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ON THE FIXED POINTS OF A FUNCTION AND THE FIXED POINTS OF ITS COMPOSITE FUNCTIONS

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Abstract: In this paper we study the relationship between the fixed point(s) of a function $f(x)$ (if they exist), and the fixed point(s) of its composite functions $f^n(x)$ for $n \geq 2$. We start by showing that if x_p is a fixed point of the function $f(x)$, then x_p is a fixed point of the composite function $f^n(x)$ $(n \geq 2)$, and if $f(x)$ is fixed point free, then $f^n(x)$ $(n \geq 2)$ is fixed point free as well. The functions under our discussion will include polynomials and natural trigonometric functions as well as hyperbolic functions. Finally, we pose ten questions for the reader.

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1. Introduction

This paper is motivated by a problem proposal by the author which appeared in the January 2007 issue of the *College Mathematics Journal* [1], p. 60. In that problem proposal we assumed that the quadratic polynomial $f(x) = ax^2+bx+c$ $(a \neq 0)$ had two fixed points x_1 and x_2 $(x_1 \neq x_2)$, and we were asked to find the exact values of x_1 and x_2, provided 1 and -1 were given as two fixed points of $f(f(x))$, but not of $f(x)$. This paper is a broad generalization of that problem. We study the relationship between the fixed point(s) of a function $f(x)$ (if

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they exist) and the fixed point(s) of its composite functions $f^n(x)$ for $n \geq 2$. By a fixed point in this article, as in numerical analysis, we mean simply a solution of the equation $f(x) = x$ if it exists [5]. Hence, we will not be dealing with other types of fixed points such as elliptic fixed points, group fixed points, hyperbolic fixed points, parabolic fixed points, etc. We hope that our discussion here will serve as a catalyst for further study of this topic in other branches of mathematics as well as other areas of science and technology such as biology, chemistry, economics, engineering, game theory, and physics [4], [2].

We call a function $f(x)$ *fixed point free* if $f(x)$ does not have any fixed point. We keep in mind that a function may have no fixed point, k fixed points ($k \geq 1$) or infinitely many fixed points. A function $f(x)$ is *idempotent* if $f(f(x)) = f(x)$ for all x in the domain of f and a fixed point x_p of f is an *attractive fixed point* if $f^n(x_p) = x_p$ for all $n \geq 1$. A trigonometric function $t(x)$ is called a *natural trigonometric function* if x is measured in radian only. In this paper all trigonometric functions will be natural trigonometric functions. Moreover, if f is a function, then we define the *iterates* of f as $f^1(x) = f(x), f^2(x) = f(f(x))$,..., and $f^n(x) = \underbrace{f(f(f(...f(x)...)))}_{n \text{ times}}, n \geq 1$.

2. Results

Proposition 2.1. *(i) If x_p is a fixed point of the function $f(x)$, then x_p is a fixed point of the composite function $f^n(x)$ $(n \geq 2)$, and (ii) if $f(x)$ is fixed point free, then $f^n(x)$ $(n \geq 2)$ is fixed point free as well.*

Proof. (i) For proof we use mathematical induction and we show that if x_p is a fixed point of $f(x)$, then x_p is a root of the equation $f^n(x) - x = 0$, $(n \geq 1)$.

Let x_p be a fixed point of $f(x)$. If $n = 1$, then, by definition

$$f(x_p) - x_p = 0. \tag{1}$$

Also, when $n = 2$,

$$f(f(x_p)) - x_p = f(x_p) - x_p = x_p - x_p = 0. \tag{2}$$

Next, we need to show that if x_p is a fixed point of $f(x)$, such that

$$f^k(x_p) - x_p = 0, \tag{3}$$

then

$$f^{k+1}(x_p) - x_p = 0 \tag{4}$$

for an arbitrary positive integer k.

Now, equation (4) is a direct consequence of the definition of $f^n(x)$, equations (2), (3), and (1) as follows:

$$f^{k+1}(x_p) - x_p = f(f^k(x_p)) - x_p = f(x_p) - x_p = 0.$$

Therefore, we have shown that a fixed point of a function $f(x)$ is a fixed point of the composite function $f^n(x)$ $(n \geq 2)$.

