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KELVIN-HELMHOLTZ INSTABILITY MODEL FOR A SUPERSONIC ADIABATIC JET

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Abstract: The linear model for Kelvin-Helmholtz instability for supersonic adiabatic jet of low viscosity is developed, and the dispersion relation based on conservation equations (continuity, Navier-Stokes and energy equations) is deduced. This formula involving Heun functions in simplified cases coincides with various forms of dispersion relation obtained by other authors, and in resonance cases reduces to the simple algebraic equations. Based on the dispersion relation it is shown that an expanded adiabatic jet is unstable to Kelvin-Helmholtz modes which can grow with nonlinear amplitudes. To demonstrate further applications of this equation, the instability modes are found and investigated without detailed numerical calculations for complicated hydrodynamic model.

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1. Introduction

Kelvin-Helmholtz instability (KHI) can occur when the velocity shear is present within a continuous fluid or when there is sufficient velocity difference across the interface of two fluids, Drazin [1]. The existence of the surface tension stabilizes the short wavelength instability, and theory then predicts stability until a velocity threshold is reached. Supersonic jets in pressure balance with an external medium are unstable to surface perturbation analogous to KHI of vortex sheet, Hardee [4], Drazin [3]. For the high Mach number ($M > 1$) the interesting shocks form a Mach disk (in axial jets), and the bright spots ("brilliants") appear in the high temperature regions following the internal shocks.

The present work studies the behavior of short and long wavelength perturbations to a jet, the effect of the resulting growing wave modes. It also analyzes the energy equation by comparison with the methodology for isothermal jet Drazin [3]. Due to low viscosity the viscid term is accounted using the perturbation's methods. The dispersion equation is obtained under assumption that the sound velocity at the boundary layer has Gaussian (bell) form; hence the perturbed velocity radial component satisfies to Heun equation. Similarly to Gaissinski et al [3], where the simple case of constant sound velocity was studied, the present study is limited on the behavior of pinching and helical ($n = 0, 1$) perturbed modes.

The main results of the paper are:

1. The equivalence of using continuity and energy conservation perturbed equations to deduce the dispersion relation;

2. For $n = 1$ and low viscosity, the instability decreases rapidly; it is shown the modes dissipation with decreasing of dimensionless wave number $X = kR_0$ (starting from $X \geq 0.6$).

The illustrative example with rocket jet properties is presented. The analytical estimations (i.e., asymptotic solutions) show a good consistence of the proposed model with numerical simulations.

2. Dispersion Relation

Assume that a sharp discontinuity in velocity exists at the boundary between two fluids and that the pressure is continuous across the boundary. Hence the results are accurate in the more general case if the boundary layer has characteristic size much less than the wave length of perturbations. Our starting point are the *continuity and momentum equations* Zel'dovich et al [6]:

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{v}) = 0, \tag{1}$$

$$\rho\Big[\frac{\partial \overrightarrow{v}}{\partial t} + \big(\overrightarrow{v} \cdot \overrightarrow{\nabla}\big)\overrightarrow{v}\Big] = -\overrightarrow{\nabla} P + \mu \overrightarrow{\nabla}^2 \overrightarrow{v}, \tag{2}$$

where ρ – density, $\overrightarrow{v}$ – fluid velocity, P – pressure. For gaseous (low viscosity) media one may assume $\mu = \mu_0$ =const., and neglect the terms in the *energy equation* responsible for the viscous energy dissipation, Zel'dovich et al [6]:

$$\frac{\partial}{\partial t}\Big(\frac{A}{\gamma-1}\rho T + \frac{1}{2}\rho v^2\Big) = -\overrightarrow{\nabla} \cdot \Big[\rho \overrightarrow{v}\Big(\frac{A}{\gamma-1}T + \frac{1}{2}v^2\Big) + P\overrightarrow{v}\Big] + \mu c_P(\overrightarrow{\nabla}^2 T), \tag{3}$$

where μ – dynamic viscosity, T – temperature, c_P – heat capacity under constant pressure, γ – isentropic exponent. To analyze a diverging flow of a zero opening angle, we use cylindrical coordinates (r, ϕ, z) with the axis along the centerline of the jet, and the boundary $r = R_0$ between jet and external medium. In aim to apply the perturbation theory, we denote $\overrightarrow{v} = \overrightarrow{v}_0 + \overrightarrow{v}'$, $\rho = \rho_0 + \rho'$, $P = P_0 + P'$, $T = T_0 + T'$, and $\overrightarrow{v}_0 = (v_{0r}, v_{0\phi}, v_{0z})$, $\overrightarrow{v}' = (v'_r, v'_\phi, v'_z)$ and assume $\rho_0 = \rho_0(r, z)$, $P_0 = P_0(r, z)$, $\overrightarrow{v}_0 = \overrightarrow{v}_0(r, z)$, $T_0 = T_0(r, z)$. The linearized equations (1), (2) (see Gaissinski et al [3]) and equation (3) in cylindrical coordinates are

$$\frac{\partial \rho'}{\partial t} + \big(v'_r \frac{\partial \rho_0}{\partial z} + v'_z \frac{\partial \rho_0}{\partial z}\big) + \rho_0\big(\frac{v'_r}{r} + \frac{\partial v'_r}{\partial r} + \frac{1}{r}\frac{\partial v'_\phi}{\partial \phi} + \frac{\partial v'_z}{\partial z}\big) = 0, \tag{4}$$

$$\rho_0\big(\frac{\partial v'_r}{\partial t}\big) = -\frac{\partial P'}{\partial r} + \mu_0\big(\frac{\partial^2 v'_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v'_r}{\partial \phi^2} + \frac{\partial^2 v'_r}{\partial z^2} + \frac{1}{r}\frac{\partial v'_r}{\partial r} - \frac{2}{r^2}\frac{\partial v'_\phi}{\partial \phi} - \frac{v'_r}{r^2}\big), \tag{5}$$

$$\rho_0\big(\frac{\partial v'_\phi}{\partial t}\big) = -\frac{1}{r}\frac{\partial P'}{\partial \phi} + \mu_0\big(\frac{\partial^2 v'_\phi}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 v'_\phi}{\partial \phi^2} + \frac{\partial^2 v'_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial v'_\phi}{\partial r} + \frac{2}{r^2}\frac{\partial v'_r}{\partial \phi} - \frac{v'_\phi}{r^2}\big),$$

$$\rho_0\big(\frac{\partial v'_z}{\partial t}\big) = -\frac{\partial P'}{\partial z} + \mu_0\big(\frac{\partial^2 v'_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v'_z}{\partial \phi^2} + \frac{\partial^2 v'_z}{\partial z^2} + \frac{1}{r}\frac{\partial v'_z}{\partial r}\big),$$

