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MULTIPARAMETER QUANTUM PROCESSES OVER THE CLIFFORD SHEET

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Abstract: Quantum analogues of classical stochastic constructions have been developed in which a quantum stochastic base consists of a Hilbert space $\mathcal{H}$, a von Neumann algebra $\mathcal{A}$, a filtration $\{\mathcal{A}_z\}$, a gauge m and a parameter space I. In this discussion we employ such a base to introduce multiparameter Clifford processes, quantum stochastic integrals, their relationships and properties.

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1. Introduction

Given a complete probability space $(\Omega, \mathcal{F}, \mu)$, one may construct a classical stochastic scheme $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in T}, \mu, T)$, in which a filtration $\{\mathcal{F}_t\}_{t\in T}$ of sub σ-fields of $\mathcal{F}$ is defined together with adapted processes $\{X_t\}_{t\in T}$ for which each variate X_t forms a μ-measurable function with associated σ-field $\mathcal{F}_t$. Quantum analogues of such constructions $(\mathcal{H}, \mathcal{A}, \{\mathcal{A}_i\}_{i\in I}, m, I)$, may be formed whereby the sample space Ω is replaced by a Hilbert Space $\mathcal{H}$, σ-field $\mathcal{F}$ is replaced by a von Neumann algebra $\mathcal{A}$, the filtration $\{\mathcal{F}_t\}_{t\in T}$ by a filtration $\{\mathcal{A}_z\}$ of von Neumann subalgebras of the von Neumann algebra $\mathcal{A}$ the probability measure μ with gage m [4] and parameter set T with parameter set I.

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In this paper we introduce geneneral quantum stochastic integrals with Hilbert space $\mathcal{F}(\mathcal{H})$ and n dimensional parameter space $\mathbb{R}^n_+$. The stochastic base thus takes the form $(\mathcal{F}(\mathcal{H}), \mathcal{A}, \{\mathcal{A}_z\}, m, \mathbb{R}^n_+)$. Consistent with classical stochastic integration theory our quantum stochastic integrals employ martingales as integrators [5].

2. Clifford Construction

Let $\mathcal{H}$ denote the Hilbert space $L^2(\mathbb{R}^n_+)$, in which $\mathbb{R}^n_+$ denotes the positive n-dimensional subspace of $\mathbb{R}^n$. For the stochastic base we take the Hilbert space $\mathcal{F}(\mathcal{H}) = \bigoplus_{r=0}^{\infty} \mathcal{H}_r$ this being the antisymmetric fermion Fock space generated by r-fold tensor products of $\mathcal{H}$, the one-particle fermion space, [2, 3]. For our von Neumann algebra $\mathcal{A}$ we work with fermion fields $\psi(f) = a^*(f) + a(f)$, $\mathcal{A}$ being the von Neumann algebra generated by such fields with a^* and a denoting the creation and annihilation operators acting on $\mathcal{F}(\mathcal{H})$, with f real valued elements in $\mathcal{H}$. ψ, satisfies the CAR properties [2] and the following result holds.

Lemma 1. *Let f and g be real valued functions in $L^2(\mathbb{R}^n_+)$. Then $\{\psi(f), \psi(g)\} = 2(f, g)_{L^2(\mathbb{R}^n_+))}$.*

Proof.

$$\begin{aligned}
\{\psi(f), \psi(g)\} &= \psi(f)\psi(g) + \psi(g)\psi(f) \\
&= (b^*(f) + b(f))(b^*(g) + b(g)) + (b^*(g) + b(g))(b^*(f) + b(f)) \\
&= b^*(f)b^*(g) + b^*(f)b(g) + b(f)b^*(g) + b(f)b(g) \\
&\quad + b^*(g)b^*(f) + b^*(g)b(f) + b(g)b^*(f) + b(g)b(f) \\
&= \{b^*(f), b^*(g)\} + \{b^*(f), b(g)\} + \{b^*(g), b(f)\} + \{b^*(f), b^*(g)\} \\
&= \{b^*(f), b(g)\} + \{b^*(g), b(f)\} \quad \text{by CAR's} \\
&= 2Re(f, g)\mathbb{I} \quad \text{by CAR's} \\
&= 2(f, g)\mathbb{I} \quad \text{since } f, g \text{ real valued}.
\end{aligned}$$

