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FUZZY QUASI-IDEAL SUBSETS AND FUZZY QUASI-FILTERS OF ORDERED SEMIGROUPS

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Abstract: In this paper, we define quasi-ideal subsets, fuzzy quasi-ideal subsets, quasi-filters and fuzzy quasi-filters in ordered semigroups and characterize ordered semigroups in terms of fuzzy quasi-ideal subsets and fuzzy quasi-filters.

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1. Introduction and Preliminaries

The concept of fuzzy sets was introduced by Zadeh in 1965 [10]. The algebraic approach of fuzzy sets was studied by some authors, for example, Rosenfeld [7], Kuroki [6], Kehayopulu and Tsingelis [2]-[5], Shabir and Khan [8], Xie [5], [9], Yan [9], etc.

An ordered semigroup $(S, \cdot, \leq)$ is a nonempty set S together with a binary operation $\cdot$ and an order $\leq$ such that $(S, \cdot)$ is a semigroup, $(S, \leq)$ is a partially ordered set and for all $a, b, x \in S, a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. Let $(S, \cdot, \leq)$ be an ordered semigroup. For $A \subseteq S$, we denote

$$(A] := \{t \in S \mid t \leq h \text{ for some } h \in A\}.$$

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Let $(S, \cdot, \leq)$ be an ordered semigroup. A function f from S to the unit interval $[0, 1]$ is called a *fuzzy subset* of S. The ordered semigroup S itself is a fuzzy subset of S such that $S(x) = 1$ for all $x \in S$, denoted also by S. If $A \subseteq S$, the characteristic function f_A of A is a fuzzy subset of S defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

In [8], Shabir and Khan defined fuzzy bi-ideal subsets and fuzzy bi-filters in ordered semigroups and characterized ordered semigroups in terms of fuzzy bi-ideal subsets and fuzzy bi-filters. In this paper, we define quasi-ideal subsets, fuzzy quasi-ideal subsets, quasi-filters and fuzzy quasi-filters in ordered semigroups and characterize ordered semigroups in terms of fuzzy quasi-ideal subsets and fuzzy quasi-filters.

2. Main Results

We define quasi-ideal subsets, fuzzy quasi-ideal subsets, quasi-filters and fuzzy quasi-filters in ordered semigroups.

Definition 2.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset Q of S is called a *quasi-ideal subset* of S if:

(1) $(Q] \subseteq Q$;

(2) $xS \cap Sx \subseteq Q$ for all $x \in Q$.

Definition 2.2. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy quasi-ideal subset* of S if for all $x, y, z \in S$

(1) $x \leq y$ implies $f(x) \geq f(y)$;

(2) $xy = zx$ implies $f(xy) \geq f(x)$.

Definition 2.3. Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset F of S is called a *quasi-filter* of S if

(1) $F^2 \subseteq F$;

(2) for all $x, y \in S, x \leq y$ and $x \in F$ imply $y \in F$;

(3) for all $x, y, z \in S, xy = zx \in F$ implies $x \in F$.

Definition 2.4. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy quasi-filter* of S if for all $x, y, z \in S$

(1) $x \leq y$ implies $f(x) \leq f(y)$;

(2) $f(xy) \geq \min\{f(x), f(y)\}$;

(3) $xy = zx$ implies $f(xy) \leq f(x)$.

We characterize quasi-ideal subsets of ordered semigroups in terms of fuzzy quasi-ideal subsets.

Theorem 2.1. *Let Q be a nonempty subset of an ordered semigroup S. Then Q is a quasi-ideal subset of S if and only if the characteristic function f_Q of Q is a fuzzy quasi-ideal subset of S.*

Proof. Assume Q is a quasi-ideal subset of S.

Let $x, y \in S$ such that $x \leq y$.

Case 1. $y \in Q$. Since $x \leq y$ and $y \in Q, x \in (Q]$. Since Q is a quasi-ideal subset of $S, x \in Q$. So $f_Q(x) = 1$. Therefore $f_Q(x) \geq f_Q(y)$.

Case 2. $y \notin Q$. Then $f_Q(y) = 0$. Therefore $f_Q(x) \geq f_Q(y)$.

Next, let $x, y, z \in S$ such that $xy = zx$.