$$\begin{aligned} \frac{A}{\gamma-1}\big(\rho_0\frac{\partial T'}{\partial t} + T_0\frac{\partial \rho'}{\partial t}\big) &= -\frac{P_0 v'_r}{r} - v'_r\frac{\partial P_0}{\partial r} - P_0\frac{\partial v'_r}{\partial r} - \frac{P_0}{r}\frac{\partial v'_\phi}{\partial \phi} - v'_z\frac{\partial P_0}{\partial z} \\ &+ \rho_0 v'_z\frac{\partial T_0}{\partial z} + \rho_0 T_0\frac{\partial v'_z}{\partial z}\big) + \mu_0 c_P\big(\frac{1}{r}\frac{\partial T'}{\partial r} + \frac{\partial^2 T'}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 T'}{\partial \phi^2} + \frac{\partial^2 T'}{\partial z^2}\big) \\ &- P_0\frac{\partial v'_z}{\partial z} - \frac{A}{\gamma-1}\big(\frac{\rho_0 T_0 v'_r}{r} + T_0 v'_r\frac{\partial \rho_0}{\partial r} + \rho_0 v'_r\frac{\partial T_0}{\partial r} + \rho_0 T_0\frac{\partial v'_r}{\partial r}\frac{\rho_0 T_0}{r}\frac{\partial v'_\phi}{\partial \phi} + T_0 v'_z\frac{\partial \rho_0}{\partial z}. \end{aligned} \tag{6}$$

We assume the *plane wave expansion form* for the quantities

$$\{v'_r, v'_\phi, v'_z\} \quad = \quad \{v_{r1}(r), v_{\phi 1}\Psi(r), v_{z1}\Psi(r)\}e^{i(kz+n\phi-\omega t)}, \tag{7}$$

$$\rho' = \rho_1 \Psi(r) e^{i(kz+n\phi-\omega t)}, \quad T' = T_1 \Psi(r) e^{i(kz+n\phi-\omega t)},$$

where $n = 0, 1, \dots$, k is the wave number, ω is the frequency, ϕ is the initial phase, $v_{r1}(r), \Psi(r)$ are smooth functions. Equations (4) − (6), in plane wave expansion become

$$\Psi(r)(v_{z1}\frac{\partial \rho_0}{\partial z} - i\omega\rho_1) + v_{r1}(r)\frac{\partial \rho_0}{\partial r}$$
$$+ \rho_0\big[\Psi(r)\big(ikv_{z1}+in\frac{v_{\phi 1}}{r}\big) + \frac{v_{r1}(r)}{r}+\frac{dv_{r1}(r)}{dr}\big] = 0, \quad (8)$$

$$v_{r1}(r) + \frac{\partial P'}{\partial \rho'}\frac{\rho_1}{\rho_0}\frac{i}{\omega}\Psi'(r) + \frac{\mu_0}{\rho_0}\frac{1}{i\omega}\big(\overrightarrow{\nabla}^2\overrightarrow{v}'\big)_r = 0, \quad (9)$$

$$v_{\phi 1} - \frac{\partial P'}{\partial \rho'}\big(\frac{\rho_1}{\rho_0}\big)\frac{1}{r}\frac{n}{\omega} + \frac{\mu_0}{\rho_0}\frac{1}{i\omega}\frac{1}{\Psi(r)}\big(\overrightarrow{\nabla}^2\overrightarrow{v}'\big)_\phi = 0, \quad (10)$$

$$v_{z1} - \frac{\partial P'}{\partial \rho'}\big(\frac{\rho_1}{\rho_0}\big)\frac{k}{\omega} + \frac{\mu_0}{\rho_0}\frac{1}{i\omega}\frac{1}{\Psi(r)}\big(\overrightarrow{\nabla}^2\overrightarrow{v}'\big)_z = 0, \quad (11)$$

$$\mu_0 c_P T_1\big(\Psi''(r)+\frac{1}{r}\Psi'(r)-\Psi(r)\big(\frac{n^2}{r^2}+k^2\big)\big)+\frac{Ai\omega}{\gamma-1}\Psi(r)\big(\rho_0 T_1+T_0\rho_1\big) \quad (12)$$
$$- c_0\rho_0^\gamma\frac{\gamma}{\gamma-1}\big[\frac{v_{r1}(r)}{r} + \frac{d}{dr}v_{r1}(r) + \Psi(r)i\big(\frac{v_{\phi 1}n}{r} + v_{z1}k\big)\big] = 0,$$

where a constant c_0 relates the units of pressure and density. Equations (8)–(11) were obtained in Gaissinski et al [3]. Substituting (7) in the energy equation (6), we get

$$\mu_0 c_p T_1\big(\Psi''(r)+\frac{1}{r}\Psi'(r)\big)+\Psi(r)\big[\frac{i\omega A}{\gamma-1}\big(\rho_0 T_1+\rho_1 T_0\big)-\mu_0 c_p T_1\big(\frac{n^2}{r^2} + k^2\big)$$
$$- v_{\phi 1}\frac{in}{r}\big(P_0 + \frac{A}{\gamma-1}\rho_0 T_0\big)-v_{z1}ik\big(P_0 + \frac{A}{\gamma-1}\rho_0 T_0\big) - T_0\frac{\partial \rho_0}{\partial z} + \rho_0\frac{\partial T_0}{\partial z}\big]$$
$$-v_{r1}\big[\frac{1}{r}(P_0+\frac{A}{\gamma-1}\rho_0 T_0)+\frac{\partial P_0}{\partial r}+T_0\frac{\partial \rho_0}{\partial r}+\rho_0\frac{\partial T_0}{\partial r}\big]-\frac{dv_{r1}}{dr}\big(P_0+\frac{A}{\gamma-1}\rho_0 T_0\big)=0. \quad (13)$$