(ii) If $f(x)$ is fixed point free, then $f^2(x)$ is fixed point free as well. For if there exists an x_0 in the domain of f such that $f^2(x_0) = x_0$, then $f(f(x_0)) = x_0$. This implies that $f(x_0)$ is mapped to x_0 which is impossible. For an arbitrary $n > 2$, one could use mathematical induction to finish the proof. □

We note that the converse of Proposition 2.1 may not be true. One could find many counter examples among polynomials of order 2 or higher. The function $f(x) = x^2 - 1$ is an easy example. Also, it is clear that $f^n(x)$ $(n \geq 2)$ may have more fixed points than $f(x)$.

Theorem 2.2. *Let $f(x) = ax + b$, where a and b are arbitrary real numbers. A number x_p is a fixed point of $f(x)$ if and only if x_p is the root of the equation $f^n(x) - x = 0, (n \geq 1)$.*

Proof. First we observe that if $a = 1$, then $f(x)$ is fixed point free. Thus, we may assume that $a \neq 1$. Also, if x_p is a fixed point of $f(x)$, then by Proposition 2.1, x_p is a root of the equation $f^n(x) - x = 0, (n \geq 2)$. Now, if x_r is a root of $f^n(x) - x = 0, (n \geq 1)$, then

$$\begin{aligned} f^n(x_r) - x_r &= \underbrace{f(f(f(...f(ax_r + b)...)))}_{n \text{ times}} - x_r \\ &= f(f(f(...a(ax_r + b) + b...))) - x_r \\ &\vdots \\ &= f(a^{n-1}x_r + a^{n-2}b + ... + ab + b) - x_r \\ &= a^n x_r + a^{n-1}b + ... + ab + b - x_r \\ &= (a^n - 1)x_r + (a^{n-1} + a^{n-2} + ... + a + 1)b \\ &= 0. \end{aligned}$$

Therefore,

$$x_r = \frac{-(a^{n-1} + a^{n-2} + ... + a + 1)b}{a^n - 1} = \frac{b}{1 - a},$$

which is the fixed point of $f(x) = ax + b$, and hence the proof is complete.

We note that if $b = 0$, then $f(x)$ is an idempotent function, and furthermore the fixed point x_p is an attractive fixed point. Moreover, we note that if $a = 0$, then $x_p = b$ is the only fixed point of $f^n(x)$ for $n \geq 1$. $\square$

Theorem 2.3. *If $f(x) = ax^2 + bx + c$ is a quadratic polynomial $(a \neq 0)$ and if x_1 and x_2 are two fixed points of $f(x)$, then*

$$f(f(x)) - x = a(x - x_1)(x - x_2)[a^2x^2 + a(b + 1)x + ac + b + 1].$$

Proof. Without loss of generality we may assume that $x_1 \neq x_2$. Now, by Proposition 2.1 both x_1 and x_2 are roots of the equation $f(f(x)) - x = 0$. Also, since both x_1 and x_2 are roots of the equation $f(x) - x = 0$, we have

$$f(x) - x = ax^2 + (b - 1)x + c = a(x - x_1)(x - x_2) = a(x^2 - (x_1 + x_2)x + x_1x_2).$$

That is, $a(x_1 + x_2) = 1 - b$. Now,

$$\begin{aligned}
&f(f(x)) - x = f(f(x)) - f(x) + f(x) - x \\
&= a[f(x) - x_1][f(x) - x_2] + a(x - x_1)(x - x_2) \\
&= a[f(x) - x + x - x_1][f(x) - x + x - x_2] + a(x - x_1)(x - x_2) \\
&= a[a(x - x_1)(x - x_2) + (x - x_1)][a(x - x_1)(x - x_2) + (x - x_2)] \\
&+ a(x - x_1)(x - x_2) \\
&= a(x - x_1)(x - x_2)[a^2(x - x_1)(x - x_2) + a(x - x_1) + a(x - x_2) + 2] \\
&= a(x - x_1)(x - x_2)q(x),
\end{aligned}$$

where

$$\begin{aligned}
q(x) &= a^2(x - x_1)(x - x_2) + a(x - x_1) + a(x - x_2) + 2 \\
&= a(f(x) - x) + ax - ax_1 + ax - ax_2 + 2.
\end{aligned}$$

But, since $a(x_1 + x_2) = 1 - b$, we can rewrite $q(x)$ as

$$\begin{aligned}
q(x) &= a(ax^2 + bx + c) + b - 1 + ax + 2 \\
&= a^2x^2 + a(b + 1)x + ac + b + 1.
\end{aligned}$$

Therefore,

$$f(f(x)) - x = a(x - x_1)(x - x_2)[a^2x^2 + a(b + 1)x + ac + b + 1],$$

as desired. $\square$

The following corollary is an immediate application of Theorem 2.3.

