical case, the divergence is along two lines parallel to the corresponding eigenline.

5. Finally if $\lambda = 0$, the system takes the form: $x_{n+1} = y_n$ and $y_{n+1} = 0$. Therefore, the iteration converges to the origin in two steps: Starting with (x_0, y_0), $x_1 = y_0, y_1 = 0$, and $x_2 = y_1 = 0, y_2 = 0$.

Figure 3 summarizes all the possibilities in case one cannot find a second independent eigenvector.

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