Using the expression $P_0 + \frac{A}{\gamma-1}\rho_0 T_0 = c_0\rho_0^\gamma\gamma/(\gamma-1)$ for steady state solution, and assuming $\ln\rho_0$ to be a weak function (Section 4), we obtain (12). We apply (similarly to Gaissinski et al [3]) the perturbation method according to the low value parameter

$$(\mu_0 k^2)/(\rho_0\omega) = \varepsilon_1 \ll 1. \quad (14)$$

We will present the solution to (9) in the following form:

$$v_{r1}(r) = v_{r1}^{(0)} + \varepsilon_1\big(\overrightarrow{\nabla}^2\overrightarrow{v}^{(0)}\big)_r,$$

$$\big(\overrightarrow{\nabla}^2\overrightarrow{v}^{(0)}\big)_r = \frac{\partial^2 v_{r1}^{(0)}}{\partial(kr)^2} + \frac{1}{(kr)^2}\frac{\partial^2 v_{r1}^{(0)}}{\partial\phi^2} + \frac{\partial^2 v_{r1}^{(0)}}{\partial(kz)^2} + \frac{1}{kr}\frac{\partial v_{r1}^{(0)}}{\partial(kr)} - \frac{2}{(kr)^2}\frac{\partial v_{\phi 1}^{(0)}}{\partial\phi} - \frac{1}{(kr)^2}v_{r1}^{(0)} \tag{15}$$

and $\overrightarrow{v}^{(0)} = \big(v_{r1}^{(0)}, v_{\phi1}^{(0)}, v_{z1}^{(0)}\big)$. Here $v_{r1}^{(0)}$, $v_{\phi1}^{(0)}$, $v_{z1}^{(0)}$ are the solutions of the momentum equation with zero viscosity

$$v_{\phi1} - \frac{\partial P'}{\partial\rho'}\big(\frac{\rho_1}{\rho_0}\big)\frac{1}{r}\frac{n}{\omega} + \frac{\mu_0}{\rho_0}\frac{1}{i\omega}\frac{1}{\Psi(r)}\big(\overrightarrow{\nabla}^2\overrightarrow{v}'\big)_\phi = 0, \tag{16}$$

$$v_{z1} - \frac{\partial P'}{\partial\rho'}\big(\frac{\rho_1}{\rho_0}\big)\frac{k}{\omega} + \frac{\mu_0}{\rho_0}\frac{1}{i\omega}\frac{1}{\Psi(r)}\big(\overrightarrow{\nabla}^2\overrightarrow{v}'\big)_z = 0. \tag{17}$$

In aim to deduce linear ODE for the function $\Psi(r)$, we find that

$$v_{r1}^{(0)} = -\frac{i}{\omega}\big(\frac{\rho_1}{\rho_0}\big)\frac{\partial P'}{\partial\rho'}\Psi'(r), \quad v_{\phi1}^{(0)} = \frac{n}{\omega}\big(\frac{\rho_1}{\rho_0}\big)\frac{1}{r}\frac{\partial P'}{\partial\rho'}, \quad v_{z1}^{(0)} = \frac{k}{\omega}\big(\frac{\rho_1}{\rho_0}\big)\frac{\partial P'}{\partial\rho'}, \tag{18}$$

where $v_{r1}^{(0)}, v_{\phi1}^{(0)}, v_{z1}^{(0)}$ can be determined easily from (9), (16), (17). Substituting (18) in (8) and (12) with $\mu_0 = 0$ we get two equations

$$\Psi''(r) + \Psi'(r)\big[\frac{1}{r} + \frac{\partial}{\partial r}\big(\ln\frac{\partial P'}{\partial\rho'}\big)\big] + \Psi(r)\big(\omega^2/\frac{\partial P'}{\partial\rho'} - k^2 - \frac{n^2}{r^2}\big) = 0, \tag{19}$$

$$c_0\rho_0^{\gamma-1}\frac{\gamma\rho_1}{\omega}\frac{\partial P'}{\partial\rho'}\big(\frac{1}{r}\Psi'(r) + \Psi''(r)\big)$$
$$+ \Psi(r)\big[A\omega^2\big(\rho_0 T_1 + T_0\rho_1\big) - \big(\frac{n^2}{r^2} + k^2\big)c_0\rho_0^{\gamma-1}\frac{\gamma\rho_1}{\omega}\frac{\partial P'}{\partial\rho'}\big] = 0. \tag{20}$$

Proposition 1. *The ODEs (19), (20) for $\Psi(r)$ present the same.*

Proof. Using this relation $\frac{\partial P'}{\partial\rho'} = \gamma AT_0$ and the definition $a = \sqrt{\gamma AT_0}$ of sound velocity we conclude from (19) that

$$\Psi''(r) + \frac{1}{r}\Psi'(r) + \big(\beta^2(r) - \frac{n^2}{r^2}\big)\Psi(r) = 0, \tag{21}$$

where $\beta(r) = \sqrt{\omega^2/a^2(r) - k^2}$. Dividing (20) by $c_0\rho_0^{\gamma-1}\gamma\rho_1\dfrac{\partial P'}{\partial\rho'}$ gives

$$\Psi''(r) + \frac{1}{r}\Psi'(r) + \big[\frac{A\omega^2(\rho_0 T_1 + T_0\rho_1)}{c_0\rho_0^{\gamma-1}\,\gamma\,\rho_1\,(\partial P'/\partial\rho')} - \big(\frac{n^2}{r^2} + k^2\big)\big]\Psi(r) = 0. \tag{22}$$

From $P_1 = A(\rho_0 T_1 + T_0\rho_1)$ and $P_1/\rho_1 = \gamma c_0\rho_0^{\gamma-1}$ we find that $A(\rho_0 T_1 + T_0\rho_1) = c_0\rho_0^{\gamma-1}\,\gamma\,\rho_1$. Hence, (22) reduces to (21). □

Let us assume the sound velocity $a(r)$ to be of Gaussian form

$$a(r) = a_0\big[1 + (1/2)e^{-(r-R_0)^2/\sigma^2}\big] \tag{23}$$

at the jets boundary layer and to be constant otherwise. This form with $\sigma \approx 0.1R_0$ is a good approximation for the real radial distribution Gaissinski et al [3], [2]. We select $\sigma^2 = R_0^2/100\ln(50)$ to make the function $a(r)$ be nearly constant inside all jet except boundary layer. Thus, $a(r) = a_0\big[1 + (1/2)e^{-100\ln(50)(r/R_0-1)^2}\big]$, and

$$1/a^2(r) \approx \frac{4}{9a_0^2}\big[1 + \frac{1}{3}200\ln(50)\big(r/R_0 - 1\big)^2\big]. \tag{24}$$

Substituting (24) in (21) leads to the *biconfluent Heun equation* (see Ronveaux [5]) for Ψ:

$$\Psi'' + \frac{1}{r_1}\Psi' + \big(\frac{b\,\omega^2}{a_0^2R_0^2\alpha^4\beta_1^4}r_1^2 - \frac{2\,b\,\omega^2}{a_0^2R_0\alpha^3\beta_1^3}r_1 + \frac{1}{\alpha^2} - \frac{n^2}{r_1^2}\big)\Psi = 0. \tag{25}$$

Here $b = 800\ln(50)/27$, $c = \frac{4}{9} + b$, $\beta_1 = k\sqrt{c\,\omega^2/(a_0^2k^2) - 1}$, and $r_1 = \alpha\beta_1 r$, $A_0' = -n^2$, $A_1' = 0$, $A_2' = 1/\alpha^2$, $A_3' = -\frac{2\,b\,\omega^2}{a_0^2R_0\alpha^3\beta_1^3}$, $A_4' = \frac{b\,\omega^2}{a_0^2R_0^2\alpha^4\beta_1^4} = 1$, and $\alpha = (-A_4)^{1/4}$, where $A_4 = b\,\omega^2/(a_0^2R_0^2\beta_1^4)$.