Corollary 2.4. *Let $f(x) = x^2 + bx + c$ where b and c are arbitrary real numbers. Also, let x_1 and x_2 be two fixed points of $f(x)$. If r_1 and r_2 are two roots of the equation $f(f(x)) - x = 0$ which are not the fixed points of $f(x)$, then*

$$b = -(1 + r_1 + r_2),$$

$$c = r_1r_2 + r_1 + r_2,$$

and x_1 and x_2 are the roots of the equation

$$x^2 - (2 + r_1 + r_2)x + r_1 r_2 + r_1 + r_2 = 0.$$

Theorem 2.5. *If $f(x)$ is a function such that its inverse $f^{-1}(x)$ exists in an interval, then a fixed point of $f(x)$ is a fixed point of $f^{-1}(x)$ in that interval, and vice versa. If $f(x)$ is fixed point free in an interval so is $f^{-1}(x)$ and vice versa.*

The proof is immediate from the fact that $f(x)$ and $f^{-1}(x)$ are symmetric with respect to the line $y = x$.

We note that the above theorem is valid for $f^n(x)$ and $(f^n(x))^{-1}$ $(n \geq 2)$ as well.

Theorem 2.6. *Let $a = \pm 1$.*

(i) If $f(x) = \cos(ax)$, then $f(x)$ and all its composite functions $f^n(x)$ $(n \geq 2)$ have the same fixed point, and if $a = 0$, then $x_p = 1$ is the only fixed point of $f^n(x)$ for all $n \geq 1$.

(ii) If $f(x) = \sin(ax)$, then $f(x)$ and all its composite functions $f^n(x)$ $(n \geq 2)$ have the same fixed point $x_p = 0$.

(iii) If $f(x) = \tan(ax), \cot(ax), \sec(ax)$, or $\csc(ax)$, then all $f^n(x)$ $(n \geq 1)$ have infinitely many fixed points.

Proof. It is very clear that $f(x) = \cos(ax) = \cos x$ has only one fixed point x_p which is the only solution of the equation $\cos x = x$. By Proposition 2.1 this x_p is also a fixed point of $f^n(x)$ for all $n \geq 2$. Moreover, x_p is the only solution to $f^n(x) = f(x) = \cos x = x$, for $n \geq 2$. Furthermore, if $a = 0$, then the proof is trivial. The proof of (ii) follows from the fact that the only solution to equations $x = f(x)$ and $x = f^n(x)$ is $x_p = 0$, and the proof of (iii) is immediate from the fact that the line $y = x$ intersects the graph of each of these four functions and their corresponding composite functions $f^n(x)$ $(n \geq 2)$ at infinitely many points. □

Theorem 2.7. *(i) If $f(x) = \cosh(ax)$, then all $f^n(x)$ $(n \geq 1)$ are fixed point free for all real numbers a such that $|a| \geq 1$ and if $a = 0$, then $x_p = 1$ is the only fixed point of $f^n(x)$ for all $n \geq 1$.*

(ii) If $f(x) = \sinh(ax)$, where $|a| \geq 1$ or $a = 0$, then $x_p = 0$ is the only fixed point $f^n(x)$ $(n \geq 1)$.

(iii) If $f(x) = \tanh(ax)$, where $|a| \leq 1$, then $x_p = 0$ is the only fixed point of $f^n(x)$ $(n \geq 1)$.

Proof. If $|a| \geq 1$, then clearly, $\cosh(ax) \geq 1 > x$, and thus $f(x)$ is fixed point free. Similarly, $f^2(x) = f(\cosh(ax)) \geq 1 > x$. Thus, $f^2(x)$ is fixed point free as well. Now, one could use mathematical induction on n to finish the rest of the proof. If $a = 0$, then the proof is trivial. The proof of (ii) and (iii) clearly follows from the fact that the only solution to equations $x = f(x)$ and $x = f^n(x)$ in both cases is $x_p = 0$. □

3. Questions

In this section we pose ten questions for the reader. We recall from Proposition 2.1, that (i) any fixed point of $f(x)$ is a fixed point of $f^n(x)$ $(n \geq 2)$, (ii) if $f(x)$ is fixed point free, then so is $f^n(x)$ $(n \geq 2)$, and (iii) $f^n(x)$ $(n \geq 2)$ may have fixed points other than the fixed points of $f(x)$. Before attempting to answer a question it would be very helpful to first graph the function under the discussion.

