PERIOD ANNULI IN THE LIÉNARD TYPE EQUATION

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Abstract: We consider the equation
\[ x'' + \frac{2x}{1 + x^2}x'^2 + g(x) = 0, \] (i)
where \( g(x) = -x(x^2 - p^2)(x^2 - q^2) \). Comparison of phase portraits for equations (i) and
\[ x'' + g(x) = 0 \] (ii)
is made. We describe decomposition of the first quadrant of the \((q, p)\)-plane into regions where equations (i) and (ii) have or have not a period annulus, that is, a set of concentric cycles enclosing several critical points of equivalent planar systems.

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1. Introduction

In this paper we consider the specific Liénard type equation of the form
\[ x'' + f(x)x'^2 + g(x) = 0, \] (1)
where \( g(x) \) is a polynomial. We are looking for the so called period annuli. Let
equation (1) be written as a planar system

\[ x' = y, \quad y' = -f(x)y^2 - g(x). \]  \hspace{1cm} (2)

Critical points of this system are points \((p_i, 0)\), where \(p_i\) are zeros of \(g(x)\). If all zeros of \(g(x)\) are simple (in the meaning that \(g'(p_i) \neq 0\)) then only two types of critical points are possible, namely centers and saddle points.

Recall that a critical point \(O\) of (2) is a center if it has a punctured neighborhood covered with nontrivial cycles. Due to terminology in Sabatini [4], the largest connected region covered with cycles surrounding \(O\) is called central region. Every connected region covered with nontrivial concentric cycles is usually called a period annulus.

2. Conservative Equation

Consider equation

\[ x'' + g(x) = 0, \]  \hspace{1cm} (3)

where \(g(x)\) is a polynomial of order 5 with five simple zeros. A sample of \(g(x)\) is depicted in Figure 1 together with the primitive \(G(x) = \int_0^x g(s) \, ds\). Zeros of \(g(x)\) are \(p_1 < p_2 < p_3 < p_4 < p_5\).

The equivalent system has three saddle points at \((p_1, 0)\), \((p_3, 0)\), \((p_5, 0)\) and centers at \((p_2, 0)\) and \((p_4, 0)\).

One of the typical phase portrait is given in Figure 2. There are two central regions filled with “small” amplitude periodic solutions located in neighborhoods of \((p_2, 0)\) and \((p_4, 0)\).

No other nontrivial periodic solutions exist if three local maxima of the
primitive $G(x)$ are such that either $G(p_1) > G(p_3) > G(p_5)$ or $G(p_1) < G(p_3) < G(p_5)$ or $G(p_5) < G(p_1) < G(p_3)$ or $G(p_1) < G(p_5) < G(p_3)$.

Situation is quite different if $G(p_3)$ is less (strictly) than $G(p_1)$ and $G(p_5)$. Then appear period annulus like in Figure 4.

**Theorem 1.** (see [2]) If $G(p_3)$ is less (strictly) than $G(p_1)$ and $G(p_5)$, then equation (3) has “large”-amplitude periodic solutions (a period annulus), that is, solutions enclosing the critical points $(p_2;0)$ and $(p_4;0)$.

**Remark.** If the inequalities $G(p_1) > G(p_3) > G(p_5)$ or $G(p_1) < G(p_3) < G(p_5)$ or $G(p_5) < G(p_1) < G(p_3)$, $G(p_5) < G(p_3)$ hold then “large”-amplitude periodic solutions do not exist. This can be shown (see Figure 2, which corresponds to the latter case).

### 3. Reduction to a Conservative Equation

It was shown by Sabatini [3] that equation (1) can be reduced to the form $u'' + h(u) = 0$ by the following transformation. Let $F(x) = \int_0^x f(s) \, ds$ and $G(x) = \int_0^x g(s) \, ds$. Introduce

$$u := \Phi(x) = \int_0^x e^{F(s)} \, ds.$$  

(4)

Since $\frac{du}{dx} > 0$, this is one-to-one transformation and the inverse $x = x(u)$ is well defined.

**Lemma 1.** (see [3], Lemma 1) The function $x(t)$ is a solution to (1) if and only if $u(t) = \Phi(x(t))$ is a solution to

$$u'' + g(x(u))e^{F(x(u))} = 0.$$  

(5)
Denote $H(u) = \int_0^u g(x(s))e^{F(x(s))} \, ds$. The existence of periodic solutions depends entirely on properties of the primitive $H$.

Let us state some easy assertions (see [2], [1]) about equation (1), the equivalent system

\[
x' = y, \quad y' = -f(x)y^2 - g(x)
\]

and the system

\[
x' = y, \quad y' = -g(x).
\]

**Proposition 1.** Critical points and their character are the same for systems (7) and (6).

Consider a system

\[
u' = v, \quad v' = -g(x(u))e^{F(x(u))},
\]

which is equivalent to equation (5).

