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SOME PROPERTIES OF UNIVALENT FUNCTIONS

M. Nunokawa[1], S. Owa[2 §], T. Hayami[3], K. Kuroki[4]
[1]University of Gunnma
798-8, Hoshikuki-machi, Chuo-ku
Chiba-Shi, Chiba, 260-0808, JAPAN
e-mail: mamoru_nuno@doctor.nifty.jp
[2,3,4]Department of Mathematics
Faculty of Science and Technology
Kinki University
Higashi-Osaka, Osaka, 577-8502, JAPAN
[2]e-mail: owa@math.kindai.ac.jp
[3]e-mail: ha_ya_to112@hotmail.com
[4]e-mail: freedom@sakai.zaq.ne.jp

Abstract: For some univalent functions $f(z)$ which are normalized by $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $\mathbb{U}$, some properties for the length $L(r)$ of the image curve $C(r)$ by $w = f(z)$ of $|z| = r < 1$ are considered. It is the object of the present paper to derive properties for lengths $L(r)$ by close-to-convex functions and Bazilevič functions.

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$.

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§Correspondence author

If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{1.2}$$

then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{U}$, and denoted by $f(z) \in \mathcal{S}^*$. If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{1.3}$$

then $f(z)$ is said to be convex in $\mathbb{U}$, and denoted by $f(z) \in \mathcal{K}$. Furthermore, if $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \qquad (z \in \mathbb{U}) \tag{1.4}$$

for some $g(z) \in \mathcal{S}^*$, then $f(z)$ is called to be close-to-convex with respect to $g(z)$, and denoted by $f(z) \in \mathcal{C}$. If $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)^{1-\beta}g(z)^{\beta}}\right) > 0 \qquad (z \in \mathbb{U}) \tag{1.5}$$

for some $g(z) \in \mathcal{S}^*$ and $\beta > 0$, then we call $f(z)$ Bazilevič function of type β with respect to $g(z)$, and denote by $f(z) \in \mathcal{B}(\beta)$.

Let $C(r)$ be the closed curve which is the image of $|z| = r < 1$ by $w = f(z)$. Let $L(r)$ denote the length of $C(r)$ and let $A(r)$ be the area enclosed by $C(r)$.

Let us define $M(r)$ by

$$M(r) = \max_{|z|=r<1} |f(z)| \tag{1.6}$$

for some $f(z) \in \mathcal{A}$.

Pommerenke [7] has shown that

Theorem A. *If $f(z) \in \mathcal{C}$, then*

$$L(r) = \mathcal{O}\left\{M(r)\left(\log\frac{1}{1-r}\right)^{\frac{5}{2}}\right\} \tag{1.7}$$

as $r \to 1$, where $\mathcal{O}$ means Landou's symbol.

Thomas [8] has given

Theorem B. *If $f(z) \in \mathcal{S}^*$, then*

$$L(r) = \mathcal{O}\left\{A(r)^{\frac{1}{2}}\log\frac{1}{1-r}\right\} \tag{1.8}$$

as $r \to 1$.

Furthermore, Nunokawa [4,5] has derived

Theorem C. *If $f(z) \in \mathcal{K}$, then*

$$L(r) = \mathcal{O}\left(A(r) \log \frac{1}{1-r}\right)^{\frac{1}{2}} \tag{1.9}$$

as $r \to 1$.

In the present paper, we consider the length $L(r)$ of $C(r)$ by $w = f(z)$ for $f(z) \in \mathcal{C}$ and for $f(z) \in \mathcal{B}(\beta)$. To discuss our problems, we need the following lemmas.

Lemma 1. *If $f(z) \in \mathcal{C}$, then*

$$\int_0^{2\pi} |g(z)| \operatorname{Re}(h(z)) d\theta \leqq 2\pi M(r) \qquad (z = re^{i\theta} \in \mathbb{U}), \tag{1.10}$$

where $h(z) = \dfrac{zf'(z)}{g(z)}$ and $M(r) = \max\limits_{|z|=r<1} |f(z)|$.

This lemma can be found in Clunie and Pommerenke [2, Theorem 1]. Next lemma was given by Pommerenke [7, Lemma 2].

Lemma 2. *If $f(z) \in \mathcal{S}$, then*

$$M(r) \leqq 4\left(\frac{A(r)}{\pi} \log \frac{3}{1-r}\right)^{\frac{1}{2}} \qquad (|z| = r < 1), \tag{1.11}$$

or

$$M(r) = \mathcal{O}\left(A(r) \log \frac{1}{1-r}\right)^{\frac{1}{2}} \tag{1.12}$$

as $r \to 1$.