Question 3.1. Is the analog of Theorem 2.3 true for a polynomial of degree $n(n \geq 3)$? That is, if $a_n \neq 0$, and $x_1, x_2, ..., x_n$ are n fixed points of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,$$

then is it possible to have

$$f(f(x)) - x = (x - x_1)(x - x_2)...(x - x_n)q(x),$$

where $q(x)$ is a polynomial of order $n^2 - n$? Can this question be generalized further for $f^k(x) - x$ $(k \geq 3)$?

Question 3.2. (i) If $a \neq \pm 1$, then what can be said about the fixed point(s) of $f(x) = \sin(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$? (ii) If $a \neq \pm 1$ or $a \neq 0$, then what can be said about the fixed point(s) of $f(x) = \cos(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$? (iii) By Theorem 2.6, if $a = \pm 1$ and if $f(x) = \tan(ax), \cot(ax), \sec(ax)$, or $\csc(ax)$, then $f^n(x)$ $(n \geq 1)$ in all four cases have infinitely many fixed points. What are the values of these fixed points?

Question 3.3. (i) If $0 < |a| < 1$, then what can be said about the fixed points of $f(x) = \cosh(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$? (ii) If $0 < |a| < 1$, then what can be said about the fixed points of $f(x) = \sinh(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$? (iii) What can be said about the fixed points of $f(x) = \tanh(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$ provided $|a| > 1$?

Question 3.4. What can be said about the fixed points of $f(x) = \coth(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$? (Note that if $a \geq 0$, then $f(x)$ and all its composite functions $f^n(x)$ $(n \geq 2)$ have two fixed points with opposite signs and all fixed points with the same sign are equal to each other. If $a < 0$, then consider two cases when n is odd and when n is even.)

Question 3.5. What can be said about the fixed points of $f(x) = \operatorname{sech}(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$? (Note that if $a \leq 2$, then all $f^n(x)$ $(n \geq 1)$ have only one fixed point which is the same for all. If $a > 2$, then consider two cases, when n is odd and when n is even.)

Question 3.6. What can be said about the fixed points of $f(x) = \operatorname{csch}(ax)$ and the fixed point(s) of $f^n(x)$ $(n \geq 2)$? (Note that if $a = 0$, then all $f^n(x)$ $(n \geq 1)$ are fixed point free. If $a \neq 0$, then see what happens when $a > 0$ and when $a < 0$, based on n being even or odd.)

Question 3.7. Is there a relationship between the root(s) of $f(x)$ and the root(s) of $f^n(x)$ $(n \geq 2)$? Is there a relationship between their fixed point(s) and their root(s)?

Question 3.8. For an arbitrary function $f(x)$ there does not seem to be an obvious relationship between the fixed point(s) of $f(x)$ and the fixed point(s) of the derivatives of $f(x)$ or $\int f(x)dx$. Are there some special functions where there are some relationships between the fixed point(s) of $f(x)$ and the fixed point(s) of the derivatives of $f(x)$ or its indefinite integral? (One may assume that the constant of integration is zero.)

Question 3.9. Let $f(x)$ be an arbitrary function.

(i) If k is any nontrivial and nonzero real number, then what can be said about the fixed point(s) of $f(kx)$ and $f^n(kx)$ $(n \geq 2)$ and their relationships with the fixed point(s) of $f(x)$?

(ii) If k is any nonzero real number, then what can be said about the fixed point(s) of $f(x+k)$, $f(x)+k$, $f^n(x+k)$, and $f^n(x)+k$ $(n \geq 2)$ and their relationships with the fixed point(s) of $f(x)$?

(iii) If k is any positive integer, then what can be said about the fixed point(s) of $f(x^k)$, and $f^n(x^k)$ $(n \geq 2)$ and their relationships with the fixed point(s) of $f(x)$?

Question 3.10. What can be said about the fixed points of $f(x) = \log_a x$ and the fixed points of $f^n(x)$ $(n \geq 2)$? (Note that for example if $a \geq 1.5$, then $x > f^i(x) > f^j(x)$, for all i and j such that $j > i \geq 1$, and hence $f(x)$ and all its composite functions $f^n(x)$ $(n \geq 2)$ are fixed point free.)

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