Remark 1. Using $k > 1$ order approximations one may obtain more general than (25) representative equation (Ronveaux [5]) that depends on $m = 2k$.

Using substitution $\Psi(r_1) = r_1^{B_0/2}e^{-\frac{1}{2}(B_1r_1+r_1^2)}\widetilde{\Psi}(r_1)$, we reduce (25) to canonical form, Ronveaux [5] (with $B_3 = 0$):

$$\widetilde{\Psi}''+\big(\frac{1+B_0}{r_1}-B_1-2r_1\big)\widetilde{\Psi}' + \big[\,(B_2-B_0-2) - \frac{B_1}{2r_1}\,(1+B_0)\,\big]\widetilde{\Psi} = 0, \tag{26}$$

where $A_0' = -n^2 = -B_0^2/4$, $A_1' = 0 = -B_3/2$, $A_2' = 1/\alpha^2 = B_2 - B_1^2/4$, $A_3' = -2\,b\,\omega^2/(a_0^2R_0\alpha^3\beta_1^3) = -B_1$, $A_4' = b\,\omega^2/(a_0^2R_0^2\alpha^4\beta_1^4) = -1$. The general solution of (26) is, see Ronveaux [5]:

$$\widetilde{\Psi}_n(r_1) = C_1N+C_2\big(N+\textstyle\sum_{n=1}^{\infty} d_nr_1^{n-B_0}\big),\ N{=}N_n(B_0,B_1,B_2,0;r_1). \tag{27}$$

Denote by $\widetilde{B}_n(r_1){=}r_1^{B_0/2}e^{-\dfrac{1}{2}(B_1r_1+r_1^2)}\,C_1N_n(B_0,B_1,B_2,0;r_1)$ the *biconfluent Heun function*, where N_n is the general solution of (26), and by $H_n^{(1)}$ the Hankel function.

Proposition 2. *The complex function* $\Phi = \frac{\omega}{ka_{in}}$ *(the ratio of phase velocity* $\dfrac{\omega}{k}$ *and the sound velocity in the jet) satisfies the following* dispersion relation:

$$\frac{\widetilde{B}_n'(\beta_{in}X)+\Upsilon_\nu^{in}[\beta_{in}^2\widetilde{B}_n'''(\beta_{in}X)+\frac{\beta_{in}}{X}\widetilde{B}_n''(\beta_{in}X)-\frac{1}{X^2}\widetilde{B}_n'(\beta_{in}X)]}{H_n^{(1)\prime}(\beta_{ex}X)+\Upsilon_\nu^{ex}[\beta_{ex}^2H_n^{(1)\prime\prime\prime}(\beta_{ex}X)+\frac{\beta_{ex}}{X}H_n^{(1)\prime\prime}(\beta_{ex}X)-\frac{1}{X^2}H_n^{(1)\prime}(\beta_{ex}X)]} \tag{28}$$
$$\times\frac{\beta_{in}}{\beta_{ex}}\frac{H_n^{(1)}(\beta_{ex}X)}{\widetilde{B}_n(\beta_{in}X)}=\frac{\rho_{in}}{\eta\,\rho_{ex}}\Big(\frac{\Phi-M_{in}}{\Phi/\sqrt{\eta}-M_{ex}}\Big)^2,$$

where $\Upsilon_\nu^{in}=\dfrac{\nu_{in}\,k}{a_{in}(\Phi-M_{in})}$, $\Upsilon_\nu^{ex}=\dfrac{\nu_{ex}\,k}{a_{ex}(\Phi/\sqrt{\eta}-M_{ex})}$, $X=kR_0$, $\eta=(a_{ex}/a_{in})^2$, ρ_{in},ρ_{ex} *non-perturbed jet's and ambient fluid (atmosphere) densities.*

Proof. From the boundness of solution for $r_1\to 0$ we obtain $C_2=0$. Substitution $\widetilde{\Psi}_n(r_1)=C_1N_n(B_0,B_1,B_2,0;r_1)$ in $\widetilde{B}_n(r_1)=r_1^{B_0/2}e^{-\frac{1}{2}(B_1r_1+r_1^2)}\widetilde{\Psi}_n(r_1)$ gives the solution to (25) as follows: $\Psi_n(r_1)=C_1\widetilde{B}_n(r_1)$. Substituting $\Psi_n(r_1)=C_1\widetilde{B}_n(r_1)$ in $v_{r1}^{(0)}=-\frac{i}{\omega}\big(\frac{\rho_1}{\rho_0}\big)\frac{\partial P'}{\partial\rho'}\frac{\partial\Psi(r)}{\partial r}$ for $r<R_0$, we obtain $v_{r1}^{(0)}(kr)=-i(\rho_1/\rho_0)$ $a_0\pi(kr)t_{in}\widetilde{B}_n'(r_1)$, where

$$r_1=\beta_{in}kr,\quad \beta_{in}=\alpha\sqrt{c\,\omega^2/(a_0^2k^2)-1},\quad t_{in}=\sqrt{1-a_{in}^2k^2/(c\omega_{in}^2)},$$

and $g(kr)=1+\dfrac{1}{2}e^{-100\ln(50)\left(\frac{kr}{kR_0}-1\right)^2}$. Recall that

$$v_{r1}(kr)=v_{r1}^{(0)}+\varepsilon_1(\overrightarrow{\nabla}^2\overrightarrow{v}^{(0)})_{kr},$$

where

$$\big(\overrightarrow{\nabla}^2\overrightarrow{v}^{(0)}\big)_{kr}=-i\big(\frac{\rho_1}{\rho_0}\big)C_1t_{in}a^2(kr)\big[\beta_{in}^2\widetilde{B}_n'''(r_1)+\frac{\beta_{in}}{kr}\widetilde{B}_n''(r_1)-\frac{\widetilde{B}_n'(r_1)}{(kr)^2}\big]. \tag{29}$$