Further, Pommerenke has shown

Lemma 3. *If $h(z)$ is analytic and $\operatorname{Re}(h(z)) > 0$ in $\mathbb{U}$ with $h(0) = 1$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^2 d\theta \leqq \frac{1+3r^2}{1-r^2} \qquad (|z| = r < 1). \tag{1.13}$$

2. Some Properties for $L(r)$

We discuss the length $L(r)$ of $C(r)$ by $w = f(z)$.

Theorem 1. *If $f(z) \in \mathcal{C}$, then*

$$L(r) = \mathcal{O}\left\{\left(A(r)B(r)\right)^{\frac{1}{4}} \left(\log \frac{1}{1-r}\right)^{\frac{3}{2}}\right\} \tag{2.1}$$

as $r \to 1$, *where*

$$A(r) = \int_0^r \int_0^{2\pi} \rho |f'(\rho e^{i\theta})|^2 d\theta d\rho \tag{2.2}$$

and

$$B(r) = \int_0^r \int_0^{2\pi} \rho |g'(\rho e^{i\theta})|^2 d\theta d\rho. \tag{2.3}$$

Proof. Let us define $h(z)$ by

$$h(z) = \frac{zf'(z)}{g(z)} \tag{2.4}$$

for $f(z) \in \mathcal{C}$ and $g(z) \in \mathcal{S}^*$. Then $\mathrm{Re}(h(z)) > 0$ $(z \in \mathbb{U})$.

It follows that

$$\begin{aligned} L(r) &= \int_0^{2\pi} |zf'(z)| d\theta \\ &= \int_0^{2\pi} |g(z)h(z)| d\theta \\ &\leqq \left(\int_0^{2\pi} |g(z)^2 h(z)| d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} |h(z)| d\theta \right)^{\frac{1}{2}}. \end{aligned} \tag{2.5}$$

It is well-known by Keogh [3, Theorem 1] that

$$I_1 = \int_0^{2\pi} |h(z)| d\theta \leqq 2\pi + 4 \log \left(\frac{1+r}{1-r} \right). \tag{2.6}$$

Further, we have

$$\begin{aligned} I_2 &= \int_0^{2\pi} |g(z)^2 h(z)| d\theta \\ &\leqq 2 \int_0^r \int_0^{2\pi} |g'(z)g(z)h(z)| d\theta d\rho + \int_0^r \int_0^{2\pi} |g(z)^2 h'(z)| d\theta d\rho \\ &= 2J_1 + J_2, \end{aligned} \tag{2.7}$$

where $z = \rho e^{i\theta} \in \mathbb{U}$. Note that

$$\begin{aligned} J_1 &= \int_0^r \int_0^{2\pi} |g'(z) z f'(z)| d\theta d\rho \\ &\leqq \left(\int_0^r \int_0^{2\pi} \rho |f'(z)|^2 d\theta d\rho \right)^{\frac{1}{2}} \left(\int_0^r \int_0^{2\pi} \rho |g'(z)|^2 d\theta d\rho \right)^{\frac{1}{2}} \\ &= \big(A(r) B(r) \big)^{\frac{1}{2}}. \end{aligned} \tag{2.8}$$

Since $\mathrm{Re}(h(z)) > 0$ $(z \in \mathbb{U})$ and $h(0) = 1$, from Carathéodory Theorem [1], we have that

$$|h'(z)| \leqq \frac{2\mathrm{Re}(h(z))}{1-\rho^2} \qquad (|z| = \rho < 1).$$

Therefore, applying Lemma 1 and Lemma 2, we see that

$$\begin{aligned} J_2 &= \int_0^r \int_0^{2\pi} |g(z)^2 h'(z)| d\theta d\rho \\ &\leqq \int_0^r \int_0^{2\pi} |g(z)|^2 \frac{2\mathrm{Re}(h(z))}{1-\rho^2} d\theta d\rho \qquad (z = \rho e^{i\theta} \in \mathbb{U}) \\ &\leqq 4\pi \int_0^r M(\rho)|g(z)| \frac{1}{1-\rho^2} d\rho \\ &\leqq 32\big(A(r)B(r)\big)^{\frac{1}{2}} \left(\log \frac{3}{1-r}\right)\left(\log \frac{1+r}{1-r}\right), \end{aligned} \tag{2.9}$$

where $M(\rho) = \max_{|z|=\rho<1} |f(z)|$. Therefore, we obtain that

$$L(r) \leqq 2\big(A(r)B(r)\big)^{\frac{1}{4}} \left(\pi + 2\log \frac{1+r}{1-r}\right)^{\frac{1}{2}} \left(1 + 16\left(\log \frac{3}{1-r}\right)\left(\log \frac{1+r}{1-r}\right)\right)^{\frac{1}{2}},$$