□

Let m denote the gage given by $m(\circ) = (\Omega, \circ\,\Omega)$, in which $\Omega = 1 \in \mathbb{C} = \mathcal{H}_0$ in $\mathcal{F}(\mathcal{H})$. $(\mathcal{F}(\mathcal{H}), \mathcal{A}, m)$ forms a probability gage space [7, 8, 9], m is a faithful, central state on $\mathcal{A}$ preserving independence of von Neumann algebras generated by fermion fields $\psi(f) \in \mathcal{H}_1, \psi(g) \in \mathcal{H}_2$ with $\mathcal{H}_1 \perp \mathcal{H}_2$. Noncommutative $L^p(\mathcal{A})$ spaces, for $1 \le p < \infty$ may be formed via the relation $m(|a|) = (\Omega, |a^p|^{1/p}\Omega)$ see [13] for further details, with $L^\infty(\mathcal{A})$ taking the usual operator norm.

A filtration of von Neumann algebras $(\mathcal{A}_z)$ may be defined in $\mathcal{A}$ by restricting the fermion fields ψ to functions $f \in L^2_{loc}(\mathbb{R})$ of the form $\chi_{R_z} g \in L^2(R_z) \subseteq L^2(\mathbb{R}^n_+)$, R_z denoting the n-dimensional cube with inf $R_z = 0$, the origin. Conditional expectations $\mathbb{E}(\circ|\mathcal{B}) : \mathcal{A} \longrightarrow \mathcal{B}$ folow which may be extended to $L^p(\mathcal{A}) \longrightarrow L^P(\mathcal{B})$. Processes $(\psi(\chi_{R_z} g))$ form centred martingales [4, 10].

3. The Underlying Parameter Set

We consider stochastic integrals for the case $L^2(\mathbb{R}^n_+)$ with integrands of the form $a \underset{i=1}{\overset{r}{\pi}} \chi_{\Delta_i}(z_i)$, $1 \leq r \leq n$, $a \in \mathcal{A}_{inf\Delta_1 \wedge \cdots \wedge inf\Delta_r}$ and increments $\Delta_i \subset \mathbb{R}^n_+$ each 'forward' of $R_{inf\Delta_1 \wedge \cdots \wedge inf\Delta_r}$.

For $L^2(\mathbb{R}_+)$ such integrands are of the form $a\chi_{\Delta_1}$, with $a \in \mathcal{A}_{inf\Delta_1}$. $a\chi_{\Delta_1}$ is referred to as a type 1 integrand and leads to a type 1 integral, the Ito-Clifford integral [10].

With $L^2(\mathbb{R}^2_+)$ we work with $a\chi_{\Delta_1}$, and $b\chi_{\Delta_2}\chi_{\Delta_3}$. As before $a \in \mathcal{A}_{inf\Delta_1}$, is a type 1 integrand leading to a type 1 integral whilst $b \in \mathcal{A}_{inf\Delta_2 \wedge inf\Delta_3}$ is referred to as a type 2 integrand, leading to the type 2 Wong-Zakai-Clifford integral. We refer the interested reader to [10] for further details.