Using plane wave expressions for perturbed values of the velocity radial component and the density, $v_r'=v_{r1}(r)e^{i(kz+n\phi-\omega t)}$ and $\rho_1=\frac{\rho'}{\widetilde{B}_n(r_1)}\,e^{-i(kz+n\phi-\omega t)}$, we obtain for $r<R_0$

$$v_r'=-i\big(\frac{\rho_{in}'}{\rho_{in}}\big)a_0g(kr)t_{in}\frac{\widetilde{B}_n'(r_1)+\varepsilon_{in}\big[\beta_{in}^2\widetilde{B}_n'''(r_1)+\frac{\beta_{in}}{kr}\widetilde{B}_n''(r_1)-\frac{\widetilde{B}_n'(r_1)}{(kr)^2}\big]}{\widetilde{B}_n(r_1)}. \tag{30}$$

Solution of (21) for $r>R_0$ has the form $\Psi(\beta_0kr)=C_3H_n^{(1)}(\beta_0kr)$, where $H_n^{(1)}=J_n+iY_n$ is the Hankel function describing outward propagating disturbances since the sound velocity $a(r)=a_0$ =const. For $r>R_0$ we deduce the following equation (Gaissinski et al [3]):

$$v_r' \;=\; -ia_{ex}\left(\rho_{ex}'/\rho_{ex}\right)t_{ex}\{H_n^{(1)\prime}(\beta_{ex}kr)+\varepsilon_{in}[\beta_{ex}^2H_n^{(1)\prime\prime\prime}(\beta_{ex}kr)$$

$$+\frac{\beta_{ex}}{kr}H_n^{(1)\prime\prime}(\beta_{ex}kr)-\frac{1}{(kr)^2}H_n^{(1)\prime}(\beta_{ex}kr)]\}/H_n^{(1)}(\beta_{ex}kr), \qquad (31)$$

where $t_{ex} = \sqrt{1-(ka_{ex}/\omega_{ex})^2}$. Displacement of the boundary $r = R_0$ between two fluids by an amount $\delta' = \delta_1 e^{i(kz+n\phi-\omega t)}$ is related to the velocity by $\frac{\partial\delta'}{\partial t} = -i\omega\delta' = v'_r$. Using (30) and (31) we may write the the equality for the displacement at the boundary $r = R_0$ as

$$\delta'=\frac{1}{\omega_{in}}\frac{\rho'_{in}}{\rho_{in}}\,a_{in}t_{in}\frac{\widetilde{B}'_n(\beta_{in}X)+\varepsilon_{in}[\beta_{in}^2\widetilde{B}'''_n(\beta_{in}X)+\frac{\beta_{in}}{X}\widetilde{B}''_n(\beta_{in}X)-\frac{\widetilde{B}'_n(\beta_{in}X)}{X^2}]}{\widetilde{B}_n(\beta_{in}X)}$$

$$=\frac{a_{ex}\rho'_{ex}}{\omega_{ex}\rho_{ex}}t_{ex}\times$$

$$\frac{H_n^{(1)\prime}(\beta_{ex}X)+\varepsilon_{ex}[\beta_{ex}^2H_n^{(1)\prime\prime\prime}(\beta_{ex}X)+\frac{\beta_{ex}}{X}H_n^{(1)\prime\prime}(\beta_{ex}X)-\frac{1}{X^2}H_n^{(1)\prime}(\beta_{ex}X)]}{H_n^{(1)}(\beta_{ex}X)}. \qquad (32)$$

Using the transformations for frequencies of the internal and the external medium, $\omega_{in} = \omega - ku_{in}$, $\omega_{ex} = \omega - ku_{ex}$, and the continuity of the pressure, $\rho'_{in}a_{in}^2 = \rho'_{ex}a_{ex}^2$, we obtain from (32) the desired *dispersion relation* (28). □

For zero viscosity case, i.e., $\nu_{in} = \nu_{ex} = 0$, (28) reduces to a simpler form

$$\frac{H_n^{(1)}(\beta_{ex}X)\widetilde{B}'_n(\beta_{in}X)}{\widetilde{B}_n(\beta_{in}X)H_n^{(1)\prime}(\beta_{ex}X)}\cdot\frac{\beta_{in}}{\beta_{ex}}=\frac{\rho_{in}}{\eta\,\rho_{ex}}\left(\frac{\Phi-M_{in}}{\Phi/\sqrt{\eta}-M_{ex}}\right)^2. \qquad (33)$$

3. Asymptotic Analysis of Instabilities

The *dispersion relation* (28) can be expanded in the limit of small and large arguments of the Heun and Hankel functions. It is instructive to start with the expansion for small arguments $|\xi| \ll 1$, $|\zeta| \ll 1$, where $\xi = \beta_{in}X$, $\zeta = \beta_{ex}X$, i.e., the long-wavelength limit. Asymptotic analysis of instabilities for constant sound velocity for $n = 0$ and $n = 1$ see Gaissinski et al [3].

3.1. Pinching Mode ($n = 0$)

For $n = 0$ we use expansions of the Hankel functions (Gaissinski et al [3]) and expansions of the Heun functions, see Ronveaux [5]:

$$\widetilde{B}_0(\xi) = \widetilde{B}_0(0) + \widetilde{B}_0'(0)\xi + o(\xi), \quad \widetilde{B}_0'(\xi) = \widetilde{B}_0'(0) + \widetilde{B}_0''(0)\xi + o(\xi),$$
$$\widetilde{B}_0''(\xi) = \widetilde{B}_0''(0) + \widetilde{B}_0'''(0)\xi + o(\xi), \quad \widetilde{B}_0'''(\xi) = \widetilde{B}_0'''(0) + \widetilde{B}_0^{(4)}(0)\xi + o(\xi).$$

Recall that $\widetilde{B}_0(\xi) = e^{-\frac{1}{2}\left(B_1\xi + \xi^2\right)} C_1 N_0(0, B_1, B_2, 0; \xi)$. Using the power series for $N_0(0, B_1, B_2, 0; \xi)$, Ronveaux [5], we find that

$$\begin{aligned} \widetilde{B}_0(\xi) &= C_1 + o(\xi), \quad \widetilde{B}_0'(\xi) = C_1\xi b_1 + o(\xi), \\ \widetilde{B}_0''(\xi) &= C_1(b_1 + b_2\xi) + o(\xi), \quad \widetilde{B}_0'''(\xi) = C_1(b_2 + b_3\xi) + o(\xi), \end{aligned} \tag{34}$$

where $b_1 = \frac{1}{8}B_1^2 - \frac{1}{2}B_2 = -1/2\alpha^2$, $b_2 = \frac{2B_1}{3}$, $b_3 = \frac{3}{128}B_1^4 - \frac{3}{16}B_1^2 B_2 + \frac{3}{2} + \frac{3}{8}B_2^2$, and $B_0 = 2n$, $B_1 = \frac{2b\,\omega^2}{a_0^2 R_0 \alpha^3 \beta_1^3}$, $B_2 = \frac{1}{\alpha^2} + \left(\frac{b\,\omega^2}{a_0^2 R_0 \alpha^3 \beta_1^3}\right)^2$, and $\alpha = (-A_4)^{1/4} = A_4^{1/4}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$, $A_4 = \frac{b\,\omega^2}{a_0^2 R_0^2 \beta_1^4}$, $\beta_1 = \sqrt{c\omega^2/a_0^2 - k^2}$.