which implies that

$$L(r) = \mathcal{O}\left\{\big(A(r)B(r)\big)^{\frac{1}{4}} \left(\log \frac{1}{1-r}\right)^{\frac{3}{2}}\right\}$$

as $r \to 1$. □

Remark 1. It is trivial that

$$A(r)^{\frac{1}{2}} \leqq \sqrt{\pi} M(r).$$

Next we consider

Theorem 2. *If $f(z) \in \mathcal{B}(\beta)$, then*

$$L(r) = \mathcal{O}\left\{A(r)^{\frac{1-\beta}{2}} B(r)^{\frac{\beta}{2}} \left(\log \frac{1}{1-r}\right)^2\right\} \tag{2.10}$$

as $r \to 1$.

Proof. If we define $h(z)$ by

$$h(z) = \frac{zf'(z)}{f(z)^{1-\beta} g(z)^{\beta}}$$

for $f(z) \in \mathcal{B}(\beta)$ and $g(z) \in \mathcal{S}^*$, we have that $h(0) = 1$ and $\mathrm{Re}(h(z)) > 0$ ($z \in \mathbb{U}$). Using Lemma 2 and the result by Keogh [3, Theorem 1], we have that

$$L(r) \;=\; \int_0^{2\pi} |f(z)^{1-\beta} g(z)^{\beta} h(z)| d\theta \qquad (z = \rho e^{i\theta} \in \mathbb{U})$$

$$\begin{aligned}
&\leqq CA(r)^{\frac{1-\beta}{2}}B(r)^{\frac{\beta}{2}}\int_0^{2\pi}\left(\log\frac{1}{1-\rho}\right)^{\frac{1-\beta}{2}}\left(\log\frac{1}{1-\rho}\right)^{\frac{\beta}{2}}|h(z)|d\theta\\
&= CA(r)^{\frac{1-\beta}{2}}B(r)^{\frac{\beta}{2}}\int_0^{2\pi}\left(\log\frac{1}{1-\rho}\right)|h(z)|d\theta\\
&\leqq CA(r)^{\frac{1-\beta}{2}}B(r)^{\frac{\beta}{2}}\left(\log\frac{1}{1-r}\right)^2,
\end{aligned}$$

where C is an absolute constant, not necessarily the same for each time. □

Finally, we consider

Theorem 3. *If $f(z)\in\mathcal{C}$, then*

$$A(r)=\mathcal{O}\left\{B(r)\left(\log\frac{1}{1-r}\right)^2\right\} \tag{2.11}$$

as $r\to 1$, where $A(r)$ and $B(r)$ are defined in Theorem 1.

Proof. With the help of Lemma 2 and Lemma 3, we have that

$$\begin{aligned}
A(r) &= \int_0^r\int_0^{2\pi}\rho|f'(\rho e^{i\theta})|^2 d\theta d\rho \qquad (z=\rho e^{i\theta}\in\mathbb{U})\\
&= \int_0^r\int_0^{2\pi}\left|\frac{zf'(z)}{g(z)}\right|^2|g(z)|^2\frac{1}{\rho}d\theta d\rho\\
&\leqq \int_0^{r_1}\int_0^{2\pi}\left|\frac{zf'(z)}{g(z)}\right|\left|\frac{g(z)^2}{z}\right|d\theta d\rho+\int_{r_1}^r\int_0^{2\pi}\left|\frac{zf'(z)}{g(z)}\right|^2|g(z)|^2\frac{1}{r_1}d\theta d\rho\\
&\leqq C+\frac{1}{r_1}\int_0^r\int_0^{2\pi}\left|\frac{zf'(z)}{g(z)}\right|^2|g(z)|^2 d\theta d\rho\\
&\leqq C+\frac{16}{\pi r_1}\int_0^r\int_0^{2\pi}|h(z)|^2B(\rho)\left(\log\frac{3}{1-\rho}\right)d\theta d\rho\\
&\leqq C+C\int_0^r\int_0^{2\pi}|h(z)|^2B(r)\left(\log\frac{1}{1-r}\right)d\theta d\rho\\
&= C+CB(r)\left(\log\frac{1}{1-r}\right)\int_0^r\int_0^{2\pi}|h(z)|^2d\theta d\rho\\
&\leqq C+CB(r)\left(\log\frac{1}{1-r}\right)\int_0^r\frac{1+3\rho^2}{1-\rho^2}d\rho\\
&= C\left\{B(r)\left(\log\frac{1}{1-r}\right)^2\right\},
\end{aligned} \tag{2.12}$$

where $0<r_1<r<1$ and C denotes an absolute constant, not necessarily the same for each time. □

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