Case 1. $x \in Q$. Since $xy = zx \in xS \cap Sx, xy \in Q$. So $f_Q(xy) = 1$. Therefore $f_Q(xy) \geq f_Q(x)$.

Case 2. $x \notin Q$. So $f_Q(x) = 0$. Therefore $f_Q(xy) \geq f_Q(x)$.

Conversely, assume f_Q is a fuzzy quasi-ideal subset of S. Let $x \in (Q]$. Then $x \leq y$ for some $y \in Q$. Since f_Q is a fuzzy quasi-ideal subset of $S, f_Q(x) \geq f_Q(y) = 1$. Thus $x \in Q$. So $(Q] \subseteq Q$. Next, let $x \in Q$ and $u \in xS \cap Sx$. Then $u = xy = zx$ for some $y, z \in S$. Since f_Q is a fuzzy quasi-ideal subset of $S, f_Q(u) \geq f_Q(x) = 1$. Hence $u \in Q$. Therefore $xS \cap Sx \subseteq Q$. □

We also characterize quasi-filters of ordered semigroups in terms of fuzzy quasi-filters.

Theorem 2.2. *Let F be a nonempty subset of an ordered semigroup S. Then F is a quasi-filter of S if and only if the characteristic function f_F of F is a fuzzy quasi-filter of S.*

Proof. Assume that F is a quasi-filter of S.

Let $x, y \in S$ such that $x \leq y$.

Case 1. $x \notin F$. Then $f_F(x) = 0$. Then $f_F(x) \leq f_F(y)$.

Case 2. $x \in F$. Since $x \leq y$ and F is a quasi-filter of $S, y \in F$. Thus $f_F(y) = 1$. Hence $f_F(x) \leq f_F(y)$.

Next, let $x, y \in S$.

Case 1. $x, y \in F$. Then $xy \in F$. Hence $f_F(xy) = 1$. Therefore $f_F(xy) \geq \min\{f(x), f(y)\}$.

Case 2. $x \notin F$ or $y \notin F$. So $f_F(x) = 0$ or $f_F(y) = 0$. This implies

$f_F(xy) \geq \min\{f(x), f(y)\}$.

Finally, let $x, y, z \in S$ such that $xy = zx$.

Case 1. $xy \in F$. Since f_F is a quasi-filter of S and $xy = zx \in F, x \in F$. So $f_F(x) = 1$. Therefore $f_F(xy) \leq f_F(x)$.

Case 2. $xy \notin F$. Then $f_F(xy) = 0$. Therefore $f_F(xy) \leq f(x)$.

Conversely, assume f_F is a fuzzy quasi-filter of S. Let $x, y \in F$. Then $f_F(x) = 1 = f_F(y)$. Thus $f_F(xy) \geq \min\{f_F(x), f_F(y)\} = 1$. Hence $xy \in F$. Next, let $x, y \in S$. Assume $x \leq y$ and $x \in F$. Then $f_F(x) \leq f_F(y)$ and $f_F(x) = 1$. Thus $f_F(y) = 1$, this implies $y \in F$. Finally, let $x, y, z \in S$ such that $xy = zx \in F$. So $f_F(xy) = 1$. Since f_F is a fuzzy quasi-filter of S, then $f_F(x) \geq f_F(xy)$. This implies $f_F(x) = 1$. So $x \in F$. □

Definition 2.5. (see [9]) Let f be a fuzzy subset of an ordered semigroup S. Then for any $t \in [0, 1]$, the set $f_t = \{x \in S \mid f(x) \geq t\}$ is called a *level subset* of f.

Theorem 2.3. *Let f be a fuzzy subset of an ordered semigroup S. The following statements are equivalent.*

(1) f is a fuzzy quasi-ideal subset of S.

(2) For all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a quasi-ideal subset of S.

Proof. (1) $\rightarrow$ (2) : Let $t \in [0, 1]$ such that $f_t \neq \emptyset$. Let $x \in (f_t]$. Then there exists $y \in f_t$ such that $x \leq y$. Since f is a fuzzy quasi-ideal subset of $S, f(x) \geq f(y) \geq t$. Thus $x \in f_t$. Hence $(f_t] \subseteq f_t$. Next, let $x \in f_t$ and $u \in xS \cap Sx$. Thus $u = xy = zx$ for some $y, z \in S$. Since f is a fuzzy quasi-ideal subset of $S, f(u) = f(xy) \geq f(x) \geq t$. So $u \in f_t$. Therefore $xS \cap Sx \subseteq f_t$.