Substituting long wavelength expansions of Hankel functions and (34) in (28) for $n = 0$, and in view of $|\xi| \ll 1$, $|\zeta| \ll 1$, (28) gives

$$\frac{\frac{2i}{\pi}\xi^2\left[b_1 + \frac{\Upsilon_\nu^{in}}{X^2}(2\xi b_2)\right]\left(\ln\frac{\zeta}{2} + C\right)}{\left(\frac{2i}{\pi}\left[\frac{1}{2}\ln\frac{\zeta}{2} + \frac{1}{4}(2C-1) + \frac{1}{\zeta^2}\right] - \frac{1}{2} - \frac{i}{X^2\pi}\Upsilon_\nu^{ex}\right)\zeta^2} = \frac{\rho_{in}}{\eta\,\rho_{ex}}\left(\frac{\Phi - M_{in}}{\Phi/\sqrt{\eta} - M_{ex}}\right)^2. \tag{35}$$

For the low jets viscosity we have $|\Upsilon_\nu^{in}| \ll 1$ and $|\Upsilon_\nu^{ex}| \ll 1$. For $n = 0$ and under condition $\nu_{(in,ex)} \to 0$, the equation (35) transforms to

$$\frac{1}{\eta}\frac{\rho_{in}}{\rho_{ex}}\left(\frac{\Phi - M_{in}}{\Phi/\sqrt{\eta} - M_{ex}}\right)^2 = b_1(\beta_{in}X)^2\left(\ln\frac{\beta_{ex}X}{2} + C\right), \tag{36}$$

where $\beta_{ex} = \left((\Phi/\sqrt{\eta} - M_{ex})^2 - 1\right)^{1/2}$, $\beta_{in} = b^{1/4}\sqrt{(\Phi - M_{in})\,i/X}$. In view of $\ln\frac{\zeta}{2} + C \backsimeq -\ln\frac{2}{\zeta}$, (36) becames similar to equation in Gaissinski [3]

$$\left(\frac{\Phi - M_{in}}{\Phi - \sqrt{\eta}M_{ex}}\right)^2 = \frac{1}{2}\left(\widetilde{\beta}_{in}X\right)^2\frac{\rho_{ex}}{\rho_{in}}\ln\frac{2}{\beta_{ex}X}, \tag{37}$$

where $\widetilde{\beta}_{in} = \sqrt{c\,(\Phi - M_{in})^2 - 1}$. For $\widetilde{\beta}_{in}X \ll 1$ it can be seen that the RHS of (37) is more less than 1 also due to $\ln(2/\beta_{ex}X)$ is a weak function (see Section 4) by comparison with $\widetilde{\beta}_{in}X$ and the ratio ρ_{ex}/ρ_{in} is bounded.

3.2. Helical Mode ($n = 1$)

For $n = 1$ we use the appropriate long wavelength expansions of the Hankel functions (Gaissinski et al [3]), and the expansions of the Heun function (Ronveaux [5]):

$$\widetilde{B}_1(\xi) = \widetilde{B}_1(0) + \widetilde{B}_1'(0)\xi + o(\xi), \quad \widetilde{B}_1'(\xi) = \widetilde{B}_1'(0) + \widetilde{B}_1''(0)\xi + o(\xi),$$
$$\widetilde{B}_1''(\xi) = \widetilde{B}_1''(0) + \widetilde{B}_1'''(0)\xi + o(\xi), \ \widetilde{B}_1'''(\xi) = \widetilde{B}_1'''(0) + \widetilde{B}_1^{(4)}(0)\xi + o(\xi).$$

Recall that

$$\widetilde{B}_1(\xi) = \xi e^{-\frac{1}{2}\left(B_1\xi + \xi^2\right)} C_1 N_1(2, B_1, B_2, 0; \xi). \tag{38}$$

Using power series for $N_1(2, B_1, B_2, 0; \xi)$ together with recursive formula, Ronveaux [5], we find from (38) that

$$\begin{aligned} \widetilde{B}_1(\xi) &= C_1\xi + o(\xi), \quad \widetilde{B}_1'(\xi) = C_1 + o(\xi), \\ \widetilde{B}_1''(\xi) &= C_1\widehat{b}_1\xi + o(\xi), \quad \widetilde{B}_1'''(\xi) = C_1(\widehat{b}_1 + \widehat{b}_2\xi) + o(\xi), \end{aligned} \tag{39}$$

where $\widehat{b}_1 = \frac{3}{16}B_1^2 - \frac{3}{4}B_2$, $\widehat{b}_2 = \frac{8}{5}B_1$. Substituting appropriate long wavelength expansions of the Hankel functions and (39) in *dispersion relation* (28) for $n = 1$, and in view of $|\xi| \ll 1$ and $|\zeta| \ll 1$, equation (28) becomes

$$\frac{1}{\eta}\frac{\rho_{in}}{\rho_{ex}}\Big(\frac{\Phi - M_{in}}{\Phi/\sqrt{\eta} - M_{ex}}\Big)^2 = -\frac{1 - \Upsilon_\nu^{in}/X^2}{1 + 3\Upsilon_\nu^{ex}/X^2}. \tag{40}$$

For the low jets viscosity we have $|\Upsilon_\nu^{in}| \ll 1$ and $|\Upsilon_\nu^{ex}| \ll 1$. In case $n = 1$ the equation (40) under condition $\nu_{(in,ex)} \to 0$ transforms to

$$\frac{1}{\eta}\frac{\rho_{in}}{\rho_{ex}}\left(\frac{\Phi - M_{in}}{\Phi/\sqrt{\eta} - M_{ex}}\right)^2 = -1. \tag{41}$$

For long-wavelength limit $X \ll 1$, equation (41) has the solution

$$\Phi^{\mathrm{Re}} = \frac{M_{in} + (\rho_{ex}/\rho_{in})\sqrt{\eta}M_{ex}}{1 + \rho_{ex}/\rho_{in}}, \qquad \Phi^{\mathrm{Im}} = (\rho_{ex}/\rho_{in})^{\frac{1}{2}}\,\frac{M_{in} - \sqrt{\eta}M_{ex}}{1 + \rho_{ex}/\rho_{in}}.$$