In the case of $L^2(\mathbb{R}^3_+)$ we work with $a\chi_{\Delta_1}$, $b\chi_{\Delta_2}\chi_{\Delta_3}$ and $c\chi_{\Delta_4}\chi_{\Delta_5}\chi_{\Delta_6}$. Throughout $a \in \mathcal{A}_{inf\Delta_1}, b \in \mathcal{A}_{inf\Delta_2 \wedge inf\Delta_3}$ and $c \in \mathcal{A}_{inf\Delta_4 \wedge inf\Delta_5 \wedge inf\Delta_6}$. For $a \in \mathcal{A}_{inf\Delta_1}$ points in Δ are seen to be 'forward' of all points in $R_{inf\Delta_1}$ for each coordinate. For $b \in \mathcal{A}_{inf\Delta_2 \wedge inf\Delta_3}$ each point lying in an increment is forward of all points in $R_{z \wedge z'}$ by just two of the three available coordinates, and each Δ is forward of the other Δ in in at least one coordinate. With $c \in \mathcal{A}_{inf\Delta_4 \wedge inf\Delta_5 \wedge inf\Delta_6}$, each point in an increment is forward of all points in $R_{inf\Delta_4 \wedge inf\Delta_5 \wedge inf\Delta_6}$ and the other Δ's by just one coordinate. Each Δ is therefore mutually exclusive and said to be 'cock-eyed' [14]. We write $\Delta_z {}^{\wedge}_{\wedge} \Delta_{z'}$ and $\Delta_z {}^{\wedge}_{\wedge} \Delta_{z'} {}^{\wedge}_{\wedge} \Delta_{z''}$ to indicate the above. In the case of $L^2(\mathbb{R}^2_+)$, [10], Type I and Type II integrals resulted from partially ordered sets generated by such 'forward' and 'cock-eyed' [14] increments $z \in A$ and $z' \in B$, respectively. For the case of $n = 3$ the richer selection of increments result in new Type II and Type III integrals. We defer to [11] for further details.

4. Integrals

Definition 2. A map $h : \mathbb{R}^n_+ \times \cdots \times \mathbb{R}^n_+ \longrightarrow L^2(\mathcal{A})$ is said to be a $\mathcal{A}$ valued elementary r adapted process if there exist $\Delta_1, \ldots \Delta_r$ with $\Delta_1 {}^\wedge_\wedge \ldots {}^\wedge_\wedge \Delta_r$, and h is of the form $h(z_1, \ldots, z_r) = a \overset{r}{\underset{i=1}{\pi}} \chi_{\Delta_i}(z_i)$, with $a \in \mathcal{A}_{inf \Delta_1 \wedge \cdots \wedge inf \Delta_r}$

Definition 3. Let $h(z_1, \ldots, z_r) = a \overset{r}{\underset{i=1}{\pi}} \chi_{\Delta_i}(z_i)$, denote elementary r adapted processes with $a \in \mathcal{A}_{inf \Delta_1 \wedge \cdots \wedge inf \Delta_r}$ and each $z_i \in \mathbb{R}^n_+$. We define the type r integral $\mathcal{S}_r$ of h over R_z to be

$$\begin{aligned}
\mathcal{S}_r(h, z, f_1, \ldots, f_r) &= \int\limits_{R_z} \ldots \int\limits_{R_z} h(z_1, \ldots, z_r) d\psi_{z_1}(f_1) \ldots d\psi_{z_r}(f_r) \\
&= a \overset{r}{\underset{i=1}{\pi}} \psi(\chi_{\Delta_i \cap R_z} f_i)
\end{aligned}$$

We extend to simple adapted processes on $\mathbb{R}^n_+ \times \cdots \times \mathbb{R}^n_+$ and their respective integrals via linearity.

Theorem 4. *Each of the integrals given above satisfy isometry properties.*

Proof. We outline the case for type r integrals, $1 \leq r \leq n$ extending the approach taken with type I and type II integrals [10] over $L^2(\mathbb{R}^2_+)$ to more general quantum stochastic integrals for the case of $L^2(\mathbb{R}^n_+)$.

