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ON THE K-THEORY OF VECTOR BUNDLES ON CURVES

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be a smooth genus $g \geq 2$ curve. Let $\mathcal{P}(X)$ be the exact category of vector bundles on X $I \subset \mathbb{R}$ an interval and $\mathcal{P}(I)$ the exact subcategory generated by the bundles with slopes in I. Here we prove that the inclusion $\mathcal{P}(I) \to \mathcal{P}(X)$ induces isomorphisms of K-groups $K_i(\mathcal{P}(I)) \to K_i(\mathcal{P}(X))$ for all integers $i \geq 0$.

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1. Introduction

Let X be a smooth and connected projective curve of genus $g \geq 2$ defined over an algebraically closed field $\mathbb{K}$. For any vector bundle E on X, let $\mu(E) := \deg(E)/\mathrm{rank}(E)$ denote the slope of E. If $E_1 \subsetneqq E_2 \subsetneqq \cdots \subsetneqq E_s = E$, $s \geq 1$, is the Harder-Narasimhan filtration of E set $\mu_+(E) := \mu(E_1)$ and $\mu_-(E) := \mu(E_s/E_{s-1})$ (with the convention $E_0 := 0$). Let $\mathcal{P}(X)$ denote the exact category of all vector bundles on X. For each non-empty connected interval $I \subseteq \mathbb{R}$ (possibly of length zero) let $\mathcal{P}(I)$ denote the full subcategory of $\mathcal{P}(X)$ consisting of all vector bundles E on X such that $\mu_-(E) \in I$ and $\mu_+(E) \in I$ ([1], §2). Here we will follows some of ideas contained in [3] to answer a question raised in that paper ([3], Question 1.2) and prove the following result.

Theorem 1. *Assume $g \geq 2$. Fix any interval $I \subseteq \mathbb{R}$ of strictly positive length. Then the inclusion $\mathcal{P}(I) \to \mathcal{P}(X)$ induces isomorphisms of K-groups $K_i(\mathcal{P}(I)) \to K_i(\mathcal{P}(X))$ for all integers $i \geq 0$.*

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The case $g = 1$ of Theorem 1 is true ([3], Theorem 1.1).

2. The Proofs

Proposition 1. *Fix a real number $\epsilon > 0$, a vector bundle E on X, an integer d such that $\mu_-(E) - d \geq \max\{2g, g/\epsilon\}$, and any $L \in \mathrm{Pic}^d(X)$. Set $r := \mathrm{rank}(E)$, $a := \deg(E)$ and $b := a + r(1-g)$. Then $b \geq r$ and there is a surjection $\tau : L^{\oplus b} \to E$ such that $d-1-\epsilon \leq \mu_-(\mathrm{Ker}(\tau)) \leq \mu_+(\mathrm{Ker}(\tau)) < d-1$. If E is semistable, then $\mathrm{Ker}(\tau)$ is semistable.*

Proof. For any vector bundle A on X le $e_A : \mathcal{O}_X \otimes H^0(X,A) \to A$ denote the evaluation map. If e_A is surjective, i.e. if A is spanned, set $M_A := \mathrm{Ker}(e_A)$. Notice that $h^1(X,A) = 0$ if $\mu_-(A) > 2g-2$ and that e_A is surjective if $\mu_-(A) > 2g-1$. If $\mu_-(A) > 2g-1$, then $\deg(M_A) = -\deg(A)$ and $\mathrm{rank}(M_A) = \deg(A) - g \cdot \mathrm{rank}(A)$. Hence, if $\mu_(A) > 2g-1$, then $\mu(A) = -1/(1-g/\mu(A))$. Taking $E \otimes L^*$ instead of E we reduce to the case $d = 0$ and $L \cong \mathcal{O}_X$. In this case one of our assumptions on d becomes the inequality $\mu_-(E) \geq 2g$. Hence $h^1(X,E) = 0$, $h^0(X,E) = b$ and evaluation map e_E is surjective. Set $F := \mathrm{Ker}(e_E)$. Let $E_1 \subsetneqq E_2 \subsetneqq \cdots \subsetneqq E_s = E$, $s \geq 1$ denote the Harder-Narasimhan filtration of E. Set $E_0 := 0$ and $F_i := E_i/E_{i-1}$, $1 \leq i \leq s$. Set $G := \oplus_{i=1}^s F_i$. Hence G is the graded object obtained from the Harder-Narasimhan filtration of E. Since $\mu_-(F_i) \geq \mu_-(E) \geq 2g$ for all i, $h^1(X,F_i) = 0$ for all i, $b = h^0(X,G)$ and the evaluation maps e_G and e_{F_i}, $1 \leq i \leq s$, are surjective. Each M_{F_i} is semistable ([2], Theorem 1.2). We stress that this part of [2] is characteristic-free. Since $M_G \cong \oplus_{i=1}^s M_{F_i}$, we get $\mu_+(e_G) = \mu(e_{F_1}) = -1/(1-g/\mu_-(E))$ and $\mu_-(M_G) = \mu_(M_{F_s}) = -1/(1-g/\mu_+(E))$. Notice that $\mu_+(G) < -1$. Since $\mu_+(E) \geq g/\epsilon$, we have $-1/(1-g/\mu_+(E)) \geq -1-\epsilon$. Hence all inequalities in the statement of Proposition 1 are true if we take G instead of E. Consider any extension of vector bundles on X

$$0 \to A \to B \to D \to 0 \tag{1}$$

and call α the extension class of (1). For each $t \in \mathbb{K}\backslash\{0\}$ the extension of D by A given by the extension class $t\alpha$ has a vector bundle isomorphic to B as its middle term. Hence we we see that $A \oplus D$ is a flat limit of family of vector bundles isomorphic to B. Iterating this trick we see that G is a flat limit of a family T of vector bundles isomorphic to E. Since $h^1(X,G) = h^1(X,E) = 0$ and both G and E are spanned, the family T shows that M_G is a flat limit of a family of vector bundles isomorphic to M_E. The semicontinuity of the Harder-Narasimhan polygon gives $-1-\epsilon \leq \mu_-(M_E) \leq \mu_+(M_E) < 1$. The last assertion of Proposition 1 is just [2], Theorem 1.2 (after the reduction to the case $d = 0$ and $L \cong \mathcal{O}_X$). $\square$

Remark 1. For any interval I the category $\mathcal{P}(I)$ is an exact category whose short exact sequences are the short exact sequences in $\mathcal{P}(X)$ whose terms are in $\mathcal{P}(I)$ ([3], Lemma 2.1).

Proof of Theorem 1. Without losing generality we may assume $I = (a, b)$.

(a) Fix any $c \in \mathbb{R}$. The proof in Step II of the proofs of [3], Theorem 1.1, does not depend from the genus of X and gives that the inclusion $\mathcal{P}((c, +\infty)) \to \mathcal{P}(X)$ induces isomorphisms of all K-groups.

(b) Set $\epsilon := (b-a)/2$ and take as c any real number such that $c > 2g(1+1/\epsilon)$. Here we will prove that the inclusion $\mathcal{P}(I) \to \mathcal{P}((c, +\infty))$ induces isomorphisms of all K-groups. By part (a) this will be sufficient to prove Theorem 1. It is sufficient to prove that every semistable vector bundle E on X with $\mu(E) > c$ has a finite resolution by elements in $\mathcal{P}(I)$. This is true by Proposition 1. $\square$

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