**Proposition 2.** Critical points $(x, 0)$ and $(u(x), 0)$ of systems (7) and (8) respectively are in 1-to-1 correspondence and their characters are the same.

**Proposition 3.** Periodic solutions $x(t)$ of equation (1) turn to periodic solutions $u(t) = \Phi(x(t))$ of equation (5) by transformation (4).

**Proposition 4.** Homoclinic solutions of equation (1) turn to homoclinic solutions of equation (5) by transformation (4).

**Proposition 5.** Let $p_i$ be a zero of $g(x)$. The equality

\[
g_x(p_i) = g_u(x(u))e^{F(x(u))}|_{u=p_i}
\]

is valid.

**Proof.** By calculation of the derivative. \hfill \Box

### 4. Period Annuli

Consider

\[
x'' + \frac{2x}{1 + x^2}x^2 - x(x^2 - p^2)(x^2 - q^2) = 0.
\]

One has that

\[
H(u) = \int_0^{x(u)} g(s)(1 + s^2)^2 \, ds, \quad g(x) = -x(x^2 - p^2)(x^2 - q^2),
\]

\[
G(x) = \int_0^x g(s) \, ds = \frac{1}{2}p^2q^2x^2 + \frac{1}{4}p^2x^4 + \frac{1}{4}q^2x^4 - \frac{1}{6}x^6.
\]
Since \( G(x) \) is an even function we may consider only positive values of \( x \). The primitive \( G \) has local minimum at \( x = p \) and local maximum at \( x = q \). Since the existence of a period annulus in the equation
\[
x'' - x(x^2 - p^2)(x^2 - q^2) = 0
\] depends on the value \( G(q) \) we compute it and got the result
\[
G(q) = -\frac{1}{12}(3p^2q^4 - q^6).
\]

If \( G(q) > 0 \) then equation (11) has a period annulus. We arrive thus to the following assertion.

**Proposition 7.** Equation (11), where \( 0 < p < q \), has a period annulus, if
\[
p < \frac{1}{\sqrt{3}}q.
\]

One has that
\[
F(x) = \int_0^x \frac{2s}{1 + s^2} ds = \ln(1 + x^2), \quad e^{F(x)} = 1 + x^2,
\]
\[ u = \int_0^x e^{F(s)} \, ds = \int_0^x (1 + s^2) \, ds = x + \frac{1}{3} x^3. \]

Evidently the inverse function \( x = x(u) \) exists. Equation (1) turns to
\[ u'' + h(u) = 0, \quad (12) \]
where
\[ h(u) = g(x(u)) e^{F(x(u))} = -x(u)(x^2(u) - p^2)(x^2(u) - q^2)(1 + x^2(u)). \]

Standard computation gives
\[ H(u) = \int_0^u h(s) \, ds = \int_0^{x(u)} g(\xi) e^{2F(\xi)} \, d\xi \]
\[ = -\int_0^{x(u)} -\xi(\xi^2 - p^2)(\xi^2 - q^2)(1 + \xi^2)^2 \, d\xi. \]

The primitive \( H(u) \) is a 10-th order polynomial with a unique positive point of minimum at \( u(p) \) and a unique positive point of maximum at \( u(q) \). Introduce \( I(p, q, x) := \int_0^x -\xi(\xi^2 - p^2)(\xi^2 - q^2)(1 + \xi^2)^2 \, d\xi. \)

Integrating one gets that
\[ H(u)|_{u=q} = I(p, q, q) = \frac{1}{120} q^4 (-30p^2 + 10q^2 - 20p^2q^2 + 10q^4 - 5p^2q^4 + 3q^6). \]

Therefore equations (12) and (10) have a period annulus if
\[ -30p^2 + 10q^2 - 20p^2q^2 + 10q^4 - 5p^2q^4 + 3q^6 > 0, \quad 0 < p < q. \]

Equations (12) and (10) do not have period annuli if
\[ -30p^2 + 10q^2 - 20p^2q^2 + 10q^4 - 5p^2q^4 + 3q^6 < 0, \quad 0 < p < q. \]

We arrive to the following result.

**Theorem 2.** Equation (1), where \( 0 < p < q \), has a period annulus if
\[ p < q \sqrt{\frac{3q^4 + 10q^2 + 10}{5q^4 + 20q^2 + 30}}. \]

The line between solid and dashed lines shows the graph of
\[ p = q \sqrt{\frac{3q^4 + 10q^2 + 10}{5q^4 + 20q^2 + 30}}. \]

The derivative of this function at \( q = 0 \) is \( \frac{1}{\sqrt{3}} \) and at infinity it tends to the straight line \( p = \sqrt{\frac{2}{5}} q \). Therefore we got three cases:

- \((q, p) \in \Omega_1 \) – equations (11) and (10) do not have period annuli.
— \((q, p) \in \Omega_2\) – equation (11) does not have and equation (10) has a period annulus (see Figures 6 and 7).

— \((q, p) \in \Omega_3\) – equations (11) and (10) both have period annuli.

References