(2) $\rightarrow$ (1) : Let $x, y \in S$ such that $x \leq y$. Let $t = f(y)$. Then $y \in f_t$. Since f_t is a quasi-ideal subset of S and $x \leq y, x \in f_t$. So $f(x) \geq t = f(y)$. Next, let $x, y, z \in S$ such that $xy = zx$. Let $t = f(x)$. So $x \in f_t$. Since f_t is a quasi-ideal subset of S and $xy = zx \in xS \cap Sx, xy = zx \in f_t$. Thus $f(xy) \geq t = f(x)$. □

We define prime quasi-ideal subsets and prime fuzzy quasi-ideal subsets of ordered semigroups.

Definition 2.6. (see [1]) Let $(S, \cdot, \leq)$ be an ordered semigroup and P a nonempty subset of S. Then P is called a *prime subset* of S if for all $a, b \in S, ab \in P$ implies $a \in P$ or $b \in P$.

Definition 2.7. Let Q be a quasi-ideal subset of an ordered semigroup S. Then Q is called a *prime quasi-ideal subset* of S if Q is a prime subset of S.

Definition 2.8. (see [4]) Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *prime fuzzy subset* of S if $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$.

Definition 2.9. Let f be a fuzzy quasi-ideal subset of an ordered semigroup S. Then f is called a *prime fuzzy quasi-ideal subset* of S if f is a prime fuzzy subset of S.

Theorem 2.4. *Let f be a fuzzy subset of an ordered semigroup S. The following statements are equivalent.*

(1) f is a prime fuzzy quasi-ideal subset of S.

(2) For all $t \in [0, 1]$, if $f_t \neq \emptyset$, then f_t is a prime quasi-ideal subset of S.

Proof. $(1) \rightarrow (2)$: Let $t \in [0, 1]$ such that $f_t \neq \emptyset$. By Theorem 2.3, f_t is a quasi-ideal subset of S. Let $x, y \in S$ such that $xy \in f_t$. So $f(xy) \geq t$. Since f is a prime fuzzy quasi-ideal subset of $S, f(xy) \leq \max\{f(x), f(y)\}$. Therefore $f(x) \geq f(xy) \geq t$ or $f(y) \geq f(xy) \geq t$. So $x \in f_t$ or $y \in f_t$.

$(2) \rightarrow (1)$: By Theorem 2.3, f is a fuzzy quasi-ideal subset of S. Let $x, y \in S$ and let $t = f(xy)$. So $xy \in f_t$. Thus $x \in f_t$ or $y \in f_t$. Then $f(x) \geq t = f(xy)$ or $f(y) \geq t = f(xy)$. Thus $f(xy) \leq \max\{f(x), f(y)\}$. □

Now we study the relation between prime quasi-ideal subsets and prime fuzzy quasi-ideal subsets of ordered semigroups.

Theorem 2.5. *Let Q be a nonempty subset of an ordered semigroup S. Then Q is a prime quasi-ideal subset of S if and only if f_Q is a prime fuzzy quasi-ideal subset of S.*

Proof. Assume Q is a prime quasi-ideal subset of S. By Theorem 2.1, f_Q is a fuzzy quasi-ideal subset of S. Let $x, y \in S$.

Case 1. $xy \in Q$. Since Q is prime, $x \in Q$ or $y \in Q$. Then $f_Q(x) = 1$ or $f_Q(y) = 1$. Therefore $f_Q(xy) \leq \max\{f_Q(x), f_Q(y)\}$.

Case 2. $xy \notin Q$. Then $f_Q(xy) = 0$. Therefore $f_Q(xy) \leq \max\{f_Q(x), f_Q(y)\}$.