For very low viscosity when $X^2 \ll \Upsilon_\nu^{in}$, $X^2 \ll \Upsilon_\nu^{ex}$, the equation (40) becomes $\frac{\rho_{in}}{\eta\,\rho_{ex}}\big(\frac{\Phi - M_{in}}{\Phi/\sqrt{\eta} - M_{ex}}\big)^2 = \frac{\Upsilon_\nu^{in}}{3\Upsilon_\nu^{ex}}$. Its solution is

$$\Phi = \frac{M_{in} - M_{ex}[3\,(\rho_{ex}/\rho_{in})^2\,(\mu_{in}/\mu_{ex})\,\eta^{1/4}]^{1/3}}{1 - [3\,(\rho_{ex}/\rho_{in})^2\,(\mu_{in}/\mu_{ex})\,\eta^{1/2}]^{1/3}}.$$

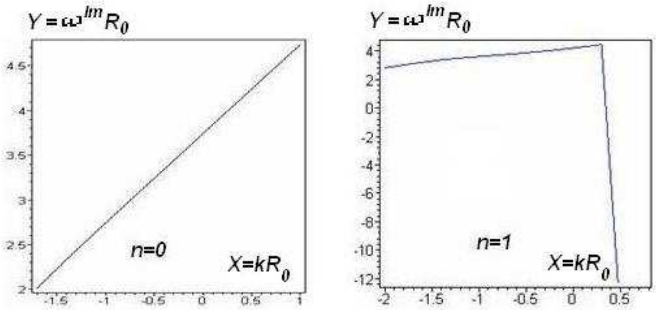

Figure 1: Growth of pinching and helical modes as a function of $X = kR_0 = (\gamma/2\pi R_0)^{-1}$ for $\mu_0 = 9.64 \cdot 10^{-5}$ kg/m^2; $\gamma/2\pi R_0$ is a ratio of the waves length to the jets radius at the inlet.

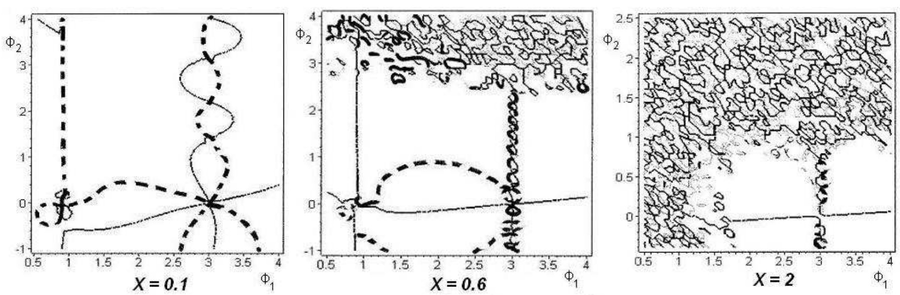

Figure 2: Spectral graphs for roots of (28); $n = 0$ and $\mu_0 = 9.64{\cdot}10^{-5}\,\text{kg/m}^2$.

4. Results and Discussion

We investigate pinching and helical wave modes by numerical approach in order to find wave-number at which growth of these modes is a maximal, and how the maximum growth is related to viscosity of the jet and of external medium. The study of modes with $n > 1$ is more complicated, the jet's radial cross section has a form of n-petal's rose. In numerical examples (see Figures 1, 2) we choose the pressures $P_{in} = 1.6$atm and $P_{ex} = 0.35\,$atm, the total densities $\rho_{in} = 0.28\,\text{kg/m}^3$ and $\rho_{ex} = 0.55\,\text{kg/m}^3$, and the specific heat ratios $\gamma_{in} = 1.29$ and $\gamma_{ex} = 1.4$, these conditions might exist, respectively, in the rocket plume's medium and in the external medium at altitude $H = 8\,$km. The sound velocity values are $a_{in} = 1000\,$m/sec and $a_{ex} = 375\,$m/sec, respectively, and $\eta = 0.141$.

The internal and external Mach numbers $M_{in} = 3$, $M_{ex} = 2.4$ are typical for the chosen altitude $H = 8\,\mathrm{km}$. Equation (28) is the complex one of the form $f_n(X, \Phi) = 0$ with $f_n = f_n^{\mathrm{Re}} + i f_n^{\mathrm{Im}}$; it should be calculated due two cases: inviscid and with low viscosity $\nu = \mu/\rho$. One may consider $\Phi = \Phi^{\mathrm{Re}} + i\Phi^{\mathrm{Im}}$ as an implicit function of $X \in I$, where $I = [0.01, 10]$. Let us rewrite $f_n(X, \Phi) = 0$ as the nonlinear system of two real equations solved numerically, see Gaissinski et al [3]:

$$f_n^{\mathrm{Re}}(X, \Phi^{\mathrm{Re}}, \Phi^{\mathrm{Im}}) = 0, \quad f_n^{\mathrm{Im}}(X, \Phi^{\mathrm{Re}}, \Phi^{\mathrm{Im}}) = 0. \tag{42}$$

The dispersion equation (36) is obtained for a linear approach to the viscous term. Its resonance cases were studied only for constant sound velocity, Gaissinski et al [3]. It will be interesting to develop the resonance case for non constant sound velocity. We analyze (36) (using *Maple* program) to study dependence of the non-dimensional phase velocity $\Phi = \omega/ka_{in}$ (complex function) on the parameter X. The numerical results allows to conclude that the pinching and helical modes increase and the maximum of their oscillations corresponds to the wavelengths longer than or on the order of the instantaneous jet's radius.

Figures 1 show that behavior of the value v^{ω}_{ph} weakly depends on the jets viscosity (under consideration the low viscosity jets). Figure 1(a) shows the behavior of the growth rate (under condition that we consider low viscosity jets) of pinching mode for the internal Mach number $M_{in} = 3$ and for external Mach number $M_{ex} = 2.4$. It can be seen that for pinching mode the value of the perturbed phase velocity $v^{\omega}_{ph} = \omega^{\mathrm{Im}} R_0$ has asymptotic $\sim X$ (36) at $X \ll 1$ and perturbed phase velocity increases in almost straight line when the wave number k increases. Figure 1(b), shows the value of phase velocity v^{ω}_{ph} for $n = 1$ under low viscosity jet ($\mu_{in} = 9.64 \cdot 10^{-5}\,\mathrm{kg/(m \cdot sec)}$) in rocket plume since there can occur about of 40% of alumina (Al_2O_3). The phase velocity weakly depends on the jet's viscosity (for low viscosities) has maximum at $X \approx 0.4$ and decreases rapidly for $X \gg 1$, i.e., short wavelengths oscillations die down in case of low viscosities. Spectral graphs (Figure 2) show that for low viscosity, (28) has asymptotic solutions (36). Hence, the asymptotic equation's root are $\Phi^{\mathrm{Re}} \approx 3$ and $\Phi^{\mathrm{Im}} \approx 0.04$ at the point $X = 0.01$, that obtained by iterative method is preserved at the interval $0.01 \lesssim X \lesssim 0.1$, i.e., the topology of the curves is preserved. These graphs show that the number of roots increases rapidly with X, here Φ^{Im} increases while Φ^{Re} remain ≈ 3, i.e., a dissipation of the perturbed waves to high number microwaves occurs. This process begin from $X \gtrsim 0.6$. The point $X \backsimeq 0.6$ may be considered as the point of wave dissipation. The microwave developed flow may be considered as turbulence of the jets boundary layer (Figure 2 at point $X = 0.6$). We can conclude that for pinching mode the low viscosity does not affect on the topology of the roots; it influences on time of the beginning of dissipation.