Let $h = \sum\limits_{i=1}^{m} a_i \prod\limits_{j=1}^{r} \chi_{\Delta_{ij}}$ with $\Delta_{ij} \cap \Delta_{kl} = \emptyset,\ \forall i \neq j, \forall k \neq l$. This is acceptable via linearity of the ψ. Each Δ is forward or cock-eyed according to the value of r. The gage/trace are both cyclic [7], and independent [4] the a_i's are by hypothesis generated by sums and products of ψ 's each satisfying the CAR's and Lemma 1. It follows that:

$$\begin{aligned}
&\| \mathcal{S}_r(h, z, f_1, \ldots, f_r)\|_2^2 \\
&\quad = (\Omega, (\sum_{i=1}^{m} a_i \prod_{j=1}^{r} \psi(\chi_{\Delta_{ij} \cap R_z} f_j))^* (\sum_{k=1}^{m} a_k \prod_{l=1}^{r} \psi(\chi_{\Delta_{kl} \cap R_z} f_l))\Omega) \\
&\quad = \sum_{i=1}^{m} \sum_{k=1}^{m} (\Omega, \prod_{j=1}^{r} \psi(\chi_{\Delta_{i(r+1-j)} \cap R_z} f_{(r+1-j)}) a_i^* a_k \prod_{l=1}^{r} \psi(\chi_{\Delta_{kl} \cap R_z} f_l)\Omega) \\
&\quad = \sum_{i=1}^{m} (\Omega, \prod_{j=1}^{r} \psi(\chi_{\Delta_{i(r+1-j)} \cap R_z} f_{(r+1-j)}) \prod_{l=1}^{r} \psi(\chi_{\Delta_{il} \cap R_z} f_l) a_i^* a_i \Omega) \\
&\quad = \sum_{i=1}^{m} (\Omega, \prod_{j=1}^{r} \psi(\chi_{\Delta_{ij} \cap R_z} f_j) \psi(\chi_{\Delta_{ij} \cap R_z} f_j) a_i^* a_i \Omega) \\
&\quad = \sum_{i=1}^{m} (\Omega, a_i^* a_i \Omega) \prod_{j=1}^{r} (f_j, f_j)
\end{aligned}$$

$$= \int_{R_z} \dots \int_{R_z} \| h\|_2^2 \prod_{j=1}^{r} |f_j|^2 dz_1 dz_2 \dots dz_r$$

Theorem 5. (Orthogonality) *Each of the integrals above are orthogonal to each other.*

Proof. We outline the case for type r and type s integrals with $1 \leq r,\ s \leq n$.

By linearity of ψ we may take each increment in $\mathbb{R}_+^n$ to be the same size (or zero). Choose a direction in which there exists a non-zero increment Δ_i. Any increment(s) in the direction chosen will either be the only such increment (case 1) or there will be two such increments. For the case of two such increments these will either be the same (case 2) or disjoint. If they are disjoint then their will either exist one increment in advance of the other (case 3) or they will be parallel but disjoint (case 4).

For case 2 we may choose another increment which is of case 1, 3 or 4 since the integrals are not of the same type.

For case 3 we use the independence of the gage [4] to obtain a product of gages in which one acts on ψ of the increment. Since ψ is a centred martingale the product of gages is zero and independence holds.

For case 4 we proceed as in case 3 this time obtaining the gage of a product of ψ's. If Δ_1 and Δ_2 denote the increments then case 4 is of the form $m(ab) = m(a)m(b)$ with $b = \psi(\Delta_1)\psi(\Delta_2)$ which since $\Delta_1 \cap \Delta_2 =$ is equal to $m(\psi(\Delta_1))m(\psi(\Delta_2))$. These again are zero since the ψ's are centred martingales.

For case 1. If the increment lies in advance of that that the other integral acts on then we apply the approach taken with case 3. If not, then there exists such an increment in the other integral and we may apply the approach taken in case 3. □

The centred martingale property established in [10] for the two parameter case extends to the r-parameter case again predominantley by independence of the gage m. As a result it is seen that general r-parameter quantum stochastic integrals for simple adapted processes each satisfy isometry and orthogonality as centred martingales.

Theorem 6. *Each of the above integrals form centred martingales*

Proof. This follows independence of the gage. The details appear in [11]. □

5. Completion

The integrals described above extend via the isometry property to an appropriate completion of the simple adapted processes in $L^2(\mathcal{A})$ where they continue to satisfy the same isometry and martingale properties. Processes belonging to the respective completions of type r simple processes are themselves found to be orthogonal, isometric centred martingales.

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