Conversely, assume f_Q is a prime fuzzy quasi-ideal subset of S. By Theorem 2.1, Q is a quasi-ideal subset of S. Let $x, y \in S$ such that $xy \in Q$. Then $f_Q(xy) = 1$. Since f_Q is prime, $f_Q(x) = 1$ or $f_Q(y) = 1$. So $x \in Q$ or $y \in Q$. □

Definition 2.10. (see [4]) Let $(S, \cdot, \leq)$ be an ordered semigroup and f a fuzzy subset of S. The fuzzy subset f' defined by

$$f'(x) = 1 - f(x) \text{ for all } x \in S$$

is called the *complement* of f in S.

Lemma 2.1. (see [8]) *Let f be a fuzzy subset of an ordered semigroup S.*

The following statements are equivalent.

(1) $f(xy) \leq \max\{f(x), f(y)\}$ *for all* $x, y \in S$*;*

(2) $f'(xy) \geq \min\{f(x), f(y)\}$ *for all* $x, y \in S$*.*

Theorem 2.6. *Let* f *be a fuzzy subset of an ordered semigroup* S*. Then* f *is a fuzzy quasi-filter of* S *if and only if the complement* f' *of* f *is a prime fuzzy quasi-ideal subset of* S*.*

Proof. Assume f is a fuzzy quasi-filter of S. Let $x, y \in S$ such that $x \leq y$. Since f is a fuzzy quasi-filter of $S, f(x) \leq f(y)$. This implies $f'(x) \geq f'(y)$. Next, let $x, y, z \in S$ such that $xy = zx$. Since f is a fuzzy quasi-filter of $S, f(xy) \leq f(x)$. Thus $f'(xy) \geq f'(x)$. Finally, let $x, y \in S$. Since f is a fuzzy quasi-filter of $S, f(xy) \geq \min\{f(x), f(y)\}$. By Lemma 2.1, $f'(xy) \leq \max\{f'(x), f'(y)\}$.

Conversely, assume f' is a prime fuzzy quasi-ideal subset of S. Let $x, y \in S$ such that $x \leq y$. Since f' is a fuzzy quasi-ideal subset of $S, f'(x) \geq f'(y)$. Therefore $f(x) \leq f(y)$. Next, let $x, y \in S$. Since f' is prime, $f'(xy) \leq \max\{f'(x), f'(y)\}$. By Lemma 2.1, we have $f(xy) \geq \min\{f(x), f(y)\}$. Finally, let $x, y, z \in S$ such that $xy = zx$. Since f' is a fuzzy quasi-ideal subset of $S, f'(xy) \geq f'(x)$. Hence $f(xy) \leq f(x)$. □

Corollary 2.1. *Let* F *be a nonempty subset of an ordered semigroup* S*. Then* F *is a quasi-filter of* S *if and only if* f'_F *is a prime fuzzy quasi-ideal subset of* S*.*

Proof. It follows by Theorem 2.2 and Theorem 2.6. □

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References

[1] Y. Cao, Chain decompositions of ordered semigroups, *Semigroup Forum*, **65** (2002), 83-106.

[2] N. Kehayopulu, M. Tsingelis, A note on fuzzy sets in semigroups, *Sci. Math.*, **2** (1999), 411-413.

[3] N. Kehayopulu, M. Tsingelis, A note on fuzzy sets of groupoids-semigroups, *Sci. Math.*, **3** (2000), 247-250.

[4] N. Kehayopulu, M. Tsingelis, Fuzzy sets in ordered groupoids, *Semigroup Forum*, **65** (2002), 128-132.

[5] N. Kehayopulu, X.Y. Xie, M. Tsingelis, A characterization of prime and semiprime ideals of groupoids in terms of fuzzy subsets, *Soochow J. Math.*, **27** (2001), 139-144.

[6] N. Kuroki, On fuzzy semigroups, *Inform. Sci.*, **53** (1991), 203-236.

[7] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, **35** (1971), 512-517.

[8] M. Shabir, A. Khan, Fuzzy filters in ordered semigroups, *Lobachevskii J. Math.*, **29**(2008), 82-89.

[9] X.Y. Xie, F. Yan, Fuzzy ideals extensions of ordered semigroups, *Lobachevskii J. Math.*, **19** (2005), 29-40.

[10] L.A. Zadeh, Fuzzy sets, *Inform. Cont.*, **8** (1965), 338-353.