References

[1] P.G. Drazin, *Hydrodynamics Stability*, Cambridge University Press (2004).

[2] I.M. Gaissinski, M. Lev, J. Stricker, Spatial distribution calculations of soot particles in supersonic rocket plumes, In: *Proceed. 43-rd Israel Annual Conference on Aerospace Sciences*, February (2003).

[3] I.M. Gaissinski, V.Yu. Rovenski, Kelvin-Helmholtz instability model for supersonic viscid jet, *J. of Pure and Applied Mathematics*, **24**, No. 3 (2005), 537-551.

[4] P. Hardee, Helical and pinching instability of supersonic expanding jets in extragalactic radio sources, *Astrophysical J.*, **257** (1982), 509-526.

[5] A. Ronveaux, *Heun's Differential Equations*, Oxford University Press (1995).

[6] Ya.B. Zel'dovich, Yu.P. Raizer, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, Academy of Sciences, USSR, Moscow, Volume 1 (1966).

Appendix – Physical Estimations

We estimate the terms in brackets in the 3-rd line of (13),

$$\frac{1}{r}\big(P_0+\frac{A}{\gamma-1}\rho_0 T_0\big)+\frac{\partial P_0}{\partial r}+T_0\frac{\partial \rho_0}{\partial r}+\rho_0\frac{\partial T_0}{\partial r}. \tag{43}$$

Since the three terms in (43) are $\sim \rho_0^\lambda \partial(\ln\rho_0)/\partial r$, we estimate only 1 ratio

$$\frac{T_0(\partial\rho_0/\partial r)\, r}{P_0+\rho_0 T_0 A/(\gamma-1)}=\frac{r\, c_0/A}{1+A/(\gamma{-}1)}\frac{\rho_0^{\gamma-2}}{a_0^2}\frac{\partial\rho_0}{\partial r}\le\gamma r\frac{\rho_0^{\gamma-2}}{a_0^2}\frac{\partial\rho_0}{\partial r}, \tag{44}$$

because of $c_0\approx 83300$, $A=8340/\mathcal{M}_{J,a}\approx 290$, and, consequently, $\frac{c_0/A}{1+A/(\gamma-1)}<1$; here $\mathcal{M}_{J,a}\approx 28\,(29)$ – molecular weight of the jets medium and the atmosphere. The aim of our estimation is to find the characteristic length l of the boundary layer where the dumping of the density is maximal. The reason of the boundary layer appearance is the inner dissipative processes – viscosity and thermoconductivity of the gas. Let us compare inertial forces with the forces caused by viscosity μ and thermoconductivity $\varkappa$. If U be a velocity scaling, and R be a characteristic size of the flow (in the radial direction), then the inertial term is $\sim \rho U^2/R$. The viscous term $\mu\nabla^2\mathbf{u}$ in the equations for the momentum may be estimated as $\mu U^2/R^2$; and so their ratio is $1/Re = \mu/\rho UR = \nu/UR \sim lc/UR$. Similarly, by comparing the conductive heat transfer with the mechanical one, we obtain $1/\mathrm{Pe} = \varkappa/\rho c_p UR = \chi/UR \sim la/UR$, where c_P– the heat capacity under constant pressure; Pe is the Peckle number, which is, in gases, to be nearly equal to Reynolds number Re, since, for gases, molecular thermoconductivity $\chi = \varkappa/\rho c_P$ is nearly equal to kinematic viscosity $\nu = \mu/\rho$. Hence, $l \sim (U/a_0)R/Re = M_J\,R/Re$. Substituting this into (44) gives

$$\frac{(\partial P_0/\partial r)\,r}{P_0+\rho_0 T_0\,A/(\gamma-1)} \le \gamma r\frac{\rho_0^{\gamma-2}}{a_0^2}\frac{\partial\rho_0}{\partial r} \approx \gamma R\frac{\rho_0^{\gamma-1}}{a_0^2}\frac{\delta\rho_0}{l} = \frac{1}{a_0}\gamma\frac{\rho_0^{\gamma}\delta\rho_0}{\mu}R \approx 0.02. \tag{45}$$

Let us estimate the value $\delta\rho_0$. From Bernoulli equation $P_0+\rho_0 u_0^2 = P_a+\rho_a u_a^2$ and the adiabatic relation $P = c_0\rho^\gamma$, we obtain $\rho_0^\gamma - \rho_a^{\gamma_a} = \gamma_a M_a^2\rho_a^{\gamma_a} - \gamma M^2\rho_0^\gamma$, the subscript '$a$' signs the ambient atmospheric properties. Hence the damp of the density at the jet's interface may be estimated under assumption $\gamma \approx \gamma_a$, as $\delta\rho_0 = |\rho_a - \rho_0| \simeq \rho_0\big|1-\big(\frac{1+\gamma M^2}{1+\gamma M_a^2}\big)^{1/\gamma}\big|$. For supersonic high temperature jets: $a_0 \simeq 2000\mathrm{m}\cdot\mathrm{c}^{-1}$, $\gamma = 1.2-1.5$, $R < 0.5\,\mathrm{m}$, $M \simeq 3.0$, $M_a \simeq 2.4-2.6$, $\rho_0 \approx 0.1\,\mathrm{kg}\cdot\mathrm{m}^{-3}$, $\mu_0 \simeq 1.7\times10^{-5}\,\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{c}^{-1}$, and we finally obtain

$$\frac{(\partial P_0/\partial r)\,r}{P_0+\rho_0 T_0\,A/(\gamma-1)} \le \frac{1}{a}\gamma R\frac{\rho_0^{\gamma+1}}{\mu}\Big|1-\Big(\frac{1+\gamma M^2}{1+\gamma M_a^2}\Big)^{1/\gamma}\Big| \approx 0.01. \tag{46}$$